Homework # 25. Due to Wednesday, May 27, 11:00 am

(1) Prove that any $K(\mathbb{Z}_2, n)$ is infinite dimensional space for each $n \geq 1$.

(2) Let $M$ be a simply-connected compact closed manifold with $\dim M = 3$. Prove that $M$ is homotopy equivalent to $S^3$.

(3) Construct a space $X$ with $H_*(X) = H_*(pt)$, which is not homotopy equivalent to a point.

(4) Let $h : S^3 \to S^2$ be the Hopf map. Let $\lambda \geq 1$ be an integer. Define a map

$$f_\lambda : S^3 \xrightarrow{h} S^3 \lor \cdots \lor S^3 \xrightarrow{h \lor \cdots \lor h} S^2.$$ 

Prove that the space $X_\lambda = S^2 \cup f_\lambda D^4$ is homotopy equivalent to a closed compact manifold of dimension four if and only if $\lambda = 1$.

(5) Let $D^3 \subset T^3$ and $c : T^3 \to S^3$ be a map which collapses a complement of $D^3 \subset T^3$ to a point. Prove that the map $g : T^3 \xrightarrow{c} S^3 \xrightarrow{h} S^2$ (where $h : S^3 \to S^2$ is the Hopf map) induces trivial homomorphism on homology and homotopy, but is not homotopic to a constant map.