QUESTIONS FOR THE MIDTERM, WINTER 2020

1. Basic spaces: $\mathbb{R}^n$, $S^n$, stereographic projection. The space $S^\infty$.

2. Projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$: definitions, local coordinate system, the Hopf maps $S^n \rightarrow \mathbb{RP}^n$, $S^{2n+1} \rightarrow \mathbb{CP}^n$, $S^{4n+3} \rightarrow \mathbb{HP}^n$.

3. Prove the homeomorphisms: $\mathbb{RP}^1 \cong S^1$, $\mathbb{CP}^1 \cong S^2$, $\mathbb{HP}^1 \cong S^4$.

4. Define Grassmannian manifolds $G_k(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$: and construct local coordinate systems, in particular, find their dimensions.

5. Prove that the Grassmannian manifolds $G_k(\mathbb{R}^n)$, and $G_k(\mathbb{C}^n)$ are compact and connected.

6. Define classic Lie groups $GL(\mathbb{R}^n)$, $GL(\mathbb{C}^n)$, $O(k)$, $SO(k)$, $U(k)$, $SU(k)$. Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact. How many connected components does each of these spaces have?

7. Prove that $SO(2)$ and $U(1)$ are homeomorphic to $S^1$, $SO(3)$ is homeomorphic to $\mathbb{RP}^3$, and $SU(2)$ is homeomorphic to $S^3$.

8. Define $SO(4) \cong SO(3) \times S^3$.

9. Define Stiefel manifolds $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$, $V_k(\mathbb{H}^n)$. Prove the following homeomorphisms:

$$V_n(\mathbb{R}^n) \cong O(n), \quad V_{n-1}(\mathbb{R}^n) \cong SO(n),$$

$$V_n(\mathbb{C}^n) \cong U(n), \quad V_{n-1}(\mathbb{C}^n) \cong SU(n),$$

$$V_1(\mathbb{R}^n) \cong S^{n-1}, \quad V_1(\mathbb{C}^n) \cong S^{2n-1}, \quad V_1(\mathbb{H}^n) \cong S^{4n-1}.$$

10. Define $V_k(\mathbb{R}^n)/O(k) \cong G_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)/U(k) \cong G_k(\mathbb{C}^n)$.

11. Prove the following homeomorphisms:

$$S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1),$$

$$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1),$$

$$G_k(\mathbb{R}^n) \cong O(n)/O(n-k) \times O(n-k), \quad G_k(\mathbb{C}^n) \cong U(n)/U(k) \times U(n-k).$$

12. Define action of the groups $O(k)$, $U(k)$ on the Stiefel manifolds $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$. Prove the following homeomorphisms:

$$S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1),$$

$$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1),$$

$$G_k(\mathbb{R}^n) \cong O(n)/O(n-k) \times O(n-k), \quad G_k(\mathbb{C}^n) \cong U(n)/U(k) \times U(n-k).$$

13. Prove that the Klein bottle $K^2$ is homeomorphic to the union of two M"ebius bands along the circle.

14. Prove that $K^2 \# \mathbb{RP}^2$ is homeomorphic to $\mathbb{RP}^2 \# T^2$.

15. Define a cylinder and a cone of a map $f : X \rightarrow Y$. Prove that the cones of the maps $c : S^n \rightarrow \mathbb{RP}^n$ and $h : S^{2n+1} \rightarrow \mathbb{CP}^n$ are homeomorphic to $\mathbb{RP}^{n+1}$ and $\mathbb{CP}^{n+1}$ respectively.

16. Define suspension. Prove that $\Sigma(S^0) \cong S^{n+1}$.

17. Define a compact-open topology on $C(X,Y)$. Prove the homeomorphism: $C(X,C(Y,Z)) \cong C(X \times Y,Z)$ for Hausdorff and locally compact spaces $X$, $Y$, $Z$. Prove that this homeomorphism is natural.
18. Define the spaces of paths \( \mathcal{E}(X, x_0, x_1) \), \( \mathcal{E}(X, x_0) \), and loops \( \Omega(X, x_0) \). Prove that the spaces \( \Omega(S^n, x_0) \) and \( \Omega(S^n, x_1) \) are homeomorphic for any points \( x_0, x_1 \in S^n \).

19. Let \( X, Y \) be pointed spaces. Prove the homeomorphism \( C(\Sigma(X), Y) \cong C(X, \Omega(Y)) \) for Hausdorff and locally compact spaces \( X, Y \). Prove that this homeomorphism is natural.

20. Define smash-product \( X \wedge Y \). Prove that \( S^n \wedge S^k \cong S^{n+k} \) (as pointed spaces).

21. Define homotopy of two maps. Prove that the maps \( \phi^*: [X', Y] \to [X, Y] \), \( \psi_*: [X, Y] \to [X, Y'] \) induced by maps \( \phi: X \to X' \), \( \psi: Y \to Y' \) are well-defined.

22. Give three definitions of homotopy equivalence. Prove that they are equivalent.

23. Prove that \( X \sim Y \) implies \( \Sigma(X) \sim \Sigma(Y) \) and \( \Omega(X) \sim \Omega(Y) \).

24. Give a definition of a contractible space. Prove that \( \mathcal{E}(X, x_0) \) is a contractible.

25. Prove that a space \( X \) is contractible if and only if it is homotopy equivalent to a point.

26. Prove that a space \( X \) is contractible if and only if every map \( f: Y \to X \) is null-homotopic.

27. Give definition of a retract and deformational retract. Examples. Prove that \( \{0\} \cup \{1\} \) is not a retract of \( I = [0,1] \). Define map of pairs. Examples.

28. Define a \( CW \)-complex. Give examples of cell decomposition. Show that the axiom \( (W) \) does not imply the axiom \( (C) \) and wise-versa.

29. Construct a cellular decomposition of the wedge \( X = S^1 \lor S^2 \) (with a single 2-cell \( e^2 \)) such that a closure of the cell \( e^2 \) is not a \( CW \)-subcomplex of \( X \).

30. Construct a cellular decomposition of the wedge \( X = \Sigma(S^n \lor S^k) \). Prove that \( \Sigma(S^n \lor S^k) \sim S^{n+1} \lor S^{k+1} \).

31. Prove that a \( CW \)-complex compact if and only if it is finite.

32. Construct a cellular decomposition of \( S^n \), \( D^n \), \( RP^n \), \( CP^n \), \( HP^n \).

33. Construct a cellular decomposition of the oriented 2-manifold of genus \( g \).

34. Define the Schubert cells \( e(\sigma) \) corresponding to the Schubert symbol \( \sigma \). Give examples.

35. Define the spaces \( H^2, \overline{H^2} \). Prove that a \( k \)-plane \( \pi \) belongs to \( e(\sigma) \) if and only if there exists its basis \( v_1, \ldots, v_k \), such that \( v_i \in H^{\sigma_i}, \ldots, v_k \in H^{\sigma_k} \).

36. Prove the following statement: Let \( \pi \in e(\sigma) \), where \( \sigma = (\sigma_1, \ldots, \sigma_n) \). Then there exists a unique orthonormal basis \( v_1, \ldots, v_k \) of \( \pi \), so that \( v_1 \in H^{\sigma_1}, \ldots, v_k \in H^{\sigma_k} \).

37. Define the sets \( E(\sigma), \overline{E(\sigma)} \subset V_k(R^n) \). Prove that the set \( E(\sigma) \subset V_k(R^n) \) is homeomorphic to the closed cell of dimension \( d(\sigma) = (\sigma_1-1) + (\sigma_2-2) + \cdots + (\sigma_k-k) \). Furthermore the map \( q: e(\sigma) \to E(\sigma) \) is a homeomorphism.

38. Define the transformations \( T_{u,v} \), prove its properties. Explain how the transformations \( T_{u,v} \) are used to prove that \( E(\sigma) \subset V(n, k) \) is homeomorphic to a closed cell of dimension \( d(\sigma) \).

39. Prove the statement: a collection of \( \binom{k}{n} \) Schubert cells \( e(\sigma) \) gives \( G_k(R^n) \) a cell-decomposition.

40. Outline a construction of Schubert cells of the complex Grassmannian \( G_k(C^n) \).

41. Define when a pair \( (X, Y) \) is a Borsuk pair. Prove that a \( CW \)-pair \( (X, Y) \) is a Borsuk pair (in the case when \( X, Y \) are finite complexes).
42. Let \((X, A)\) be a Borsuk pair. Prove that \(A\) is a deformation retract of \(X\) if and only if the inclusion \(A \rightarrow X\) is a homotopy equivalence.

43. Prove the statement: let \(X\) be a CW-complex and \(A \subset X\) be its contractible subcomplex. Then \(X\) is homotopy equivalent to the complex \(X/A\).

44. Prove that for a CW-pair \((X, A)\) \(X/A \sim X \cup C(A)\).


46. State and prove Free Point Lemma.

47. Define homotopy groups \(\pi_n(X)\). Prove that \(\pi_n(X)\) is commutative group for \(n \geq 2\). Prove that \(\pi_k(S^n)\) is a trivial group for \(k \leq n\).

48. Prove the statement: Let \(X\) be a CW-complex with only one zero-cell and without cells of dimension \(q < n\), and \(Y\) be a CW-complex of dimension \(q < q\). Then any map \(Y \rightarrow X\) is homotopic to a constant map.

49. Define \(n\)-connected space. Prove the statement: Any \(n\)-connected CW-complex homotopy equivalent to a CW-complex with a single zero cell and without cells of dimensions \(1, 2, \ldots, n\).

50. Prove that if \(f, g : X \rightarrow Y\) are homotopic maps, then the homomorphisms \(f_*, g_* : \pi_n(X) \rightarrow \pi_n(Y)\) coincide.

51. Prove that if \(X\) is a path-connected space, then \(\pi_1(X, x_0) \cong \pi_1(X, x_1)\). Describe all isomorphisms here.

52. Prove that \(\pi_1S^1 \cong \mathbb{Z}\).

53. Prove that \(\pi_1(\bigvee_{a \in A} S^1_a)\) is a free group.

54. Prove that \(\pi_1(X, x_0) \cong \pi_1(X^{(2)}, x_0)\), where \(X\) is a connected CW-complex and \(X^{(2)}\) its 2-skeleton.

55. Compute \(\pi_1(M^2)\) for two-dimensional oriented closed manifold of genus \(g\), the sphere with \(g\) handles.

56. Compute \(\pi_1(M^2)\) for two-dimensional non-oriented closed manifold of genus \(g\), the projective plane or the Klein bottle with \(g\) handles.

57. Let \(M = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2\) \((n\) times). Compute \(\pi_1(M)\).

58. Compute \(\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2)\) and \(\pi_1(K^2 \# \mathbb{R}P^2)\).

59. Define \(G_1 * G_2\). Give examples. Prove that \(\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)\).

60. Define \(G_1 *_{H} G_2\). Give examples. State and prove Van Kampen Theorem.

61. Define covering space. Give examples. Construct \(n\)-fold covering of \(S^1 \vee S^1\) \((including \ n = \infty)\).

62. State and prove Theorem on Covering Homotopy.

63. Prove that covering \(p : T \rightarrow X\) induces a monomorphism \(p_* : \pi_1(T, \bar{x}_0) \rightarrow \pi_1(X, x_0)\).

64. Prove that a loop \(\alpha_1 \cdots \alpha_k\), where \(\alpha_j\) is a loop going along the \(j\)-th circle in the wedge \(\bigvee_{j=1}^k S^1_j\), is not homotopic to zero.

65. Let \(p : T \rightarrow X\) be a covering, and \(f, g : Z \rightarrow T\) be two maps so that \(p \circ f = p \circ g\), where \(Z\) is path-connected. Assume that \(f(z) = g(z)\) for some point \(z \in Z\). Prove that \(f = g\).

66. Prove that \(\pi_k(\mathbb{R}P^n) = 0\) if \(1 < k < n\).

67. Prove that any map \(f : \mathbb{R}P^2 \rightarrow S^1\) is homotopic to a constant map.
68. Let $Kl^2$ be the Klein bottle. Construct two-folded covering space $T^2 \to Kl^2$. Compute $\pi_n(Kl^2)$ for all $n$.

69. Let $p : T \to X$ be a covering, $p(\tilde{x}_0) = x_0$. Prove that there is one-to-one correspondence

$$\pi_1(X(x_0)/p_*(\pi_1(T,\tilde{x}_0))) \iff p^{-1}(x_0).$$

Prove that $p^{-1}(x_0) \cong p^{-1}(x_1)$ for any points $x_0, x_1 \in X$.

70. Let $p : T \to X$ be a covering map, and let $\Gamma = p^{-1}(x_0)$. Prove that $\Gamma$ is a transitive right $G$-set for $G = \pi_1(X,x_0)$.

71. Let $X$ by ”good” space and $G = \pi_1(X,x_0)$. Prove that there is a bijection between isomorphism classes of covering spaces of $X$ and transitive right $G$-sets given by

$$\{p : Y \to X\} \mapsto p^{-1}(x_0).$$

72. Let $p : T \to X$ be a covering, and $f : Z \to X$ be a map, $f(z_0) = x_0$, and $\tilde{x}_0 \in T$ so that $p(\tilde{x}_0) = x_0$ (here $Z$ is path-connected space). Prove that there exists a lifting $\tilde{f} : Z \to T$ of the map $f$ so that $\tilde{f}(z_0) = \tilde{x}_0$ if and only if $f_*([\pi_1(Z,z_0)]) \subseteq p_*(\pi_1(T,\tilde{x}_0))$.

73. Define morphism of two covering spaces $T_1p_1 \xrightarrow{p_1} X$ and $T_2p_2 \xrightarrow{T_2} X$. Prove that two morphisms $\phi, \phi' : T_1 \to T_2$ coincide if there is a point $\tilde{x} \in T_1$ so that $\phi(\tilde{x}) = \phi'(\tilde{x})$.

74. Define a group of automorphisms (deck transformations) $\text{Aut}(T \xrightarrow{p} X)$ of a covering $p : T \to X$. Prove that the group $\text{Aut}(T \xrightarrow{p} X)$ acts on $T$ without fixed points.

75. Let $p : T \to X$ be a covering, $p(\tilde{x}_0) = p(\tilde{x}'_0) = x_0$, where $\tilde{x}_0 \neq \tilde{x}'_0$. Prove that there exists an automorphism $\phi \in \text{Aut}(T \xrightarrow{p} X)$ such that $\phi(\tilde{x}_0) = \tilde{x}'_0$ if and only if $p_*(\pi_1(T,\tilde{x}_0)) = p_*(\pi_1(T,\tilde{x}'_0))$.

76. Prove the following statement: Two covering spaces $T_1p_1 \xrightarrow{p_1} X$, $T_2p_2 \xrightarrow{T_2} X$ are isomorphic if and only if for any two points $\tilde{x}_1, \tilde{x}_2 \in T$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$ the groups $(p_1)_*(\pi_1(T_1,\tilde{x}_1)), (p_2)_*(\pi_1(T_2,\tilde{x}_2))$ belong to the same conjugacy class in $\pi_1(X,x)$.

77. Let $N(H)$ be a normalizer for a subgroup $H$ of $G$. Prove the following statement: Let $p : T \to X$ be a covering space. Then the group of automorphisms of this covering space is isomorphic to the group $N(p_*(\pi_1(T,\tilde{x}_0)))/p_*(\pi_1(T,\tilde{x}_0))$.

78. Define universal covering space over $X$. Prove the following statement: Let $X$ be a path-connected CW-complex, $x_0 \in X$. Then for any subgroup $G \subseteq \pi_1(X,x_0)$ there exists a covering $T \xrightarrow{p} X$ and a point $\tilde{x}_0 \in T$ so that $p_*(\pi_1(T,\tilde{x}_0)) = G$.

79. Define homotopy groups $\pi_n(X,x_0)$, in particular define the group operation and inverse. Prove that the groups $\pi_n(X,x_0)$ are abelian if $n \geq 2$.

80. Prove that $\pi_n(X \times Y, x_0 \times y_0) \cong \pi_n(X,x_0) \times \pi_n(Y,y_0)$. Compute $\pi_n(T^k)$ for all $n$.

81. Let $X$ be a path-connected space, and $x_0, x_1 \in X$ be two different points. Let $\gamma : I \to X$ be a path so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Define a homomorphism $\gamma_# : \pi_n(X,x_0) \to \pi_n(X,x_1)$. Prove that $\gamma_#$ is an isomorphism.

82. Let $M_g^2$ be a two-dimensional surface of genus $g \geq 1$ (oriented). Compute the homotopy groups $\pi_1(M_g^2)$.

83. Define relative homotopy groups $\pi_n(X,A;x_0)$. Describe the group operation and the inverse element. Prove that the group $\pi_n(X,A;x_0)$ is commutative for $n \geq 3$. 

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84. Define the homomorphisms in the following sequence:

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A; x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

(1)

Prove that the sequence (1) is exact.

85. Let $A \subset X$ be a retract. Prove that

- $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is monomorphism,
- $j_* : \pi_n(X, x_0) \to \pi_n(X, A; x_0)$ is epimorphism,
- $\partial : \pi_n(X, A; x_0) \to \pi_{n-1}(A, x_0)$ is zero homomorphism.

86. Let $A$ be contractible in $X$. Prove that

- $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is zero homomorphism,
- $j_* : \pi_n(X, x_0) \to \pi_n(X, A; x_0)$ is monomorphism,
- $\partial : \pi_n(X, A; x_0) \to \pi_{n-1}(A, x_0)$ is epimorphism.

87. State and prove Five-Lemma.

88. Let $0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$ be an exact sequence of finitely generated abelian groups. Prove that $\sum_{i=1}^n (-1)^i \text{rank} A_i = 0$.

89. Define locally trivial fiber bundle. Give several examples of non-trivial fiber bundles.

90. Prove that any locally–trivial fiber bundle over the cube $I^2$ is trivial.

91. Define the covering homotopy property. Outline a proof that the covering homotopy property holds for a locally-trivial fiber bundle $E \to B$.

92. Define a Serre fiber bundle. Let $Y$ be an arbitrary path-connected space, $\mathcal{E}(Y, y_0)$ be the space of paths starting at $y_0$. Prove that the map $p : \mathcal{E}(Y, y_0) \to Y$, where $p(s : I \to Y) = s(1) \in Y$ is a Serre fiber bundle.

93. Let $A \subset X$, and $(X, A)$ be a Borsuk pair (for example, a CW-pair). Let $E = C(X, Y)$, $B = C(A, Y)$, and the map $p : E \to B$ be defined as $p(f : X \to Y) = (f|_A : A \to Y)$. Prove that the map $p : E \to B$ is a Serre fiber bundle.

94. Define weak homotopy equivalence. Prove that finite CW-complexes $X$, $Y$ are weak homotopy equivalent if and only if they are homotopy equivalent.

95. Let $p : E \to B$ be Serre fiber bundle, where $B$ be a path-connected space. Prove that the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent for any two points $x_0, x_1 \in B$.

96. Prove that for any continuous map $f : X \to Y$ there exists homotopy equivalent map $f_1 : X_1 \to Y_1$, such that $f_1 : X_1 \to Y_1$ is Serre fiber bundle.

97. Let $f : X \to Y$ be a continuous map. Prove that there exists a homotopy equivalent map $g : X \to Y'$, so that $g$ is an inclusion.

98. Let $p : E \to B$ be Serre fiber bundle, $y \in E$ be any point, $x = p(y)$, $F = p^{-1}(x)$. Prove that the homomorphism

$$p_* : \pi_n(E, F; y) \to \pi_n(B, x)$$

is an isomorphism for all $n \geq 1$.

99. Apply the homotopy exact sequence of Serre fibration to prove that (a) $\pi_2(S^2) = \pi_1(S^1) = \mathbb{Z}$; (b) $\pi_n(S^1) = \pi_n(S^2)$.

100. Let $S^\infty \to \mathbb{C}P^\infty$ be the Hopf fibration. Using the fact $S^\infty \sim \ast$, prove that $\pi_n(\mathbb{C}P^\infty) = 0$ for $n \neq 2$, and $\pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$. 

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101. Prove that \( \pi_n(\Omega(X)) \cong \pi_{n+1}(X) \) for any \( X \) and \( n \geq 0 \).

102. Prove that if the groups \( \pi_*(B) \), \( \pi_*(F) \) are finite (finitely generated), then the groups \( \pi_*(E) \) are finite (finitely generated) as well.

103. Assume that a fiber bundle \( p : E \to B \) has a section, i.e., a map \( s : B \to E \), such that \( p \circ s = \text{Id}_B \).

104. Give a construction of a space \( Y \) that \( \pi_n(X, A; x_0) \cong \pi_{n-1}(Y, y_0) \).

105. Outline a proof of the following statement:

106. Let \( K, L \subset \mathbb{R}^p \) be two finite simplicial complexes of dimensions \( k \), \( l \) respectively. Let \( k + l + 1 < p \).

107. Prove that \( \pi_n(S^n) \cong \mathbb{Z} \) for each \( n \geq 1 \).

108. Prove that \( \pi_3(S^2) \cong \mathbb{Z} \), and the Hopf map \( S^3 \to S^2 \) is a representative of the generator of \( \pi_3(S^2) \).

109. Define Whitehead product. State basic properties. Prove that if \( \alpha \in \pi_n(X) \), \( \beta \in \pi_k(X) \) then

110. Define the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \). Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) has infinite order.

111. Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) is in a kernel of each of the following homomorphisms:

112. Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) is in the kernel of the suspension homomorphism

113. Prove the isomorphism

114. Let \( \alpha \in \pi_n(X) \), \( \beta \in \pi_k(X) \). Prove that \( [\alpha, \beta] \in \text{Ker} \Sigma \), where

115. Let \( \iota_{2q} \in \pi_{2q}(S^{2q}) \) be a generator represented by the identity map \( S^{2q} \to S^{2q} \). Prove that the Whitehead product \( [\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q}) \) is a nontrivial element of infinite order.

116. Prove that the suspension \( \Sigma(S^n \times S^k) \) is homotopy equivalent to the wedge \( S^{n+1} \vee S^{k+1} \vee S^{n+k+1} \).

117. Outline a proof of the following statement:

   Let \( X \) be a connected space (not necessarily a CW-complex) with a base point \( x_0 \in X \), \( f : S^n \to X \) be a map such that \( f(s_0) = x_0 \), where \( s_0 \) is a base point of \( S^n \). Let \( Y = X \cup_f D^{n+1} \), and \( i : X \to Y \) be the inclusion. Then the induced homomorphism \( i_* : \pi_q(X, x_0) \to \pi_q(Y, x_0) \)

   (1) is an isomorphism if \( q < n \),

   (2) is an epimorphism if \( q = n \), and
(3) the kernel $\ker i_* : \pi_n(X,x_0) \to \pi_n(Y,x_0)$ is generated by $\gamma^{-1}[f] \gamma \in \pi_n(X,x_0)$ where $\gamma \in \pi_1(X,x_0)$.

118. Let $X$ be an $n$-connected CW-complex, and $Y$ be a $k$-connected CW-complex. Prove that
   - $\pi_q(X \lor Y) \cong \pi_q(X) \oplus \pi_q(Y)$ if $q \leq n + k$;
   - the group $\pi_q(X \lor Y)$ contains a subgroup $\pi_q(X) \oplus \pi_q(Y)$ as a direct summand.

119. Let $X$ be an $n$-connected CW-complex, and $Y$ be a $k$-connected CW-complex. Prove that
   $$\pi_{n+k+1}(X \lor Y) \cong \pi_{n+k+1}(X) \oplus \pi_{n+k+1}(Y) \oplus [\pi_n(X), \pi_k(Y)].$$

120. Let $X$ be an $(n - 1)$-connected CW-complex. Describe the homotopy group $\pi_n(X)$.

121. Compute the homotopy group $\pi_3(S^2 \lor S^2)$.

122. Define when a map $f : X \to Y$ is a weak homotopy equivalence. Outline the proof that the following two statements are equivalent.
   (1) The map $f : X \to Y$ is weak homotopy equivalence.
   (2) The induced homomorphism $f_* : \pi_n(X,x_0) \to \pi_n(Y,x_0)$ is isomorphism for all $n$ and $x_0 \in X$.

123. Let $X$, $Y$ be CW-complexes. Prove that if a map $f_* : X \to Y$ induces isomorphism
   $$f_* : \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$$
   for all $n \geq 0$ and $x_0 \in X$, then $f$ is a homotopy equivalence.

124. Let $X$ be a Hausdorff topological space. Prove that there exists a CW-complex $K$ and a weak homotopy equivalence $f : K \to X$. Show that the CW-complex $K$ is unique up to homotopy equivalence.

125. Let $X$, $Y$ be two weak homotopy equivalent spaces. Prove that there exist a CW-complex $K$ and maps $f : K \to X$, $g : K \to Y$ which weak homotopy equivalences.

126. Define an Eilenberg-McLane space. Prove that it does exists and unique up to weak homotopy equivalence.

127. Construct the space $K(\pi, 1)$, where $\pi$ is a finitely generated abelian group.

128. Let $X = K(\pi, n)$. Prove that $\Omega X = K(\pi, n - 1)$.

129. Let $X$ be a CW-complex, and $n \geq 1$. Construct a CW-complex $X_n$ and a map $f_n : X \to X_n$ such that
   (1) $\pi_q(X_n) = \begin{cases} \pi_q(X) & \text{if } q \leq n \\ 0 & \text{else} \end{cases}$
   (2) $(f_n)_* : \pi_q(X) \to \pi_q(X_n)$ is isomorphism if $q \leq n$.

130. Let $X$ be a CW-complex, and $n \geq 1$. Construct a CW-complex $X|_n$ and a map $g_n : X|_n \to X$ such that
   (1) $\pi_q(X|_n) = \begin{cases} \pi_q(X) & \text{if } q \geq n \\ 0 & \text{else} \end{cases}$
   (2) $(g_n)_* : \pi_q(X|_n) \to \pi_q(X)$ is isomorphism if $q \geq n$.

131. Let $X = S^2$. Prove that $X|_3 = S^3$.

132. Let $X = \mathbb{CP}^n$. Prove that $X|_3 = X|_{2n+1} = S^{2n+1}$. 
