More on the principle of inclusion and exclusion.

**Theorem.** Let $S$ be a finite set, and $c_1, \ldots, c_k$ be some conditions on elements of $S$. Then

$$N(\bar{c}_1 \cdots \bar{c}_k) = N + \sum_{\ell=1}^{k} (-1)^{\ell} \sum_{1 \leq i_1 < \cdots < i_{\ell} \leq k} N(c_{i_1} \cdots c_{i_{\ell}}),$$

where $N = |S|$, $N(c_{i_1} \cdots c_{i_{\ell}}) = |S_{i_1} \cap \cdots \cap S_{i_{\ell}}|$, and $N(\bar{c}_1 \cdots \bar{c}_k) = |S_1 \cup \cdots \cup S_k|$.

**Important Examples.** (1) Let $A = \{1, 2, \ldots, 999, 999\}$. Count how many elements $n \in A$ have the property that a sum of digits of $n$ is equal to 35?

**Solution.** Let $x_1, \ldots, x_6$ denote digits of $n = x_1x_2x_3x_4x_5x_6$. Then the condition on $n$ is equivalent to the following question. Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 35$, and the integers $x_i$ are such that $0 \leq x_i \leq 9$, $i = 1, \ldots, 6$. How many integral solutions (i.e. when all $x_i$ are integers) are there?

First, we consider all solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 35$ such that $0 \leq x_i$, $i = 1, \ldots, 6$. We denote the set of all such solutions by $S$. For each $i = 1, 2, 3, 4, 5, 6$, we say that a solution $x_1 \ldots x_6$ satisfies the property $c_i$ if $x_i \geq 10$. We denote by $S_i$ the set of solutions satisfying $c_i$. Then we compute:

$$N = |S| = \binom{35 + 6 - 1}{6 - 1} = \binom{40}{5},$$

$$N(c_i) = |S_i| = \binom{25 + 6 - 1}{6 - 1} = \binom{30}{5},$$

$$N(c_{i_1}c_{i_2}) = |S_{i_1} \cap S_{i_2}| = \binom{15 + 6 - 1}{6 - 1} = \binom{20}{5},$$

$$N(c_{i_1}c_{i_2}c_{i_3}) = |S_{i_1} \cap S_{i_2} \cap S_{i_3}| = \binom{5 + 6 - 1}{6 - 1} = \binom{10}{5},$$

$$N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}) = N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}c_{i_5}) = N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}c_{i_5}c_{i_6}) = 0.$$

Then we compute the answer:

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4 \bar{c}_5 \bar{c}_6) = \binom{40}{5} - \binom{6}{5} \binom{30}{5} + \binom{6}{5} \binom{20}{5} - \binom{6}{5} \binom{10}{5}.$$

(2) Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$. A function $f : A \to B$ is a rule which for each element $a_i \in A$ assigns an element $f(a_i) \in B$. Let $\mathcal{F}(A,B)$ be the set of all functions $f : A \to B$.

**Exercise.** Prove that $|\mathcal{F}(A,B)| = n^m$.

**Definition.** Let $f : A \to B$ be a function. We denote by $f(A) = \{ f(a) \mid a \in A \} \subset B$ the image of $f$. We say that a function $f : A \to B$ is onto if $f(A) = B$. Let $\mathcal{F}_{\text{onto}}(A,B) \subset \mathcal{F}(A,B)$ be the set of all functions $f : A \to B$ which are onto.

**Question:** Let $|A| = m$ and $|B| = n$. What is the size of the set $\mathcal{F}_{\text{onto}}(A,B)$?

**Solution.** We denote $\mathcal{F} := \mathcal{F}(A,B)$. Then we say that a function $f : A \to B$ satisfies $c_i$ iff $b_i \notin f(A)$,
where \( i = 1, \ldots, n \). We denote by \( \mathcal{F}_i \) the set of all functions satisfying \( c_i \). Then we have:

\[
\begin{align*}
N &= |\mathcal{F}| = n^m \\
N(c_i) &= |\mathcal{F}_i| = (n - 1)^m \\
N(c_1 c_2) &= |\mathcal{F}_1 \cap \mathcal{F}_2| = (n - 2)^m \\
N(c_1 c_2 c_3) &= |\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3| = (n - 3)^m \\
&\quad \vdots \\
N(c_1 \cdots c_i) &= |\mathcal{F}_1 \cap \cdots \cap \mathcal{F}_i| = (n - k)^m \\
&\quad \vdots \\
N(c_1 \cdots c_{n-1}) &= |\mathcal{F}_1 \cap \cdots \cap \mathcal{F}_{n-1}| = 1^m \\
N(c_1 \cdots c_n) &= |\mathcal{F}_1 \cap \cdots \cap \mathcal{F}_n| = 0 
\end{align*}
\]

We obtain the answer:

\[
|\mathcal{F}^{\text{onto}}(A, B)| = n^m - \binom{n}{1} (n - 1)^m + \binom{n}{2} (n - 2)^m - \cdots - (-1)^k \binom{n}{k} (n - k)^m + \cdots - (-1)^{n-1} \binom{n}{n-1} 1^m 
\]

(3) Let \( A = \{a_1, a_2, a_3, a_4, a_5, a_6\} \), \( B = \{b_1, b_2, b_3, b_4, b_5\} \). Then \( N = |\mathcal{F}(A, B)| = 5^6 \),

\[
\begin{align*}
N &= 5^6 \\
N(c_i) &= 4^6 \\
N(c_1 c_2) &= 3^6 \\
N(c_1 c_2 c_3) &= 2^6 \\
N(c_1 c_2 c_3 c_4) &= 1^6 
\end{align*}
\]

We obtain the answer:

\[
|\mathcal{F}^{\text{onto}}(A, B)| = 5^6 - \left( \frac{5}{1} \right) 4^6 + \left( \frac{5}{2} \right) 3^6 - \left( \frac{5}{3} \right) 2^6 + \left( \frac{5}{4} \right) 1^6
\]

\[
= 15,625 - 5 \cdot 4,096 + 10 \cdot 729 - 10 \cdot 64 + 5 \cdot 1
\]

\[
= 15,625 - 20,480 + 7,290 - 640 + 5 = 1,800
\]

(4) **Euler function.** For given positive integer \( n \), consider the set of numbers \( m \) such that \( 1 \leq m < n \) and \( \gcd(m, n) = 1 \). Leonard Euler defined the function:

\[
\phi(n) = |\{ m \mid 1 \leq m < n, \text{ and } \gcd(m, n) = 1 \}|.
\]

Here is the values of \( \phi(n) \) for some \( n \):

| \( n \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| \( \phi(n) \) | 1 | 2 | 2 | 4 | 2 | 6 | 4 | 6 | 4 | 10 | 4 | 12 | 6 | 8 | 8 | 16 |

There is a simple formula to compute \( \phi(n) \). Recall that for every integer \( n \) there exist primes \( p_1, \ldots, p_s \) and positive \( e_1, \ldots, e_s \) such that \( n = p_1^{e_1} \cdots p_s^{e_s} \). Here is the formula:

\[
\phi(n) = n \prod_{i=1}^s \left( 1 - \frac{1}{p_i} \right)
\]

**Example:** Let \( n = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4} \), and \( S = \{1, \ldots, n\} \).
We notice that for $m < n$ with $\gcd(m, n) > 1$, $m$ has to be divisible by one of the primes $p_i$. We say that “$m$ satifies $c_i$” iff $p_i|m$. Let

$$S_i = \{ m \in S \mid p_i|m \}, \ i = 1, 2, 3, 4.$$  

Then $N = |S| = n$, $N(c_i) = |S_i| = \frac{n}{p_i}$. Then $N(c_i c_j) = \frac{n}{p_i p_j}$, $N(c_i c_j c_k) = \frac{n}{p_i p_j p_k}$, $N(c_1 c_2 c_3 c_4) = \frac{n}{p_1 p_2 p_3 p_4}$. Then

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = n - \left( \frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} + \frac{n}{p_4} \right) + \left( \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \frac{n}{p_1 p_4} + \frac{n}{p_2 p_3} + \frac{n}{p_2 p_4} + \frac{n}{p_3 p_4} + \frac{n}{p_4} \right)$$

It is easy to check:

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 c_4) = n \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)}{p_1 p_2 p_3 p_4} = n \left( 1 - \frac{1}{p_1} \right) \left( 1 - \frac{1}{p_2} \right) \left( 1 - \frac{1}{p_3} \right) \left( 1 - \frac{1}{p_4} \right).$$

**Examples:** (1) Let $p$ be a prime. Then $\phi(p) = p - 1$, and $\phi(p^k) = p^{k-1}(p - 1)$.

(2) Since $2019 = 3 \cdot 673$, where 673 is a prime number. We obtain:

$$\phi(2019) = 3 \cdot 673 \left( 1 - \frac{1}{3} \right) \left( 1 - \frac{1}{673} \right) = 2 \cdot 672 = 1342.$$  

**Recursive definitions.** There are many mathematical objects which we can define only *recursively*. We start with well-known example:

(1) **Fibonacci numbers** $F_n$. We define:

(B) $F_0 = 0$, $F_1 = 1$,

(R) $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Here are the first few values of $F_n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_n$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
<td>233</td>
<td>377</td>
<td>610</td>
</tr>
</tbody>
</table>

We prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ by induction. Indeed, it’s true if $n = 1$.

Assume $\sum_{i=1}^{k} F_i^2 = F_k F_{k+1}$. Then

$$\sum_{i=1}^{k+1} F_i^2 = F_k F_{k+1} + F_{k+1}^2$$

$$= F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}.$$  

(2) We define a sequence of numbers $a_n$ as:

(B) $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and

(R) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.  

3
Here are the first few values of \( a_n \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
<td>233</td>
<td>377</td>
<td></td>
</tr>
</tbody>
</table>

We notice that \( a_n = F_{n-1} \) for \( n \geq 3 \). We would like to prove that \( a_{n+3} \geq (\sqrt{2})^n \) for all \( n \geq 0 \). Indeed, it’s true if \( n = 0, 1 \). Assume \( a_{k+3} \geq (\sqrt{2})^k \) for all \( k = 0, 1, \ldots, n \). We should prove that \( a_{n+4} \geq (\sqrt{2})^{n+1} \). We have:

\[
a_{n+4} = a_{n+3} + a_{n+2} \geq (\sqrt{2})^n + (\sqrt{2})^{n-1} = (\sqrt{2})^{n-1}(\sqrt{2} + 1) \geq (\sqrt{2})^{n-1} \cdot 2 = (\sqrt{2})^{n+1}.
\]

Here we use that \( \sqrt{2} + 1 \geq 2 \) and \( 2 = (\sqrt{2})^2 \).

(3) We can define recursively the binomial coefficients \( \binom{n}{r} \):

- (B) \( \binom{n}{0} = 1 \), \( \binom{n}{r} = 0 \) if \( r < 0 \) and \( r > n \).
- (R) \( \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \).

(4) We define factorial \( \text{FAC}(n) \):

- (B) \( \text{FAC}(0) = 1 \)
- (R) \( \text{FAC}(n) = \text{FAC}(n-1) \cdot n \) for \( n \geq 1 \).

(5) We define the Harmonic numbers \( H_n \):

- (B) \( H_1 = 1 \)
- (R) \( H_n = H_{n-1} + \frac{1}{n} \) for \( n \geq 2 \).

(6) We define the sequence \( \text{SEC}(n) \):

- (B) \( \text{SEC}(0) = 1 \)
- (R) \( \text{SEC}(n+1) = \frac{n+1}{\text{SEC}(0)} \).

**Exercise.** Use induction to prove that the sequence \( \text{SEC}(n) \) is well-defined.

(7) We define the sequence \( T(n) \) as follows:

- (B) \( T(1) = 1 \)
- (R) \( T(n) = 2 \cdot T(\lfloor \frac{n}{2} \rfloor) \) for \( n \geq 2 \).

We compute a couple of values of \( T(n) \):

\[
T(73) = 2 \cdot T(36) = 2^2 \cdot T(18) = 2^3 \cdot T(9) = 2^4 \cdot T(4) = 2^5 \cdot T(2) = 2^6
\]

\[
T(2019) = 2 \cdot T(1009) = 2^2 \cdot T(504) = 2^3 \cdot T(252) = 2^4 \cdot T(126) = 2^5 \cdot T(63)
\]

\[
= 2^6 \cdot T(31) = 2^7 \cdot T(15) = 2^8 \cdot T(7) = 2^9 \cdot T(3) = 2^{10}
\]
Exercise. Use induction to prove that $T(n) = \max\{ 2^k \mid 2^k \leq n \}$.

Exercise. Define a sequence $S(n)$ such that $S(n) = \min\{ 2^k \mid n \leq 2^k \}$.

Exercise. Let $p$ be a prime. Define recursively a sequence $T_p(n)$ such that

$$T(n) = \max\{ p^k \mid p^k \leq n \}.$$  

Exercise. Let $p$ be a prime. Define recursively a sequence $S_p(n)$ such that

$$S_p(n) = \min\{ p^k \mid n \leq p^k \}.$$  

Exercise. Define recursively what does it mean “well-formed formula”, see Ex. 17, p. 220.