1. Results from commutative algebra

The following results about integral extensions are proved in Chapter 5 of [AM]). The setup is the following: we have an extension of domains $A \subset B$ such that B is integral over A.

Lifting ([AM, Cor. 5.8, Thm. 5.10]). Any prime ideal in A is obtained as $J \cap A$, where J is a prime ideal in B (and J is maximal in B if and only if $J \cap A$ is maximal in A).

Going-down ([AM, Thm. 5.16, Cor. 5.9]) If $I_1 \subset I_2 \subset A$ prime ideals and $I_2 = J_2 \cap B$, where J_2 is a prime ideal in B then there exists a prime ideal J_1 in B such that $J_1 \subset J_2$ and $I_1 = J_1 \cap B$. If $J_1 \subset J_2$ are prime ideals in B such that $J_1 \cap A = J_2 \cap A$ then $J_1 = J_2$.

Recall that (Krull) dimension of a commutative ring R is the maximal length of a chain of prime ideals in R (for example $\dim \mathbb{Z} = 1$, and an example of a maximal chain is $(0) \subset (3)$). For an affine variety X we have $\dim X = \dim A(X)$. Also, if $\mathcal{O}_{X,p}$ is the local ring of X at point p then $\dim \mathcal{O}_{X,p} = \dim_p X$, where the local dimension $\dim_p X$ is defined in terms of chains of irreducible closed subvarieties passing through p (this follows from the correspondence between prime ideals in $\mathcal{O}_{X,p}$ and irreducible closed subsets containing p).

We also need the following result from commutative algebra (see [AM, Cor. 11.18])

Proposition 1.1. *Let R be a Noetherian local ring, $x \in R$ a nonzero divisor. Then* $\dim R/(x) = \dim R - 1$.

2. Applications to algebraic geometry

Definition. A morphism of affine varieties $X \to Y$ is called *finite* if $A(X)$ is integral over $A(Y)$.

Proposition 2.1. *If $f : X \to Y$ is a finite dominant morphism of affine varieties then it is surjective and* $\dim X = \dim Y$. *Furthermore, for any $x \in X$ one has* $\dim_x X = \dim_{f(x)} Y$.

Proof. Surjectivity follows from the lifting property for maximal ideals. Equality of dimensions follows from the going-down property for chains of prime ideals. □

Proposition 2.2. *Let X be an affine variety. Then for any point $x \in X$ one has* $\dim_x X = \dim X$.

Proof. The method of linear projections gives a finite surjective morphism $\pi : X \to \mathbb{A}^n$. Now Proposition 2.1 implies that $\dim_x X = \dim_{\pi(x)} \mathbb{A}^n = n$ for any $x \in X$. □

Proposition 2.3. *Let X be an affine variety, $f \in A(X)$ a nonzero function. Then $Z(f) \subset X$ is either empty or has dimension* $\dim X - 1$.

Proof. Pick a point $p \in Z(f)$. Then by Proposition 1.1, $\dim_p Z(f) = \dim_p X - 1$. Thus, our assertion follows from Proposition 2.2. □

References

[AM] M. Atiyah, I. Macdonald, Introduction to commutative algebra.