First, we return to the exercises at the end of the Lecture 2.

Exercises:

(1) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the equation $x_1 + \cdots + x_n = r$.

(2) Determine number of integral solutions $x_i \geq 1$, $i = 1, \ldots, n$ of the equation $x_1 + \cdots + x_n = r$.

(3) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the inequality $x_1 + \cdots + x_n \leq r$.

(4) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the inequality $x_1 + \cdots + x_n < r$.

(5) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 9?

(6) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 10?

(7) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 14?

(8) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 21?

Next, we consider several different examples related to the previous ones:

(9) We determine all different ways we can decompose an integer into a sum of non-zero integers.

**Example** $n = 4$:

\[
\begin{align*}
4 &= 1 + 3 & 4 &= 1 + 1 + 2 \\
4 &= 4 & 4 &= 3 + 1 & 4 &= 1 + 2 + 1 & 4 &= 1 + 1 + 1 + 1 \\
4 &= 2 + 2 & 4 &= 2 + 1 + 1 \\
\end{align*}
\]

Totally we have $8 = 2^{4-1}$ different ways. In general, for given $n$, we have the cases:

- **1 summand** $n = x_1 \quad x_1 \geq 1 \quad n - 1 = y_1, \quad y_1 \geq 0,$ $1 = \binom{n-1}{1}$
- **2 summands** $n = x_1 + x_2 \quad x_1, x_2 \geq 1 \quad n - 2 = y_1 + y_2 \quad y_1, y_2 \geq 0$ $\binom{n-1}{2}$
- **3 summands** $n = x_1 + x_2 + x_3 \quad x_1, x_2, x_3 \geq 1 \quad n - 3 = y_1 + y_2 + y_3 \quad y_1, y_2, y_3 \geq 0$ $\binom{n-1}{3}$
- **\ldots**
- **k summands** $n = x_1 + \cdots + x_k \quad x_1, \ldots, x_k \geq 1 \quad n - k = y_1 + \cdots + y_k \quad y_1, \ldots, y_k \geq 0$ $\binom{n-1}{k-1}$
- **\ldots**
- **n summands** $n = x_1 + \cdots + x_n \quad x_1, \ldots, x_n \geq 1 \quad n - n = y_1 + \cdots + y_n \quad y_1, \ldots, y_n \geq 1$ $\binom{n-1}{n-1}$

The total yeilds the answer:

\[
1 + \binom{n-1}{1} + \cdots + \binom{n-1}{k-1} + \cdots + \binom{n-1}{n-1} = (1 + 1)^{n-1} = 2^{n-1}
\]
(10) Consider the following segment of a code:

```python
for i = 1 to 2019
    for j = 1 to i
        for k = 1 to j
            print(i + j + k)
```

Here the variables $i, j, k$ are integers. How many times the command `print(i + j + k)` will be executed if $1 \leq k \leq j \leq i \leq 2019$? In fact, we have to count how many triples of integers $(i, j, k)$ satisfies the condition:

$$1 \leq k \leq j \leq i \leq 2019.$$

To answer the question, we imagine 2019 empty boxes. Then any placement of 3 objects into those 2019 boxes counts exactly one execution. The answer is

$$\binom{3 + 2019 - 1}{2019 - 1} = \binom{2021}{2018} = \binom{2021}{2019} = \binom{2021}{3} = 2021 \cdot 2020 \cdot 2019 \cdot \frac{1}{1 \cdot 2 \cdot 3}.$$ 

(11) How many times the command `print(i + j + k + ℓ)` will be executed in the following segment of a code if $1 \leq k \leq j \leq i \leq 2019$?

```python
for i = 1 to n
    for j = 1 to i
        for k = 1 to j
            for ℓ = 1 to k
                print(i + j + k + ℓ)
```

(12) The Catalan numbers. Let us consider the $xy$-plane, and two types of moves:

- $R : (x, y) \mapsto (x + 1, y)$,
- $U : (x, y) \mapsto (x, y + 1)$.

We are allowed to make the moves $R$ and $U$ to get from the point $(0, 0)$ to the point $(n, n)$. A path consisting of only the moves $R$ and $U$ is called monotonic.

**Warm-up question:** How many monotonic paths are there from $(0, 0)$ to $(n, n)$?

This is easy. Indeed, any monotonic path can be recorded as a sequence of $n$ $R$’s and $n$ $U$’s. A total number of moves is $2n$; thus it is enough to choose $n$ slots for $R$’s (or $n$ $U$’s). We obtain

$$\binom{2n}{n}$$

paths.

A monotonic path from $(0, 0)$ to $(n, n)$ is dangerous if it crosses the diagonal.

**Actual question:** How many non-dangerous monotonic paths are there from $(0, 0)$ to $(n, n)$?

Let $n = 6$. Then the paths

- $RRURURURURURU$ is non-dangerous,
- $RRURURURURURR$ is dangerous.
To distinguish dangerous and non-dangerous paths, we count how many R and U moves did we make at every step:

\[
\begin{array}{cccccccccccc}
10 & 20 & 21 & 31 & 32 & 33 & 43 & 44 & 54 & 55 & 65 & 66 \\
R & R & U & R & U & R & U & R & U & R & U & \text{is non-dangerous,}
\end{array}
\]

\[
\begin{array}{cccccccccccc}
10 & 20 & 21 & 31 & 32 & 33 & 43 & 44 & 45 & 46 & 56 & 56 \\
R & R & U & R & U & R & U & R & U & U & R & R & \text{is dangerous.}
\end{array}
\]

Moreover, once the number of U-moves gets greater than the number of R-moves, we use the red color. Then, once the first red indicator appears, we write new path, where we change the path after the dangerous U-move: all R-moves we turn to U-moves, and all U-moves we turn to R-moves:

\[
\begin{array}{cccccccccccc}
10 & 20 & 21 & 31 & 32 & 33 & 43 & 44 & 45 & 46 & 56 & 56 \\
R & R & U & R & U & R & U & R & U & U & R & R & \text{is dangerous.}
\end{array}
\]

\[
\begin{array}{cccccccccccc}
10 & 20 & 21 & 31 & 32 & 33 & 43 & 44 & 46 & 56 & 56 \\
R & R & U & R & U & U & R & U & U & R & U & \text{new path.}
\end{array}
\]

In the black portion of the new path, we have 4 R-moves and 5 U-moves; in the red portion, we have 1 R-move and 2 U-moves. Totally, new path has 5 R-moves and 7 U-moves. Thus it is a path from (0, 0) to (5, 7). We claim that in this way every dangerous path turns to a path from (0, 0) to (5, 7). Thus we have the answer:

\[
\{| \text{# of all paths} \} - | \text{# of dangerous paths} | = \binom{12}{6} - \binom{12}{5}.
\]

For general \( n \), we do the same. Namely, we consider a dangerous path (first line) and we produce new path below:

\[
\begin{array}{cccccccccccc}
(k - 1) \text{ U's}, (k - 1) \text{ R's} & U & (n - k) \text{ U's}, (n - k + 1) \text{ R's} \\
(k - 1) \text{ U's}, (k - 1) \text{ R's} & U & (n - k) \text{ R's}, (n - k + 1) \text{ U's}
\end{array}
\]

The first path is dangerous since the red marker \( \downarrow \) shows that there are \( k \) U’s and \( (k - 1) \) R’s, so the path crossed the diagonal. For the new path we changed all U’s by R’s and all R’s by U’s after the red marker \( \downarrow \). Totally, for the new path, we have

\[
\begin{align*}
k + n - k + 1 &= n + 1 & \text{U's} \\
k - 1 + n - k &= n - 1 & \text{R's}
\end{align*}
\]

Thus we have the answer:

\[
b_n := \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}.
\]