Homework # 22. Due to Wednesday, May 6, 11:00 am

(1) Compute the ring structure of $H^* (\mathbb{RP}^n; \mathbb{Z}_{2k})$ for any $k \geq 1$.

(2) Prove that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ are not homotopy equivalent.

(3) Recall that $\mathbb{RP}^{2n+1} = S^{2n+1} / \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{-1, 1\}$ which acts on $S^{2n+1}$ by taking $z \in S^{2n+1}$ to $-z$. Similarly, we identify $\mathbb{CP}^n$ with the space of orbits $S^{2n+1}/S^1$, where an element $e^{i\theta} \in S^1$ sends $z = (z_1, \ldots, z_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{2n+2}$ to $e^{i\theta} z = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1})$. We identify the group $\mathbb{Z}_2 = \{-1, 1\}$ with $\{e^{i\pi}, e^{i0}\} \subset S^1$, then we have a natural map $f_n : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ which makes the following diagram commute:

\[
\begin{array}{ccc}
S^{2n+1} & \xrightarrow{Id} & S^n \\
p & & h \\
\mathbb{RP}^{2n+1} & \xrightarrow{f_n} & \mathbb{CP}^n
\end{array}
\]

where $h : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf bundle. Notice that $f_n : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle with a fiber $S^1 = S^1 / \{e^{i\pi}, e^{i0}\}$. Taking $n \to \infty$, we get a map $f_\infty : \mathbb{RP}^\infty \to \mathbb{CP}^\infty$. Finally, the assignment: compute the ring homomorphism

$$f_\infty^* : H^*(\mathbb{CP}^\infty; \mathbb{Z}_2) \to H^*(\mathbb{RP}^\infty; \mathbb{Z}_2).$$

(4) Let $M$ be an oriented compact connected manifold, $\dim M = n$, it is equivalent to the fact that the homology group $H_n (M; \mathbb{Z}) \cong \mathbb{Z}$. Then a generator of $H_n (M; \mathbb{Z})$ is called a fundamental class and is denoted by $[M]$. Then we say that a map $f : M \to M$ has degree $\lambda$ if $f_* ([M]) = \lambda [M]$. Now the assignment: let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be a map of degree 64. Find the dimension of $\mathbb{CP}^n$.

(5) Prove that any map $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ has a fixed point.