

Elliptic Curves

Exercises set II

For Exercises 1–4, let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice, $T = \mathbb{C}/\Lambda$, and $O = 0 + \Lambda$. An elliptic function is a meromorphic Λ -periodic function. We use the Weierstrass functions

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

$$\sigma(z) = z \prod_{\omega \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\omega} \right) \exp \left(\frac{z}{\omega} + \frac{z^2}{2\omega^2} \right), \quad \wp = -\zeta'.$$

Recall that $\sigma'/\sigma = \zeta$. A divisor is *principal* if it is $\text{div}(g)$, the divisor of zeros minus poles of a nonzero elliptic function g .

Exercise 1. Let $a_1, \dots, a_r$ be distinct modulo Λ , and let $c_1, \dots, c_r \in \mathbb{C}$ satisfy $\sum_i c_i = 0$. Show that

$$f(z) = \sum_{i=1}^r c_i \zeta(z - a_i)$$

is elliptic, has residue c_i at $a_i + \Lambda$, and has no other poles. Show that every elliptic function with at most these simple poles and these residues differs from f by a constant.

Exercise 2. Let $P = a + \Lambda \in T$ and $n \geq 1$. Prove that $n[P] - n[O]$ is principal exactly when $nP = O$. When $na = m\omega_1 + \ell\omega_2$, with $m, \ell \in \mathbb{Z}$, construct a function with this divisor using the chosen lift a , without changing it.

Exercise 3. Prove the sigma addition identity

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}, \quad u, v \notin \Lambda,$$

interpreting the equality meromorphically.

Hint. Fix $v \notin \Lambda$. Show that the right-hand side is elliptic in u , compare its divisor with that of $\wp(u) - \wp(v)$, and determine the constant by examining the leading Laurent coefficient at $u = 0$.

Exercise 4. For fixed $a \notin \Lambda$, logarithmically differentiate the identity in Exercise 3 with $u = z$ and $v = a$ to show

$$\zeta(z+a) + \zeta(z-a) - 2\zeta(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)}.$$

Interpret the equality meromorphically, including when a is a nonzero half-period, that is, $a \notin \Lambda$ and $2a \in \Lambda$.

For Exercises 5–7, fix $\tau \in \mathbb{C}$ with $\Im \tau > 0$, and set $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $T_\tau = \mathbb{C}/\Lambda_\tau$.

Exercise 5. Recall the theta function

$$\vartheta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau + 2\pi i k z}.$$

Suppose f is entire and satisfies

$$f(z+1) = f(z), \quad f(z+\tau) = e^{-\pi i \tau - 2\pi i z} f(z).$$

Show that f is a scalar multiple of $\vartheta(z, \tau)$ as follows: consider f/ϑ , show that it is elliptic with at most one simple pole modulo Λ_τ , and use the impossibility of an elliptic function with just one simple pole. You may use that ϑ has the same two transformation laws as f , and that its zeros are precisely $(1+\tau)/2 + \Lambda_\tau$, all simple.

Exercise 6. For $d \in \mathbb{Z}$ and $a \in \mathbb{C}$, recall the factors of automorphy

$$j_{m+n\tau}^{(d,a)}(z) = e^{-\pi i d n^2 \tau - 2\pi i d n z + 2\pi i n a}, \quad m, n \in \mathbb{Z}.$$

The corresponding line bundle $\mathcal{L}_{d,a}$ on T_τ is the quotient of $\mathbb{C} \times \mathbb{C}$ by the identifications

$$(z, v) \sim (z + \lambda, j_\lambda^{(d,a)}(z)v), \quad \lambda \in \Lambda_\tau.$$

Its holomorphic sections are represented by entire functions s with

$$s(z+1) = s(z), \quad s(z+\tau) = e^{-\pi i d \tau - 2\pi i d z + 2\pi i a} s(z).$$

We denote this vector space by $H^0(T_\tau, \mathcal{L}_{d,a})$.

Check the cocycle identity in full:

$$j_{\lambda+\mu}^{(d,a)}(z) = j_\lambda^{(d,a)}(z+\mu) j_\mu^{(d,a)}(z), \quad j_0^{(d,a)}(z) = 1.$$

For $d, e \in \mathbb{Z}$ and $a, b \in \mathbb{C}$, show that

$$\mathcal{L}_{d,a} \otimes \mathcal{L}_{e,b} \simeq \mathcal{L}_{d+e, a+b}.$$

Verify directly that multiplication of representing functions induces the corresponding multiplication of sections.

Exercise 7. Determine $H^0(T_\tau, \mathcal{L}_{0,a})$ for all $a \in \mathbb{C}$. Show that it is one-dimensional if $a \in \Lambda_\tau$ and zero otherwise. Identify a nonzero section when $a = m + n\tau$, with $m, n \in \mathbb{Z}$.