QUESTIONS FOR THE MIDTERM, SPRING 2020

1. Basic spaces: $\mathbb{R}^n$, $S^n$, stereographic projection. The space $S^\infty$.

2. Projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$: definitions, local coordinate system, the Hopf maps $S^n \to \mathbb{RP}^n$, $S^{2n+1} \to \mathbb{CP}^n$, $S^{4n+3} \to \mathbb{HP}^n$.

3. Prove the homeomorphisms: $\mathbb{RP}^1 \cong S^1$, $\mathbb{CP}^1 \cong S^2$, $\mathbb{HP}^1 \cong S^4$.

4. Prove that $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ are connected and compact spaces.

5. Define Grassmannian manifolds $G_k(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$: and construct local coordinate systems, in particular, find their dimensions.

6. Prove that the Grassmannian manifolds $G_k(\mathbb{R}^n)$, and $G_k(\mathbb{C}^n)$ are compact and connected.

7. Define classic Lie groups $GL(\mathbb{R}^k)$, $GL(\mathbb{C}^k)$, $O(k)$, $SO(k)$, $U(k)$, $SU(k)$. Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact. How many connected components does each of these spaces have?

8. Prove that $SO(2)$ and $U(1)$ are homeomorphic to $S^1$, $SO(3)$ is homeomorphic to $\mathbb{RP}^3$, and $SU(2)$ is homeomorphic to $S^3$.

9. Prove that $SO(4) \cong SO(3) \times S^3$.

10. Define Stiefel manifolds $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$, $V_k(\mathbb{H}^n)$. Prove the following homeomorphisms:

\[ V_n(\mathbb{R}^n) \cong O(n), \quad V_{n-1}(\mathbb{R}^n) \cong SO(n), \]

\[ V_n(\mathbb{C}^n) \cong U(n), \quad V_{n-1}(\mathbb{C}^n) \cong SU(n), \]

\[ V_1(\mathbb{R}^n) \cong S^{n-1}, \quad V_1(\mathbb{C}^n) \cong S^{2n-1}, \quad V_1(\mathbb{H}^n) \cong S^{4n-1}. \]

11. Define action of the groups $O(k)$, $U(k)$ on the Stiefel manifolds $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$. Prove the following homeomorphisms: $V_k(\mathbb{R}^n)/O(k) \cong G_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)/U(k) \cong G_k(\mathbb{C}^n)$.

12. Prove the following homeomorphisms:

\[ S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1), \]

\[ S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1), \]

\[ G_k(\mathbb{R}^n) \cong O(n)/O(n) \times O(n-k), \quad G_k(\mathbb{C}^n) \cong U(n)/U(k) \times U(n-k). \]

13. Prove that the Klein bottle $K^2$ is homeomorphic to the union of two M"{e}bius bands along the circle.

14. Prove that $K^2 \# \mathbb{RP}^2$ is homeomorphic to $\mathbb{RP}^2 \# T^2$.

15. Define a cylinder and a cone of a map $f : X \to Y$. Prove that the cones of the maps $c : S^n \to \mathbb{RP}^n$ and $h : S^{2n+1} \to \mathbb{CP}^n$ are homeomorphic to $\mathbb{RP}^{n+1}$ and $\mathbb{CP}^{n+1}$ respectively.

16. Define suspension. Prove that $\Sigma(S^n) \cong S^{n+1}$.

17. Define a compact-open topology on $\mathcal{C}(X,Y)$. Prove the homeomorphism: $\mathcal{C}(X,\mathcal{C}(Y,Z)) \cong \mathcal{C}(X \times Y,Z)$ for Hausdorff and locally compact spaces $X$, $Y$, $Z$. Prove that this homeomorphism is natural.
18. Define the spaces of paths $\mathcal{E}(X, x_0, x_1)$, $\mathcal{E}(X, x_0)$, and loops $\Omega(X, x_0)$. Prove that the spaces $\Omega(S^n, x_0)$ and $\Omega(S^n, x_1)$ are homeomorphic for any points $x_0, x_1 \in S^n$.

19. Let $X$, $Y$ be pointed spaces. Prove the homeomorphism $\mathcal{C}(\Sigma(X), Y) \cong \mathcal{C}(X, \Omega(Y))$ for Hausdorff and locally compact spaces $X$, $Y$. Prove that this homeomorphism is natural.

20. Define smash-product $X \wedge Y$. Prove that $S^n \wedge S^k \cong S^{n+k}$ (as pointed spaces).

21. Define homotopy of two maps. Prove that the maps $\phi^* : [X', Y] \to [X, Y]$, $\psi_* : [X, Y] \to [X, Y']$ induced by maps $\phi : X \to X'$, $\psi : Y \to Y'$ are well-defined.

22. Give three definitions of homotopy equivalence. Prove that they are equivalent.

23. Prove that $X \sim Y$ implies $\Sigma(X) \sim \Sigma(Y)$ and $\Omega(X) \sim \Omega(Y)$.

24. Define when a pair $(X, Y)$ is a Borsuk pair. Prove that a pair $(X, Y)$ is a Borsuk pair (in the case when $X$, $Y$ are finite complexes).

25. Outline a construction of Schubert cells of the complex Grassmannian $G_k(C^n)$.

26. Define the Schubert cells $e(\sigma)$ corresponding to the Schubert symbol $\sigma$. Give examples.

27. Define the spaces $H^j$, $\bar{H}^j$. Prove that a $k$-plane $\pi$ belongs to $e(\sigma)$ if and only if there exists its basis $v_1, \ldots, v_k$, such that $v_1 \in H^{\sigma_1}$, $\ldots$, $v_k \in H^{\sigma_k}$.

28. Prove the following statement: Let $\pi \in e(\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$. Then there exists a unique orthonormal basis $v_1, \ldots, v_k$ of $\pi$, so that $v_1 \in H^{\sigma_1}$, $\ldots$, $v_k \in H^{\sigma_k}$.

29. Define the sets $E(\sigma), \bar{E}(\sigma) \subset V_k(\mathbb{R}^n)$. Prove that the set $E(\sigma) \subset V_k(\mathbb{R}^n)$ is homeomorphic to the closed cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_k - k)$. Furthermore the map $q : e(\sigma) \to E(\sigma)$ is a homeomorphism.

30. Define the transformations $T_{a,v}$, prove its properties. Explain how the transformations $T_{a,v}$ are used to prove that $\bar{E}(\sigma) \subset V(n, k)$ is homeomorphic to a closed cell of dimension $d(\sigma)$.

31. Prove the statement: a collection of \( \binom{k}{n} \) Schubert cells $e(\sigma)$ gives $G_k(\mathbb{R}^n)$ a cell-decomposition.

32. Outline a construction of Schubert cells of the complex Grassmannian $G_k(C^n)$.

33. Define when a pair $(X, Y)$ is a Borsuk pair. Prove that a CW-pair $(X, Y)$ is a Borsuk pair (in the case when $X$, $Y$ are finite complexes).

34. Let $(X, A)$ be a Borsuk pair. Prove that $A$ is a deformation retract of $X$ if and only if the inclusion $A \to X$ is a homotopy equivalence.
43. Prove the statement: let $X$ be a CW-complex and $A \subset X$ be its contractible subcomplex. Then $X$ is homotopy equivalent to the complex $X/A$.

44. Prove that for a CW-pair $(X, A)$ $X/A \sim X \cup C(A)$.


46. State and prove Free Point Lemma.

47. Define homotopy groups $\pi_n(X)$. Prove that $\pi_n(X)$ is commutative group for $n \geq 2$. Prove that $\pi_k(S^n)$ is a trivial group for $k < n$.

48. Prove the statement: Let $X$ be a CW-complex with only one zero-cell and without cells of dimension $q < n$, and $Y$ be a CW-complex of dimension $< q$. Then any map $Y \to X$ is homotopic to a constant map.

49. Define $n$-connected space. Prove the statement: Any $n$-connected CW-complex homotopy equivalent to a CW-complex with a single zero cell and without cells of dimensions $1, 2, \ldots, n$.

50. Prove that if $f, g : X \to Y$ are homotopic maps, then the homomorphisms $f_*, g_* : \pi_n(X) \to \pi_n(Y)$ coincide.

51. Prove that if $X$ is a path-connected space, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. Describe all isomorphisms here.

52. Prove that $\pi_1 S^1 \cong \mathbb{Z}$.

53. Prove that $\pi_1(\bigvee_{a \in A} S^1_a)$ is a free group.

54. Prove that $\pi_1(X, x_0) \cong \pi_1(X^{(2)}, x_0)$, where $X$ is a connected CW-complex and $X^{(2)}$ its 2-skeleton.

55. Compute $\pi_1(M^2)$ for two-dimensional oriented closed manifold of genus $g$, the sphere with $g$ handles.

56. Compute $\pi_1(M^2)$ for two-dimensional non-oriented closed manifold of genus $g$, the projective plane or the Klein bottle with $g$ handles.

57. Let $M = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ ($n$ times). Compute $\pi_1(M)$.

58. Compute $\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2)$ and $\pi_1(Kl^2 \# \mathbb{R}P^2)$.

59. Define $G_1 * G_2$. Give examples. Prove that $\pi_1(X \lor Y) = \pi_1(X) * \pi_1(Y)$.

60. Define $G_1 * H \lor G_2$. Give examples. State and prove Van Kampen Theorem.

61. Define covering space. Give examples. Construct $n$-fold covering of $S^1 \lor S^1$ (including $n = \infty$).

62. State and prove Theorem on Covering Homotopy.

63. Prove that covering $p : T \to X$ induces a monomorphism $p_* : \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$.

64. Prove that a loop $\alpha_1 \cdots \alpha_k$, where $\alpha_j$ is a loop going along the $j$-th circle in the wedge $\bigvee_{j=1}^k S^1_j$, is not homotopic to zero.

65. Let $p : T \to X$ be a covering, and $f, g : Z \to T$ be two maps so that $p \circ f = p \circ g$, where $Z$ is path-connected. Assume that $f(z) = g(z)$ for some point $z \in Z$. Prove that $f = g$.

66. Prove that $\pi_k(\mathbb{R}P^n) = 0$ if $1 < k < n$.

67. Prove that any map $f : \mathbb{R}P^2 \to S^1$ is homotopic to a constant map.

68. Let $Kl^2$ be the Klein bottle. Construct two-folded covering space $T^2 \to Kl^2$. Compute $\pi_n(Kl^2)$ for all $n$. 

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Let \( p : T \to X \) be a covering, \( p(\tilde{x}_0) = x_0 \). Prove that there is one-to-one correspondence
\[
\pi_1(X(X,x_0)/p_*(\pi_1(T,\tilde{x}_0))) \iff p^{-1}(x_0).
\]

Prove that \( p^{-1}(x_0) \cong p^{-1}(x_1) \) for any points \( x_0, x_1 \in X \).

Let \( p : T \to X \) be a covering map, and let \( \Gamma = p^{-1}(x_0) \). Prove that \( \Gamma \) is a transitive right \( G \)-set for \( G = \pi_1(X,x_0) \).

Let \( X \) by "good" space and \( G = \pi_1(X,x_0) \). Prove that there is a bijection between isomorphism classes of covering spaces of \( X \) and transitive right \( G \)-sets given by
\[
\{ p : Y \to X \} \mapsto p^{-1}(x_0).
\]

Let \( p : T \to X \) be a covering, and \( f : Z \to X \) be a map, \( f(z_0) = x_0 \), and \( \tilde{x}_0 \in T \) so that \( p(\tilde{x}_0) = x_0 \)
(here \( Z \) is path-connected). Prove that there exists a lifting \( \tilde{f} : Z \to T \) of the map \( f \) so that
\[
\tilde{f}(z_0) = \tilde{x}_0 \quad \text{if and only if} \quad f_*(\pi_1(Z,z_0)) \subset p_*(\pi_1(T,\tilde{x}_0)).
\]

Define morphism of two covering spaces \( T_1 p_1 \to X \) and \( T_2 p_2 \to X \). Prove that two morphisms \( \phi, \phi' : T_1 \to T_2 \) coincide if there is a point \( \tilde{x} \in T_1 \) so that \( \phi(\tilde{x}) = \phi'(\tilde{x}) \).

Define a group of automorphisms (deck transformations) \( \text{Aut}(T \xrightarrow{P} X) \) of a covering \( p : T \to X \).
Prove that the group \( \text{Aut}(T \xrightarrow{P} X) \) acts on \( T \) without fixed points.

Let \( p : T \to X \) be a covering, \( p(\tilde{x}_0) = p(\tilde{x}_0') = x_0 \), where \( \tilde{x}_0 \neq \tilde{x}_0' \). Prove that there exists an automorphism \( \phi \in \text{Aut}(T \xrightarrow{P} X) \) such that \( \phi(\tilde{x}_0) = \tilde{x}_0' \) if and only if \( p_*(\pi_1(T,\tilde{x}_0)) = p_*(\pi_1(T,\tilde{x}_0')) \).

Prove the following statement: Two covering spaces \( T_1 \xrightarrow{P_1} X, T_2 \xrightarrow{P_2} X \) are isomorphic if and only if for any two points \( \tilde{x}_1, \tilde{x}_2 \in T \) such that \( p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x \) the groups \( (p_1)_*(\pi_1(T_1,\tilde{x}_1)), (p_2)_*(\pi_1(T_2,\tilde{x}_2)) \) belong to the same conjugacy class in \( \pi_1(X,x) \).

Let \( N(H) \) be a normalizer for a subgroup \( H \) of \( G \). Prove the following statement: Let \( p : T \to X \) be a covering space.
Then the group of automorphisms of this covering space is isomorphic to the group \( N(p_*(\pi_1(T,\tilde{x}_0)))/p_*(\pi_1(T,\tilde{x}_0)) \).

Define universal covering space over \( X \). Prove the following statement: Let \( X \) be a path-connected \( CW \)-complex, \( x_0 \in X \). Then for any subgroup \( G \subset \pi_1(X,x_0) \) there exists a covering \( T \xrightarrow{P} X \) and a point \( \tilde{x}_0 \in T \) so that \( p_*(\pi_1(T,\tilde{x}_0)) = G \).

Define homotopy groups \( \pi_n(X,x_0) \), in particular define the group operation and inverse. Prove that the groups \( \pi_n(X,x_0) \) are abelian if \( n \geq 2 \).

Prove that \( \pi_n(X \times Y, x_0 \times y_0) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0) \). Compute \( \pi_n(T^k) \) for all \( n \).

Let \( X \) be a path-connected space, and \( x_0, x_1 \in X \) be two different points. Let \( \gamma : I \to X \) be a path so that \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). Define a homomorphism \( \gamma_* : \pi_n(X,x_0) \to \pi_n(X,x_1) \). Prove that \( \gamma_* \) is an isomorphism.

Let \( M_g^2 \) be a two-dimensional surface of genus \( g \geq 1 \) (oriented). Compute the homotopy groups \( \pi_q(M_g^2) \).

Define relative homotopy groups \( \pi_n(X,A;x_0) \). Describe the group operation and the inverse element. Prove that the group \( \pi_n(X,A;x_0) \) is commutative for \( n \geq 3 \).

Define the homomorphisms in the following sequence:
\[
\cdots \to \pi_n(A,x_0) \xrightarrow{\iota_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A;x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \cdots
\]
Prove that the sequence (1) is exact.
85. Define the covering homotopy property. Outline a proof that the covering homotopy property holds.

86. Define a Serre fiber bundle. Let \( E \rightarrow B \) be a Serre fiber bundle.

87. State and prove Five-Lemma.

88. Let \( 0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow 0 \) be an exact sequence of finitely generated abelian groups. Prove that \( \sum_{i=1}^{n} (-1)^i \text{rank } A_i = 0 \).

89. Define locally trivial fiber bundle. Give several examples of non-trivial fiber bundles.

90. Prove that any locally-trivial fiber bundle over the cube \( I^2 \) is trivial.

91. Define the covering homotopy property. Outline a proof that the covering homotopy property holds for a locally-trivial fiber bundle \( E \rightarrow B \).

92. Define a Serre fiber bundle. Let \( Y \) be an arbitrary path-connected space, \( E(Y,y_0) \) be the space of paths starting at \( y_0 \). Prove that the map \( p : E(Y,y_0) \rightarrow Y \), where \( p(s : I \rightarrow Y) = s(1) \in Y \) is a Serre fiber bundle.

93. Let \( A \subset X \), and \( (X,A) \) be a Borsuk pair (for example, a \( CW \)-pair) Let \( E = C(X,Y) \), \( B = C(A,Y) \), and the map \( p : E \rightarrow B \) be defined as \( p(f : X \rightarrow Y) = (f|_A : A \rightarrow Y) \). Prove that the map \( p : E \rightarrow B \) is a Serre fiber bundle.

94. Define weak homotopy equivalence. Prove that finite \( CW \)-complexes \( X \), \( Y \) are weak homotopy equivalent if and only if they are homotopy equivalent.

95. Let \( p : E \rightarrow B \) be Serre fiber bundle, where \( B \) be a path-connected space. Prove that the fibers \( F_0 = p^{-1}(x_0) \) and \( F_1 = p^{-1}(x_1) \) are weak homotopy equivalent for any two points \( x_0, x_1 \in B \).

96. Prove that for any continuous map \( f : X \rightarrow Y \) there exists homotopy equivalent map \( f_1 : X_1 \rightarrow Y_1 \), such that \( f_1 : X_1 \rightarrow Y_1 \) is Serre fiber bundle.

97. Let \( f : X \rightarrow Y \) be a continuous map. Prove that there exists a homotopy equivalent map \( g : X \rightarrow Y' \), so that \( g \) is an inclusion.

98. Let \( p : E \rightarrow B \) be Serre fiber bundle, \( y \in E \) be any point, \( x = p(y) \), \( F = p^{-1}(x) \). Prove that the homomorphism \( p_* : \pi_n(E,F;y) \rightarrow \pi_n(B,x) \)

99. Apply the homotopy exact sequence of Serre fibration to prove that (a) \( \pi_2(S^2) = \pi_1(S^1) = \mathbb{Z} \); (b) \( \pi_n(S^1) = \pi_n(S^1) \).

100. Let \( S^\infty \rightarrow \mathbb{C}P^\infty \) be the Hopf fibration. Using the fact \( S^\infty \sim * \), prove that \( \pi_n(\mathbb{C}P^\infty) = 0 \) for \( n \neq 2 \), and \( \pi_2(\mathbb{C}P^\infty) = \mathbb{Z} \).

101. Prove that \( \pi_n(\Omega(X)) \cong \pi_{n+1}(X) \) for any \( X \) and \( n \geq 0 \).

102. Prove that if the groups \( \pi_*(B) \), \( \pi_*(E) \) are finite (finitely generated), then the groups \( \pi_*(E) \) are finite (finitely generated) as well.
103. Assume that a fiber bundle \( p : E \rightarrow B \) has a section, i.e. a map \( s : B \rightarrow E \), such that \( p \circ s = Id_B \). Prove the isomorphism \( \pi_n(E) \cong \pi_n(B) \oplus \pi_n(F) \).

104. Give a construction of a space \( Y \) that \( \pi_n(X, A; x_0) \cong \pi_{n-1}(Y, y_0) \).

104. State the Freudenthal Theorem. Give a detailed proof that \( \Sigma \) is an isomorphism.

106. Let \( K, L \subset \mathbb{R}^2 \) be two finite simplicial complexes fo dimensions \( k, l \) respectively. Let \( k + l + 1 < p \). Prove that the simplicial complexes \( K \) and \( L \) are not linked.

107. Prove that \( \pi_n(S^n) \cong \mathbb{Z} \) for each \( n \geq 1 \).

108. Prove that \( \pi_3(S^2) \cong \mathbb{Z} \), and the Hopf map \( S^3 \rightarrow S^2 \) is a representative of the generator of \( \pi_3(S^2) \).

109. Define Whitehead product. State basic properties. Prove that if \( \alpha \in \pi_n(X) \), \( \beta \in \pi_k(X) \) then \([\alpha, \beta] = (-1)^{nk}[\beta, \alpha]\).

110. Define the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \). Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) has infinite order.

111. Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) is in a kernel of each of the following homomorphisms:
   1. \( i_* : \pi_{n+k-1}(S^n \vee S^k) \rightarrow \pi_{n+k-1}(S^n \times S^k) \),
   2. \( pr_i^n : \pi_{n+k-1}(S^n \vee S^k) \rightarrow \pi_{n+k-1}(S^n) \),
   3. \( pr_i^k : \pi_{n+k-1}(S^n \vee S^k) \rightarrow \pi_{n+k-1}(S^k) \).

112. Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) is in the kernel of the suspension homomorphism
   \[ \Sigma : \pi_{n+k-1}(S^n \times S^k) \rightarrow \pi_{n+k}(\Sigma(S^n \times S^k)) \].

113. Prove the isomorphism
   \[ \pi_{n+k}(S^n \vee S^k) \cong \pi_{n+k}(S^n) \oplus \pi_{n+k}(S^k) \]

114. Let \( \alpha \in \pi_n(X) \), \( \beta \in \pi_k(X) \). Prove that \([\alpha, \beta] \in \ker \Sigma \), where
   \[ \Sigma : \pi_{n+k-1}(X) \rightarrow \pi_{n+k}(\Sigma X) \]
   is the suspension homomorphism.

115. Let \( \iota_{2q} \in \pi_{2q}(S^{2q}) \) be a generator represented by the identity map \( S^{2q} \rightarrow S^{2q} \). Prove that the Whitehead product \([\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q}) \) is a nontrivial element of infinite order.

116. Prove that the suspension \( \Sigma(S^n \times S^k) \) is homotopy equivalent to the wedge \( S^{n+1} \vee S^{k+1} \vee S^{n+k+1} \).

117. Outline a proof of the following statement:
   Let \( X \) be a connected space (not necessarily a CW-complex) with a base point \( x_0 \in X \), \( f : S^n \rightarrow X \)
   be a map such that \( f(s_0) = x_0 \), where \( s_0 \) is a base point of \( S^n \). Let \( Y = X \cup_f D^{n+1} \), and \( i : X \rightarrow Y \)
   be the inclusion. Then the induced homomorphism \( i_* : \pi_q(X, x_0) \rightarrow \pi_q(Y, x_0) \)
   
   (1) is an isomorphism if \( q < n \),
   (2) is an epimorphism if \( q = n \), and
   (3) the kernel \( \ker i_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, x_0) \) is generated by \( \gamma^{-1}[f] \gamma \in \pi_n(X, x_0) \) where \( \gamma \in \pi_1(X, x_0) \).

118. Let \( X \) be an \( n \)-connected CW-complex, and \( Y \) be a \( k \)-connected CW-complex. Prove that
   - \( \pi_q(X \vee Y) \cong \pi_q(X) \oplus \pi_q(Y) \) if \( q \leq n + k \);
   - the group \( \pi_q(X \vee Y) \) contains a subgroup \( \pi_q(X) \oplus \pi_q(Y) \) as a direct summand.
119. Let $X$ be an $n$-connected $CW$-complex, and $Y$ be a $k$-connected $CW$-complex. Prove that
\[ \pi_{n+k+1}(X \vee Y) \cong \pi_{n+k+1}(X) \oplus \pi_{n+k+1}(Y) \oplus [\pi_n(X), \pi_k(Y)]. \]

120. Let $X$ be an $(n-1)$-connected $CW$-complex. Describe the homotopy group $\pi_n(X)$.

121. Compute the homotopy group $\pi_3(S^2 \vee S^2)$.

122. Define when a map $f : X \to Y$ is a weak homotopy equivalence. Outline the proof that the following two statements are equivalent

(1) The map $f : X \to Y$ is weak homotopy equivalence.

(2) The induced homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is isomorphism for all $n$ and $x_0 \in X$.

123. Let $X, Y$ be $CW$-complexes. Prove that if a map $f_* : X \to Y$ induces isomorphism
\[ f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \]
for all $n \geq 0$ and $x_0 \in X$, then $f$ is a homotopy equivalence.

124. Let $X$ be a Hausdorff topological space. Prove that there exists a $CW$-complex $K$ and a weak homotopy equivalence $f : K \to X$. Show that the $CW$-complex $K$ is unique up to homotopy equivalence.

125. Let $X, Y$ be two weak homotopy equivalent spaces. Prove that there exist a $CW$-complex $K$ and maps $f : K \to X$, $g : K \to Y$ which weak homotopy equivalences.

126. Define an Eilenberg-McLane space. Prove that it does exists and unique up to weak homotopy equivalence.

127. Construct the space $K(\pi, 1)$, where $\pi$ is a finitely generated abelian group.

128. Let $X = K(\pi, n)$. Prove that $\Omega X = K(\pi, n-1)$.

129. Let $X$ be a $CW$-complex, and $n \geq 1$. Construct a $CW$-complex $X_n$ and a map $f_n : X \to X_n$ such that

(1) $\pi_q(X_n) = \begin{cases} \pi_q(X) & \text{if } q \leq n \\ 0 & \text{else} \end{cases}$

(2) $(f_n)_* : \pi_q(X) \to \pi_q(X_n)$ is isomorphism if $q \leq n$.

130. Let $X$ be a $CW$-complex, and $n \geq 1$. Construct a $CW$-complex $X|_n$ and a map $g_n : X|_n \to X$ such that

(1) $\pi_q(X|_n) = \begin{cases} \pi_q(X) & \text{if } q \geq n \\ 0 & \text{else} \end{cases}$

(2) $(g_n)_* : \pi_q(X|_n) \to \pi_q(X)$ is isomorphism if $q \geq n$.

131. Let $X = S^2$. Prove that $X|_3 = S^3$.

132. Let $X = \mathbb{C}P^n$. Prove that $X|_3 = X|_{2n+1} = S^{2n+1}$.

133. Define the complex $\mathcal{C}(X)$ and the homology groups $H_q(X)$. Calculate the homology groups for $X = \{pt\}$.

134. Define chain maps and chain homotopy. Prove that two chain homotopic maps $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ induce the same homomorphism in homology groups.

135. Let $g, h : X \to Y$ be homotopic maps. Prove that $g_* = h_* : H_q(X) \to H_q(Y)$.

136. Let $X$ and $Y$ be homotopy equivalent spaces. Prove that then $H_q(X) \cong H_q(Y)$ for all $q$. 

7
137. Prove that $H_0(X) \cong \mathbb{Z}$ if $X$ is a path-connected space.

138. Prove that if $f : X \to Y$ is a map of path-connected spaces, then $f_* : H_0(X) \to H_0(Y)$ is an isomorphism.

139. Define relative homology groups. State and prove the LES-Lemma.

140. Let $B \subset A \subset X$ be a triple of spaces. Prove that there is a long exact sequence in homology:

$$
\cdots \to H_q(A,B) \overset{ι_*}{\to} H_q(X,B) \overset{j_*}{\to} H_q(X,A) \overset{∂}{\to} H_{q-1}(A,B) \overset{ι_*}{\to} \cdots
$$

141. Let $(X, A)$ be a pair of spaces. Prove that the inclusion $i : (X, A) \to (X \cup C(A), C(A))$ induces the isomorphism $H_q(X, A) \cong H_q(X \cup C(A), C(A)) = H_q(X \cup C(A), v)$.

142. Define the operation $β : C(X) \to C(X)$ (induced by the barycentric subdivision). Prove that the chain map $β : C(X) \to C(X)$ induces the identity homomorphism in homology:

$$
Id = β_* : H_q(C(X)) \to H_q(C(X)) \quad \text{for each } q \geq 0.
$$

143. Define the chain complex $C^U(X)$ for a covering $U$. Prove that the inclusion $C^U(X) \subset C(X)$ induces an isomorphism in homology groups.

144. State and prove the Excision Theorem.

145. Let $X = X_1 \cup X_2$. Prove that the following sequence of complexes is exact

$$
0 \to C(X_1 \cap X_2) \overset{α^*}{\to} C(X_1) \oplus C(X_2) \overset{β}{\to} C(X_1) + C(X_2) \to 0.
$$

146. Let $X_1, X_2 \subset X$, and $X_1 \cup X_2 = X$, $\overset{α}{X}_1 \cup \overset{α}{X}_2 = X$. Prove that the chain map

$$
C(X_1) + C(X_2) \to C(X_1 \cup X_2)
$$

induces isomorphism in the homology groups.

147. State and prove the Mayer-Vietoris Theorem.

148. Compute homology groups $H_q(S^n)$.

149. Let $X$ be a space. Prove that $\tilde{H}_{q+1}(\Sigma X) \cong \tilde{H}_q(X)$ for each $q$.

150. Let $A$ be a set of indices, and $S^n_α$ be a copy of the $n$-th sphere, $α \in A$. Compute the homology groups $\tilde{H}_q\left(\bigvee_{α \in A} S^n_α\right)$.

151. Let $(X_α, x_α)$ be based spaces, $α \in A$. Assume that the pair $(X_α, x_α)$ is Borsuk pair for each $α \in A$. Prove that

$$
\tilde{H}_q\left(\bigvee_{α \in A} X_α\right) = \bigoplus_{α \in A} \tilde{H}_q(X_α).
$$

152. Let $f : S^n \to S^n$ be a map of degree $d = \deg f$. Prove that $f_* : H_n(S^n) \to H_n(S^n)$ is a multiplication by $d$.

153. Let $g : \bigvee_{α \in A} S^n_α \overset{g}{\to} \bigvee_{β \in B} S^n_β$ be a map. Prove that the homomorphism

$$
\bigoplus_{α \in A} \mathbb{Z}(α) = H_n\left(\bigvee_{α \in A} S^n_α\right) \overset{g_*}{\to} H_n\left(\bigvee_{β \in B} S^n_β\right) = \bigoplus_{β \in B} \mathbb{Z}(β)
$$

is given by multiplication with matrix $\{d_{αβ}\}_{α \in A, β \in B}$, where $d_{αβ} = \deg g_{αβ}$. (Define the maps $g_{αβ}$.)
154. Define the cellular chain complex $\mathcal{E}(X)$. Prove that the following composition is trivial

$$\mathcal{E}_{q+1}(X) \xrightarrow{\partial_{q+1}} \mathcal{E}_q(X) \xrightarrow{\partial_q} \mathcal{E}_{q-1}(X).$$

155. Prove that there is an isomorphism $H_q(\mathcal{E}(X)) \cong H_q(X)$ for each $q$ and any CW-complex $X$.

156. Let $X$ be a CW-complex, and $e^q$ be a $q$-cell and $\sigma^{q-1}$ be a $(q-1)$-cell of $X$. Define the incidence coefficient $[e^q : \sigma^{q-1}]$. Prove that the boundary operator $\partial_q : \mathcal{E}_q(X) \to \mathcal{E}_{q-1}(X)$ is given by the formula:

$$\partial_q(e^q) = \sum_{j \in \mathcal{E}_{q-1}} [e^q : \sigma_j^{q-1}] \sigma_j^{q-1}.$$

157. Let $A : S^n \to S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \to S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

158. Let $e^0, \ldots, e^n$ be the cells in the standard cell decomposition of $\mathbb{RP}^n$. Prove that

$$[e^q : e^{q-1}] = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 0 & \text{if } q \text{ is even.} \end{cases}$$

159. Compute the homology groups $H_q(\mathbb{RP}^n)$, $H_q(\mathbb{CP}^n)$.

160. Compute the homology groups $H_q((\mathbb{RP}^n)^\#k)$ and $H_q((\mathbb{CP}^n)^\#k)$

161. Compute the homology groups $H_q(\mathbb{RP}^{2n}\#\mathbb{CP}^n)$.

162. Prove that there is no map $f : D^n \to S^{n-1}$ so that the restriction $f|_{S^{n-1}} : S^{n-1} \to S^{n-1}$ has nonzero degree.

163. Let $X$ be a topological space, $\alpha \in H_q(X)$. Prove that there exist a CW-complex $K$, a map $f : K \to X$, an element $\beta \in H_q(K)$ such that $f_*([\beta]) = \alpha$.

164. Let $f : X \to Y$ be a weak homotopy equivalence. Prove that the induced homomorphism $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for all $q \geq 0$.

165. Show that the spaces $\mathbb{CP}^\infty \times S^3$ and $S^2$ have isomorphic homotopy groups and that they are not homotopy equivalent.

166. Show that the spaces $\mathbb{RP}^n \times S^m$ and $S^n \times \mathbb{RP}^m$ ($n \neq m$) have isomorphic homotopy groups and they are not homotopy equivalent.

167. Show that the spaces $S^1 \vee S^1 \vee S^2$ and $S^3 \times S^1$ have the same homology groups and different homotopy groups.

168. Show that the projection

$$S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1)/(S^1 \vee S^1) = S^2$$

induces trivial homomorphism in homotopy groups.

169. Define the Hurewicz homomorphism $h : \pi_n(X, x_0) \to H_n(X)$. Prove that $h$ is a homomorphism.
170. Let \( x_0, x_1 \in X \), and \( \gamma : I \rightarrow X \) be a path connecting the points \( x_0, x_1 : \gamma(0) = x_0 \), and \( \gamma(1) = x_1 \). The path \( \gamma \) determines the isomorphism \( \gamma_\# : \pi_n(X, x_0) \rightarrow \pi_n(X, x_1) \). Prove that the following diagram commutes:

\[
\begin{array}{ccc}
\pi_n(X, x_0) & \xrightarrow{\gamma_\#} & \pi_n(X, x_1) \\
h & \downarrow & h \\
H_n(X) & & H_n(X)
\end{array}
\]

171. (Hurewicz Theorem) Let \( (X, x_0) \) be a based space, such that

\[
\pi_0(X, x_0) = 0, \ \pi_1(X, x_0) = 0, \cdots, \pi_{n-1}(X, x_0) = 0,
\]

where \( n \geq 2 \). Prove that

\[
H_1(X) = 0, \ \ H_2(X) = 0, \cdots, H_{n-1}(X) = 0,
\]

and the Hurewicz homomorphism \( h : \pi_{n-1}(X, x_0) \rightarrow H_n(X) \) is an isomorphism.

172. Let \( X \) be a simply-connected \( CW \)-complex with \( H_n(X) = 0 \) for all \( n \). Prove that \( X \) is contractible.

173. Let \( X \) be a simply connected space, and \( H_1(X) = 0, \ H_2(X) = 0 \cdots H_{n-1}(X) = 0 \). Prove that \( \pi_1(X) = 0, \pi_2(X) = 0 \cdots \pi_{n-1}(X) = 0 \) and the Hurewicz homomorphism \( h : \pi_n(X, x_0) \rightarrow H_n(X) \) is an isomorphism.

174. Consider the map

\[
g : S^{2n-2} \times S^3 \rightarrow (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{C}P^n.
\]

Prove that \( g \) induces trivial homomorphism in homology and homotopy groups, however \( g \) is not homotopic to a constant map.

175. Let \( X \) be a connected space. Prove that the Hurewicz homomorphism \( h : \pi_1(X, x_0) \rightarrow H_1(X) \) is epimorphism, and the kernel of \( h \) is the commutator \( [\pi_1(X, x_0), \pi_1(X, x_0)] \subset \pi_1(X, x_0) \).

176. State the relative version of the Hurewicz Theorem. State and prove the Whitehead Theorem II. Let \( X, Y \) be simply connected spaces and \( f : X \rightarrow Y \) be a map which induces isomorphism \( f_* : H_q(X) \rightarrow H_q(Y) \) for all \( q \geq 0 \). Prove that \( f \) is weak homotopy equivalence.

177. Define homology and cohomology groups with coefficients in an abelian group \( G \). Compute the groups \( H_q(\mathbb{R}P^n; \mathbb{Z}/p) \), \( H^q(\mathbb{R}P^n; \mathbb{Z}/p) \) for any prime \( p \).

178. Consider the short exact sequence \( 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0 \). Compute the connecting homomorphisms

\[
\partial = \beta^m : H^2(\mathbb{R}P^n; \mathbb{Z}/2) \rightarrow H^{q+1}(\mathbb{R}P^n; \mathbb{Z})
\]

179. Let \( G \) be an abelian group, \( 0 \rightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \rightarrow 0 \), be a free resolution of \( G \), and \( H \) be an arbitrary abelian group. Prove that the sequence

\[
0 \rightarrow \text{Ker}(\beta \otimes 1) \rightarrow R \otimes H \xrightarrow{\beta \otimes 1} F \otimes H \xrightarrow{\alpha \otimes 1} G \otimes H \rightarrow 0
\]

is exact.

180. Prove that the group \( \text{Tor}(G, H) \) is well-defined, i.e. it does not depend on the choice of resolution.

181. Let \( G, H \) be abelian groups. Prove that there is a canonical isomorphism \( \text{Tor}(G, H) \cong \text{Tor}(H, G) \).
182. Let $F$ be a free abelian group. Show that $\text{Tor}(F, G) = 0$ for any abelian group $G$.

183. Let $G$ be an abelian group. Denote $T(G)$ a maximal torsion subgroup of $G$. Show that $\text{Tor}(G, H) \cong T(G) \otimes T(H)$ for finite generated abelian groups $G, H$. Give an example of abelian groups $G, H$, so that $\text{Tor}(G, H) \neq T(G) \otimes T(H)$.

184. Let $X$ be a space, $G$ be an abelian group. Prove that there is a split short exact sequence

$$0 \rightarrow H_q(X) \otimes G \rightarrow H_q(X; G) \rightarrow \text{Tor}(H_{q-1}(X), G) \rightarrow 0$$

185. Let $G$ be an abelian group, $0 \rightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \rightarrow 0$ be a free resolution, and let $H$ be an abelian group. Prove that the following sequence is exact:

$$0 \leftarrow \text{Coker} \beta^\# \leftarrow \text{Hom}(R, H) \xrightarrow{\beta^\#} \text{Hom}(F, H) \xleftarrow{\alpha^\#} \text{Hom}(G, H) \leftarrow 0.$$

186. Prove that the group $\text{Ext}(G, H)$ is well defined, i.e. it does not depend on the choice of free resolution of $G$.

187. Let $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ be a short exact sequence of abelian groups. Prove that it induces the following exact sequence:

$$0 \rightarrow \text{Hom}(G'', H) \rightarrow \text{Hom}(G, H) \rightarrow \text{Hom}(G', H) \rightarrow \text{Ext}(G'', H) \rightarrow \text{Ext}(G, H) \rightarrow \text{Ext}(G', H) \rightarrow 0$$

188. Prove that $\text{Ext}(\mathbb{Z}, H) = 0$ for any group $H$.

189. Prove the isomorphisms: $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/m \otimes \mathbb{Z}/n$, $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$.

190. Let $X$ be a space, $G$ an abelian group. Prove that there is a split exact sequence

$$0 \rightarrow \text{Ext}(H_{q-1}(X), G) \rightarrow H^q(X; G) \rightarrow \text{Hom}(H_q(X), G) \rightarrow 0$$

for any $q \geq 0$.

191. Let $X$ be a space, and $G$ an abelian group. Prove that there is a split exact sequence

$$0 \rightarrow H^q(X; \mathbb{Z}) \otimes G \rightarrow H^q(X; G) \rightarrow \text{Tor}(H^{q+1}(X; \mathbb{Z}), G) \rightarrow 0$$

for any $q \geq 0$.

192. Let $G$ be a finitely generated abelian group. Let $F(G)$ be the maximum free abelian subgroup of $G$, and $T(G)$ be the maximum torsion subgroup. Let $X$ be a space such that the groups $H_q(X)$ are finitely generated for all $q$. Prove that $H^q(X; \mathbb{Z})$ are also finitely generated and $H^q(X; \mathbb{Z}) \cong F(H_q(X; \mathbb{Z})) \oplus T(H_{q-1}(X; \mathbb{Z}))$.

193. Let $F$ be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Prove that

$$H_q(X; F) = H_q(X) \otimes F, \quad H^q(X; F) = \text{Hom}(H_q(X), F).$$

194. Let $X$ be a finite CW-complex, and $\mathbb{F}$ be a field. Prove that the number

$$\chi(X)_{\mathbb{F}} = \sum_{q \geq 0} (-1)^q \dim H_q(X; \mathbb{F})$$

does not depend on the field $\mathbb{F}$ and is equal to the Euler characteristic

$$\chi(X) = \sum_{q \geq 0} (-1)^q \{\# \text{ of } q\text{-cells of } X\}.$$
195. Let a finite CW-complex $X$ be a union of two CW-subcomplexes: $X = X_1 \cup X_2$, where $X_1 \cap X_2 \subset X$ is a CW-subcomplex as well. Prove that

$$\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

196. Let $C_\ast$ and $C'_\ast$ be two chain complexes. Define the tensor product $\bar{C}_\ast = C_\ast \otimes C'_\ast$. Prove that $\partial_{q+1} \partial_q = 0$.

197. Let $E_\ast = E_\ast(X)$, $E'_\ast = E_\ast(X')$. Define the complexes $E_\ast(r)$, $E'_\ast(s)$ and compute the homology groups of the tensor product of these chain complexes: $E_\ast(r) \otimes E'_\ast(s)$.

198. Use the result of # 197 to prove the Künneth formula for homology groups:

$$0 \to \bigoplus_{r+s=q} H_r(X) \otimes H_s(X') \to H_q(X \times X') \to \bigoplus_{r+s=q-1} \operatorname{Tor}(H_r(X), H_s(X')) \to 0$$

199. Outline the proof of the Künneth formula for cohomology groups:

$$0 \to \bigoplus_{r+s=q} H^r(X) \otimes H^s(X') \to H^q(X \times X') \to \bigoplus_{r+s=q+1} \operatorname{Tor}(H^r(X), H^s(X')) \to 0.$$

200. Let $F$ be a field. Prove that

$$H_q(X \times X'; F) \cong \bigoplus_{r+s=q} H_r(X; F) \otimes H_s(X'; F),$$

$$H^q(X \times X'; F) \cong \bigoplus_{r+s=q} H^r(X; F) \otimes H^s(X'; F).$$

201. Let $\beta_q(X) = \operatorname{Rank}H_q(X)$ be the Betti number of $X$. Prove that

$$\beta_q(X \times X') = \sum_{r+s=q} \beta_r(X)\beta_s(X').$$

202. Let $X$, $X'$ be such spaces that their Euler characteristics $\chi(X)$, $\chi(X')$ are finite. Prove that $\chi(X \times X') = \chi(X) \cdot \chi(X')$.

203. Prove the Lefschetz Theorem: Let $X$ be a finite CW-complex, $f : X \to X$ be a map such that $\operatorname{Lef}(f) = 0$. Then $f$ has a fixed point, i.e. such point $x_0 \in X$ that $f(x_0) = x_0$.

204. Let $X$ be a finite contractible CW-complex. Prove that any map $f : X \to X$ has a fixed point.

205. Define a flow of homeomorphisms $\phi_t : X \to X$. Let $X$ be a finite CW-complex with $\chi(X) \neq 0$, and $\phi_t : X \to X$ be a flow. Prove that there exists a point $x_0 \in X$ so that $\phi_t(x_0) = x_0$ for all $t \in \mathbb{R}$.

206. Let $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ be a map. Prove that $f$ always has a fixed point. Give an example that the above statement fails for a map $f : \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}$.

207. Let $n \neq k$. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^k$.

208. Let $f : S^n \to S^n$ be a map, and $\deg(f)$ be the degree of $f$. Prove that $\operatorname{Lef}(f) = 1 + (-1)^n \deg(f)$.

209. Prove that there is no tangent vector field $v(x)$ on the sphere $S^{2n}$ such that $v(x) \neq 0$ for all $x \in S^{2n}$. Construct everywhere non-zero vector field $v$ on $S^{2n+1}$.

210. Let $K \subset S^n$ be homeomorphic to the cube $I^k$, $0 \leq k \leq n$. Prove that $H_q(S^n \setminus K) = 0$ for all $q \geq 0$. 
211. Let $S^k \subset S^n$, $0 \leq k \leq n - 1$. Prove that
\[
\tilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbb{Z}, & \text{if } q = n - k - 1, \\ 0, & \text{if } q \neq n - k - 1. \end{cases}
\]

212. State and prove the Jordan-Brouwer Theorem.

213. State and prove the Brouwer Invariance Domain Theorem.

214. Let $(X, A)$ be a CW-pair. Prove that the group $H^1(X, A; \mathbb{Z})$ is a free abelian group.

215. Define the cup-product in cohomology. Prove that $\delta(\phi \cup \psi) = (\delta \phi) \cup \psi + (-1)^k \phi \cup (\delta \psi)$ where $\phi \in C^k(X)$, $\psi \in C^l(X)$.

216. Compute the cup product of $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$, $H^*(M^2_g; \mathbb{Z})$.

217. Prove that $\alpha \beta = (-1)^{kl} \beta \alpha$ if $\alpha \in H^k(X)$, $\beta \in H^l(X)$.

218. Define the external product
\[
\mu : H^*(X; R) \otimes H^*(Y; R) \to H^*(X \times Y; R).
\]
Define the ring structure on $H^*(X; R) \otimes H^*(Y; R)$. Prove that the external product $\mu : H^*(X; R) \otimes H^*(Y; R) \to H^*(X \times Y; R)$ induces a ring isomorphism provided that $H^q(Y; R)$ are free $R$-modules for all $q$.

219. Let $\Delta : X \to X \times X$ be a diagonal map. Prove that the homomorphism
\[
H^k(X; R) \otimes H^l(Y; R) \to H^*(X \times Y; R)
\]
coincides with the cup-product, i.e. that $\Delta^*(\mu(\alpha \otimes \beta)) = \alpha \cup \beta$.

220. Prove that $H^*(\mathbb{RP}^n; \mathbb{Z}) \cong \mathbb{Z}/2[x]/x^{n+1}$.

221. Prove that $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[y]/y^{n+1}$.

222. Prove that any map $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ has a fixed point.

223. Prove that if $\mathbb{R}^n$ is a real division algebra, then $n$ is a power of two.

224. State the Poincarè Duality Theorem. Compute the Poincarè Duality for $M^2_g$.

225. Prove that the odd-dimensional manifold has zero Euler characteristic.

226. Prove that $\langle \alpha \cup \beta, \mu \rangle = \langle \beta, \mu \cap \alpha \rangle$.

227. Let $M^{4k}$ be a compact oriented manifold, and $V = H^{2k}(M^{4k}; \mathbb{Z})/\Tor$. Use the Poincarè duality to prove that the pairing
\[
\mu(\alpha, \beta) = \langle \alpha \cup \beta, [M^{4k}] \rangle
\]
defines a nondegenerated quadratic form on $V$. Compute the index of this quadratic form for $\mathbb{CP}^{2n}$.

228. Use Poincarè duality to prove that $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$.

229. Use Poincarè duality to prove that $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/x^{n+1}$.

230. Let $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ be a map. Show that $f$ has a fixed point.

231. Compute the ring structure $H^*(\mathbb{RP}^n; \mathbb{Z}/2^k)$.

232. Let $n > k$. Prove that there is no map $f : \mathbb{RP}^n \to \mathbb{RP}^k$ which induces a nontrivial ring homomorphism $f^* : H^*(\mathbb{RP}^k; \mathbb{Z}/2) \to H^*(\mathbb{RP}^n; \mathbb{Z}/2)$.
233. Let a map \( h : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1} \), be such that the induced homomorphism
\[
h^* : H^*(\mathbb{RP}^{n-1}; \mathbb{Z}/2) \to H^*(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{Z}/2)
\]
takes generator \( y \in H^1(\mathbb{RP}^{n-1}; \mathbb{Z}/2) \) to the sum of generators: \( h^*(y) = x_1 \otimes 1 + 1 \otimes x_2 \). Prove that \( n \) must be a power of 2.

234. Prove that \( \mathbb{RP}^3 \) and is not homotopy equivalent to \( S^3 \vee \mathbb{RP}^2 \).

235. Define the Hopf invariant \( h(\lambda) \) of an element \( \lambda \in \pi_{4q-1}(S^{2q}) \).

236. Prove that \( h(\lambda_1 + \lambda_2) = h(\lambda_1) + h(\lambda_2) \).

237. Prove that there is an element in \( \pi_{4n-1}(S^{2n}) \) with the Hopf invariant 2. State and prove the theorem that the group \( \pi_{4n-1}(S^{2n}) \) is infinite.

238. Prove that \( h(\iota_{2q}, \iota_{2q}) = 2 \), where \( \iota_{2q} \in \pi_{2q}(S^{2q}) \) is the standard generator.

239. Define a cohomology operation. Give examples.

240. Define a canonical fundamental class \( \iota_n \in \text{Hom}(H_n(K(\pi, n); \mathbb{Z}), \pi) \).

241. Let \( \pi, \pi' \) be abelian groups. Prove that there is a bijection
\[
[K(\pi, n), K(\pi', n)] \leftrightarrow \text{Hom}(\pi, \pi').
\]

242. Let \( \pi \) be an abelian group and \( n \) be a positive integer. Prove that the homotopy type of the Eilenberg-McLane space \( K(\pi, n) \) is completely determined by the group \( \pi \) and the integer \( n \).

243. Prove that there is a bijection
\[
\mathcal{O}(\pi, n; \pi', n') \leftrightarrow H^n(K(\pi, n), \pi')
\]
given by the formula \( \theta \leftrightarrow \theta(\iota_n) \).

244. Let \( Y \) be a homotopy simple space, \((B, A)\) a CW-pair and \( X^n = B^{(n)} \cup A \) for \( n = 0, 1, \ldots \). Define the obstruction cochain
\[
c(f) \in E^{n+1}(B, A; \pi_n(Y)) = \text{Hom}(E_{n+1}(B, A), \pi_n(Y)).
\]
Prove that \( c(f) \) is a cocycle.

245. Let \( Y \) be a homotopy simple space, \((B, A)\) a CW-pair and \( X^n = B^{(n)} \cup A \) for \( n = 0, 1, \ldots \). Prove that a map \( f : X^n \to Y \) can be extended to a map \( f : X^{n+1} \to Y \) if and only if \( c(f) = 0 \).

246. Define \( d(f, g) \in E^n(B, A; \pi_n(Y)) \). Prove the formula: \( \delta d(f, g) = c(g) - c(f) \).

247. Let \( Y \) be a homotopy simple space, \((B, A)\) a CW-pair and \( X^n = B^{(n)} \cup A \) for \( n = 0, 1, \ldots \). Let \( f : X^n \to Y \) be a map, and \( d \in E^n(B, A; \pi_n(Y)) \) is a cochain. Prove that there exists a map \( g : X^n \to Y \) such that \( f|_{X^{n-1}} = g|_{X^{n-1}} \) and \( d(f, g) = d \).

248. Let \( Y \) be a homotopy simple space, \((B, A)\) a CW-pair and \( X^n = B^{(n)} \cup A \) for \( n = 0, 1, \ldots \). Assume \( f : X^n \to Y \) is a map. Prove that there exists a map \( g : X^{n+1} \to Y \) such that \( g|_{X^{n-1}} = f|_{X^{n-1}} \) if and only if \( [c(f)] = 0 \) in \( H^{n+1}(B, A; \pi_n Y) \).

249. Prove the following result

**Theorem.** Let \( f, g : K \to Y \) be two maps, where \( K \) is a CW-complex and \( Y \) is homotopy-simple space. Assume that \( f|_{K^{(n-1)}} = g|_{K^{(n-1)}} \). Then the cohomology class \( [d(f, g)] \in H^n(K, \pi_n Y) \) vanishes if and only if there exists a homotopy between the maps \( f|_{K^{(n)}} \) and \( g|_{K^{(n)}} \) relative to the skeleton \( K^{(n-2)} \).
250. Prove the following result:

**Theorem.** There is a bijection

\[ [X, K(\pi, n)] \leftrightarrow H^n(X; \pi). \]

given by the formula \( f \mapsto f^* \iota_n \).

251. Consider a \( k \)-torus \( T^k \). We identify \( T^k \) with the quotient space \( \mathbb{R}^k / \sim \), where two vectors \( \vec{x} \sim \vec{y} \) if and only if all coordinates of the vector \( \vec{x} - \vec{y} \) are integers. It is easy to see that a linear map \( \bar{f} : \mathbb{R}^k \to \mathbb{R}^\ell \) given by an \( k \times \ell \)-matrix \( A \) with integral entries descends to a map \( f : T^k \to T^\ell \). Prove that any map \( g : T^k \to T^\ell \) is homotopic to a linear map as above.

252. Prove the following result:

**Theorem.** Let \( X \) be an \( n \)-dimensional CW-complex. Then there is a bijection:

\[ H^n(X; \mathbb{Z}) \cong [X, S^n]. \]

253. Prove that any \( K(\mathbb{Z}_2, n) \) is infinite dimensional space for each \( n \geq 1 \).

254. Let \( M \) be a simply-connected compact closed manifold with \( \text{dim} \ M = 3 \). Prove that \( M \) is homotopy equivalent to \( S^3 \).

255. Let \( h : S^3 \to S^2 \) be the Hopf map. Let \( \lambda \geq 1 \) be an integer. Define a map

\[ f_\lambda : S^3 \xrightarrow{\lambda} S^3 \lor \cdots \lor S^3 \xrightarrow{h \lor \cdots \lor h} S^2. \]

Prove that the space \( X_\lambda = S^2 \cup_{f_\lambda} D^4 \) is homotopy equivalent to a closed compact manifold of dimension four if and only if \( \lambda = 1 \).

256. Let \( D^3 \subset T^3 \) and \( c : T^3 \to S^3 \) be a map which collapses a complement of \( D^3 \subset T^3 \) to a point. Prove that the map \( g : T^3 \xrightarrow{c} S^3 \xrightarrow{h} S^2 \) (where \( h : S^3 \to S^2 \) is the Hopf map) induces trivial homomorphism on homology and homotopy, but is not homotopic to a constant map.

257. Assume a CW-complex \( X \) contains \( S^1 \) such that the inclusion \( i : S^1 \subset X \) induces an injection \( i_* : H_1(S^1; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \) with image a direct summand of \( H_1(X; \mathbb{Z}) \). Prove that \( S^1 \) is a retract of \( X \).

258. Two questions:

(a) Show that there is no map from \( \mathbb{CP}^2 \) to itself of degree \(-1\).

(b) Show that there is no map from \( \mathbb{CP}^2 \times \mathbb{CP}^2 \) to itself of degree \(-1\).