Homework # 2. Due to Wednesday, October 23, 11:00 am

(1) Define the Stiefel manifolds \( V_k(\mathbb{R}^n) \). Prove the homeomorphisms:

\[
V_n(\mathbb{R}^n) \cong O(n), \quad V_{n-1}(\mathbb{R}^n) \cong SO(n), \quad V_1(\mathbb{R}^n) \cong S^{n-1}.
\]

(2) Prove the homeomorphisms:

\[
V_n(\mathbb{C}^n) \cong U(n), \quad V_{n-1}(\mathbb{C}^n) \cong SU(n).
\]

(3) Construct a homeomorphism \( SO(4) \cong SO(3) \times S^3 \).

(4) Define a natural actions of the groups \( O(k) \) and \( U(k) \) on the spaces the \( V_k(\mathbb{R}^n) \) and \( V_k(\mathbb{C}^n) \) respectively.

(a) Prove that those actions are free.

(b) Prove the homeomorphisms:

\[
V_k(\mathbb{R}^n)/O(k) \cong G_k(\mathbb{R}^n), \quad V_k(\mathbb{C}^n)/U(k) \cong G_k(\mathbb{C}^n)
\]

(5) Let \( X, Y, Z \) be Hausdorff and locally-compact topological spaces. Define the compact-open topology on \( C(X, Y) \). Prove the homeomorphism:

\[
C(X, C(Y, Z)) \cong C(X \times Y, Z).
\]

Prove that this homeomorphism is natural.

(6) Define smash-product \( X \wedge Y \). Prove that \( S^n \wedge S^k \cong S^{n+k} \) (as pointed spaces).

(7) Prove that the maps \( \phi^* : [X', Y] \to [X, Y] \), \( \psi_* : [X, Y] \to [X, Y'] \) induced by maps \( \phi : X \to X' \), \( \psi : Y \to Y' \) are well-defined.

(8) Define a cylinder and a cone of a map \( f : X \to Y \). Prove that the cone of the Hopf map \( H : S^{4n+3} \to \mathbb{HP}^n \) is homeomorphic to \( \mathbb{HP}^{n+1} \).

(9) Let \( X \) be a projective plane \( \mathbb{RP}^2 \) with \( k \) small (open) disjoint disks \( D_i^2 \) \((i = 1, \ldots, k) \) deleted. Let \( Y \) be the space obtained by gluing \( X \) with \( k \) Möbius bands along the boundary circles. Which surface is that?