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ALGEBRAIC GEOMETRY OF POISSON BRACKETS

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This paper is devoted to the study of Poisson brackets in the framework of algebraic geometry. The need for such a study arises from several sources. One is the problem of classification of quadratic Poisson structures, i.e., Poisson brackets on the polynomial algebra of n variables $x_1, \ldots, x_n$ such that $\{x_i, x_j\}$ are quadratic forms. These structures arise as tangents to noncommutative deformations of the polynomial algebra (see [2, 7]). Other examples of algebraic Poisson structures come from the representation theory, namely, the structures derived from the Kostant–Kirillov Poisson bracket on the dual space of a Lie algebra. More specifically, this topic leads to the study of *nondegenerate* Poisson structures, i.e., those which are simplectic at the general point. In either case, it seems appropriate to apply the machinery of algebraic geometry to the study of these structures. Here the next important notion after that of Poisson scheme (which is straightforward) is the notion of a *Poisson module* (see Sec. 1 below). Namely, while Poisson schemes arise naturally when considering the degeneration loci of Poisson structures, the notion of Poisson module plays an important role in our treatment of the standard types of morphisms (such as blow-up, line bundles and projective line bundles) in the Poisson category. Also, we translate into this language the classical results concerning operators acting on the De Rham complex of a Poisson variety X (see [15, 5]) to produce the canonical Poisson module structure on the canonical line bundle ω_X. We apply the developed technique to the following problems:

1. The conjecture of A. Bondal stating that if X is a Fano variety, then the locus where the rank of a Poisson structure on X is $\leq 2k$ has a component of dimension $> 2k$. We verify this conjecture for the maximal nontrivial degeneration locus in two cases: when X is the projective space and when the Poisson structure has maximal possible rank at the general point.

2. The description of the differential complex (see Sec. 1 for a nondegenerate Poisson bracket on a smooth even-dimensional variety X). It turns out that when the degeneration divisor is a union of smooth components with normal crossings, the structure of this complex is completely determined by the corresponding codimension-1 foliation of the degeneration divisor. Also, we prove that in this case the rank of the Poisson structure is constant along the stratification defined by the arrangement of components of the degeneration divisor (provided that X is projective).

3. The classification of Poisson structures on $\mathbf{P}^3$. Namely, any such structure vanishes (at least) on a curve, and we classify those structures for which the vanishing locus contains a smooth curve as a connected component.

4. The study of hamiltonian vector fields for a nondegenerate Poisson structure on $\mathbf{P}^{2n}$. Namely, we prove the absence of nonzero hamiltonian vector fields for a nondegenerate Poisson structure on $\mathbf{P}^{2n}$ which has irreducible and reduced degeneration divisor. Examples of such Poisson structures are provided by the work of B. Feigin and A. Odesskii [7].

By a scheme we always mean a scheme of finite type over $\mathbf{C}$.

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1. Basic Definitions

Definition. A Poisson algebra is a commutative associative algebra with a unit A equipped with a Lie bracket (also called a Poisson bracket) $\{\ ,\ \}$ such that the Leibnitz identity holds: $\{x, yz\} = y\{x, z\} + z\{x, y\}$.

This definition can be easily schematized: one can define what a Poisson structure on a scheme is. Also the natural notion of a Poisson morphism of Poisson schemes (or a Poisson homomorphism of Poisson algebras) allows one to speak about Poisson subschemes (resp., Poisson ideals), etc. For example, for any Poisson scheme X there is a canonical Poisson subscheme $X_0 \subset X$ such that the induced Poisson structure on X_0 is zero and X_0 is maximal with this property, i.e., the corresponding Poisson ideal sheaf is $\mathcal{O}_X\{\mathcal{O}_X, \mathcal{O}_X\} \subset \mathcal{O}_X$.

Any Poisson structure on a scheme X is given by some $\mathcal{O}_X$-linear homomorphism $H : \Omega_X \to T_X = \mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X)$ such that $H(df)(g) = \{f, g\}$. If X is smooth, we denote by G the corresponding section of $\bigwedge^2 T_X$ so that the following identity holds:

$$i(\omega)G = H(\omega) \tag{1}$$

for any $\omega \in \Omega^1_X$, where $i(\omega)$ is the operator of contraction with ω.

If X is irreducible, a Poisson structure H on X is said to be *nondegenerate* if it has maximal rank at the general point. When X is smooth and $\dim X$ is even, we define the *divisor of degeneration* $Z \subset X$ of a nondegenerate Poisson structure on X as the zero locus of the Pfaffian of H which is a section of $\det T_X \simeq \omega_X^{-1}$. In fact, Z is a Poisson subscheme of X. The "differentiable" proof is obvious: a Poisson structure is constant along any hamiltonian flow ϕ_t, so the condition of degeneracy is preserved under ϕ_t. It follows that any hamiltonian flow moves any irreducible component of Z into itself. This means that if f is a local equation of such a component, then for any hamiltonian vector field H_g the function $H_g(f) = \{g, f\}$ is zero along this component, as required. An algebraic proof of this fact will be given in Sec 2.

The following result is rather basic in order to justify the geometric intuition.

Lemma 1.1. *Let X be a Poisson scheme of finite type over $\mathbf{C}$, and X_{red} be the corresponding reduced scheme. Then X_{red} and all its irreducible components are Poisson subschemes of X.*

Proof. It is sufficient to prove the following local statement: the nil-ideal of a commutative algebra A (resp. a minimal prime ideal of a commutative algebra A_0 without nilpotents) is preserved by any derivation $v : A \to A$ (resp. $v_0 : A_0 \to A_0$). The first part is implied by the following fact: if $x^n = 0$ for $x \in A$, then $v(x)^n$ is divisible by x. Indeed, we have

$$v(x^n) = nx^{n-1}v(x) = 0.$$

Applying v to this equality, we obtain

$$(n-1)x^{n-2}v(x)^2 + x^{n-1}v^2(x) = 0,$$

that is, $x^{n-2}v(x)^2 \in x^{n-1}A$. Iterating this procedure, we get the inclusion $x^{n-i}v(x)^i \in x^{n-i+1}A$, which for $i = n$ gives the required property. Now let A_0 be a commutative algebra without nilpotents, $P_1, P_2, \ldots, P_r$ be its minimal prime ideals so that $P_1 \cap P_2 \cap \ldots \cap P_r = 0$. Let us prove, e.g., that P_1 is preserved by v_0. Let $x_1 \in P_1$, $x_i \in P_i \setminus P_1$ for $i > 1$. Then the product $x_1 x_2 \ldots x_r$ is zero, hence

$$v(x_1 x_2 \ldots x_r) = x_1 v(x_2 \ldots x_r) + x_2 \ldots x_r v(x_1) = 0,$$

which implies that $v(x_1) \in P_1$. □

Some features of Poisson structures trace back to the following more general notion.

Definition (see [1]). A *Lie algebroid* on X is an $\mathcal{O}_X$-module L equipped with a Lie algebra bracket $[\cdot, \cdot]$ and an $\mathcal{O}_X$-linear morphism of Lie algebras $\sigma : L \to T_X$ such that for $l_1, l_2 \in L$, $f \in \mathcal{O}_X$ one has

$$[l_1, fl_2] = f[l_1, l_2] + \sigma(l_1)(f)l_2. \tag{2}$$

Remark. The affine version of this notion is also called a Lie–Rinehart algebra (see [12]).

Example. As we have seen above, a Poisson structure defines an $\mathcal{O}_X$-linear homomorphism $H : \Omega_X \to T_X$. It can be extended to the unique Lie algebroid structure on Ω_X such that $[df, dg] = d\{f, g\}$ (see [12]), which is called a *Poisson–Lie algebroid.* In the case of a symplectic structure on the smooth variety, this Lie algebroid is isomorphic to the tautological one (T_X, id).

Definition (see [1]). A (left) *module* over a Lie algebroid L (or just *L-module*) is an $\mathcal{O}_X$-module M equipped with a Lie action of L such that for any $f \in \mathcal{O}_X$, $l \in L$, $x \in M$ one has $l(fx) = \sigma(l)(f)x + (fl)x$, $(fl)x = f(lx)$.

Following [1], define a universal enveloping algebra $U(L)$ of a Lie algebroid L as a sheaf of $\mathcal{O}_X$-algebras equipped with a morphism of Lie algebras $i : L \to U(L)$ which is generated by $i(L)$ as an $\mathcal{O}_X$-algebra with the defining relations $i(fl) = fi(l)$, $[i(l), f] = \sigma(l)(f)$ for any $f \in \mathcal{O}_X$, $l \in L$. Then clearly an L-module is the same as a $U(L)$-module in the usual sense.

Applying this definition to the Poisson–Lie algebroid constructed above, we obtain the notion of a *Poisson module*, which is equivalent to that of a $\mathcal{D}$-module in the case of a symplectic structure (where $\mathcal{D}$ is the sheaf of differential operators). By analogy with $\mathcal{D}$-modules, one can represent a Poisson module structure on an $\mathcal{O}_X$-module as a *flat Poisson connection* on it (see [12]).

Definition. A *Poisson connection* on an $\mathcal{O}_X$-module $\mathcal{F}$ is a $\mathbf{C}$-linear bracket $\{\ ,\ \} : \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ which is a derivation in the first argument and satisfies the Leibnitz identity

$$\{f, gs\} = \{f, g\}s + g \cdot \{f, s\}, \tag{3}$$

where $f, g \in \mathcal{O}_X$, s is a local section of $\mathcal{F}$. Equivalently, a Poisson connection is given by a homomorphism $v : \mathcal{F} \to \mathcal{H}om(\Omega_X, \mathcal{F}) = \mathrm{Der}(\mathcal{O}_X, \mathcal{F})$ which satisfies the identity

$$v(fs) = -H(df) \otimes s + f \cdot v(s), \tag{4}$$

where $f \in \mathcal{O}_X$. Namely, $v(s) \in \mathrm{Der}(\mathcal{O}_X, \mathcal{F})$ is defined by the formula

$$v(s)(f) = \{f, s\}.$$

A Poisson connection is called *flat* if the bracket above gives a Lie action of $\mathcal{O}_X$ on $\mathcal{F}$, where $\mathcal{O}_X$ is considered as a Lie algebra via the Poisson bracket. Equivalently, for a Poisson connection $v : \mathcal{F} \to \mathrm{Der}(\mathcal{O}_X, \mathcal{F})$ one can define a homomorphism $\tilde{v} : \mathrm{Der}(\mathcal{O}_X, \mathcal{F}) \to \mathrm{Der}^2(\mathcal{O}_X, \mathcal{F})$, where the target is the $\mathcal{O}_X$-module of skew-symmetric biderivations with values in M, by the formula

$$\tilde{v}(\delta)(f, g) = \{f, \delta(g)\} + \{\delta(f), g\} - \delta(\{f, g\}). \tag{5}$$

Consider the composed map $c(v) = \tilde{v} \circ v$. Then $c(v)$ is $\mathcal{O}_X$-linear and v is flat if and only if $c(v) = 0$.

Note that the usual connection ∇ on $\mathcal{F}$ defines the Poisson connection $H(\nabla)$, and obviously this is a one-to- one correspondence in the symplectic case. The difference between two Poisson connections on the same sheaf is $\mathcal{O}_X$-linear. A Poisson module structure on $\mathcal{O}_X$ is the same as a vector field preserving the Poisson bracket (the so-called *hamiltonian vector field*). Also, for Poisson modules $\mathcal{F}$ and $\mathcal{G}$ there is a natural Poisson module structure on the $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ given by the formulas $\{f, x \otimes y\} = \{f, x\} \otimes y + x \otimes \{f, y\}$ and $\{f, \phi\}(x) = \{f, \phi(x)\} - \phi(\{f, x\})$, where $f \in \mathcal{O}_X$, $x \in \mathcal{F}$, $y \in \mathcal{G}$, $\phi \in \mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Let L be a Lie algebroid. Define the $U(L)$-linear differential d on $U(L) \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} L$ by the formula

$$\begin{aligned} d(u \otimes (l_1 \wedge \ldots \wedge l_n)) &= \sum_i (-1)^{i-1} u\lambda_i \otimes (l_1 \wedge \ldots \wedge \hat{l_i} \wedge \ldots \wedge l_n) \\ &+ \sum_{i<j} (-1)^{i+j} u \otimes ([l_i, l_j] \wedge l_1 \wedge \ldots \wedge \hat{l_i} \wedge \ldots \wedge \hat{l_j} \wedge \ldots \wedge l_n). \end{aligned} \tag{6}$$

One can check that $d^2 = 0$. Furthermore, if L is locally free as an $\mathcal{O}_X$-module, then the complex K of left $U(L)$-modules below

$$\ldots \to U(L) \otimes \bigwedge^2(L) \xrightarrow{d} U(L) \otimes L \xrightarrow{d} U(L) \to 0$$

is a free resolvent of $\mathcal{O}_X$ considered as a $U(L)$-module, where the map $U(L) \to \mathcal{O}_X$ sends u to $u \cdot 1$ (see [17]).

For any L-module M, we define the complex $K_L(M) = \mathcal{H}om_{U(L)}(K, M)$. Then the hypercohomology space $H^j(K_L(M))$ is naturally isomorphic to $\mathrm{Ext}^j_L(\mathcal{O}_X, M)$. In particular, we have an analogue of the De Rham complex $K_L = K_L(\mathcal{O}_X)$:

$$\mathcal{O}_X \to L^\vee \to \bigwedge^2 L^\vee \to \dots .$$

The exterior product on K_L is compatible with the differential, so that it is a dg-algebra. If X is smooth, then by functoriality there is a natural morphism of dg-algebras $\Omega_X \to K_L$ such that the homomorphism $\Omega^1_X \to L^\vee$ is dual to $\sigma : L \to T_X$. For the Poisson–Lie algebroid this homomorphism is equal to $-H$. Hence, for the corresponding differential d_P on $\bigwedge^\bullet T_X$ one has $d_P(f) = -H(df)$, $d_P(H(\omega)) = -\bigwedge^2(H)(d\omega)$, and so on.

Definition. Let $L = \Omega_X$ be a Poisson–Lie algebroid, N be a Poisson module. Then the spaces

$$H^j_P(X, N) := H^j(K_L(N))$$

are called *Poisson cohomologies* of X with coefficients in N, and the spaces

$$H^j_P(X) := H^j_P(X, \mathcal{O}_X) = H^j(K_L) \simeq \mathrm{Ext}^j_L(\mathcal{O}_X, \mathcal{O}_X)$$

are called Poisson cohomologies of X. The complex K_L is called the *differential complex* of a Poisson variety.

2. Some Lie Algebroids Associated with a Poisson Structure

We begin with a generalization of the notion of a Poisson module. Let X be a smooth Poisson variety, $\mathcal{D}$ be the sheaf of differential operators on X.

Definition. A $\mathcal{D}$-Poisson module is an $\mathcal{O}_X$-module M with a Lie action of $\mathcal{O}_X$ (where the Lie bracket on $\mathcal{O}_X$ is the Poisson one) given by a map

$$\{\ ,\ \} : \mathcal{O}_X \times M \to M,$$

which is a differential operator in the first argument and satisfies the identity

$$\{f, gm\} = \{f, g\}m + g\{f, m\}.$$

In other words, this structure corresponds to some map

$$v : M \to \mathcal{D} \otimes_{\mathcal{O}_X} M,$$

where the $\mathcal{O}_X$-module structure on $\mathcal{D}$ is the left one, such that

$$v(fm) = -H(df) \otimes m + fv(m),$$

where $f \in \mathcal{O}_X$, $m \in M$.

Note that there is a decomposition $\mathcal{D} \simeq \mathcal{O}_X \oplus \mathcal{D}_+$, where $\mathcal{D}_+ = \mathcal{D}T_X$ is the left ideal in $\mathcal{D}$ generated by T_X. We define a $\mathcal{D}_+$-Poisson module as a $\mathcal{D}$-Poisson module M such that $\{1, M\} = 0$. In other words, this structure is defined by a map $M \to \mathcal{D}_+ \otimes M$. Now the structure of a $\mathcal{D}$-Poisson module on M is the same as a $\mathcal{D}_+$-Poisson module structure on it together with an endomorphism of M as a $\mathcal{D}_+$-Poisson module.

Example. Let $E = \bigwedge^p T_X \otimes \Omega^q_X$ be the bundle of tensor fields on X of type (p, q). Then there is the canonical $\mathcal{D}_+$-Poisson module structure on E given by the formula

$$\{f, x\} = L_{H(df)}(x), \tag{7}$$

where $f \in \mathcal{O}_X$, $x \in E$, L_v denotes the Lie derivative along the vector field v. In the case $E = \omega_X$, this gives the usual Poisson module structure on ω_X, as one can easily check (see also Lemma 4.4).

Consider the jet bundle $\mathcal{P} := \mathcal{D}^\vee_+$. We have $\mathcal{P} \simeq \varprojlim \mathcal{P}_k$, where $\mathcal{P}_k = (\mathcal{D}_{\le k-1}T_X)^\vee$. Let us denote by $s : \mathcal{O}_X \to \mathcal{P} = \mathcal{D}^\vee_+$ the canonical map such that $s(f)(D) = D(f)$. Then there is the unique Lie algebroid

structure on $\mathcal{P}$ such that $[s(f), s(g)] = s(\{f, g\})$ and the canonical projection $\mathcal{P} \to \Omega_X$ is a morphism of Lie algebroids (the analogous statement is true for $\mathcal{P}_k$). Now one can easily verify that a $\mathcal{D}_+$-Poisson module is the same as a $\mathcal{P}$-module.

Lemma 2.1. *Let $\mathcal{F}$ be a $\mathcal{D}_+$-Poisson module which is a coherent $\mathcal{O}_X$-module. Then the annihilator of $\mathcal{F}$ is a Poisson ideal sheaf in $\mathcal{O}_X$, and the locus where the rank of $\mathcal{F}$ is greater than n is a Poisson subscheme of X for any n.*

Proof. Let $f\mathcal{F} = 0$ for some $f \in \mathcal{O}_X$. Then

$$\{g, fs\} = f\{g, s\} + \{g, f\}s = 0$$

for any $g \in \mathcal{O}_X$, $s \in \mathcal{F}$, hence $\{g, f\}s = 0$, i.e. $\{g, f\}$ annihilates $\mathcal{F}$. Thus, the support of $\mathcal{F}$ is a Poisson subscheme of X. Applying this to the exterior powers of $\mathcal{F}$, we get the second statement. □

Proposition 2.2. *Let $g : M \to N$ be a morphism of $\mathcal{D}_+$-Poisson modules which are locally free $\mathcal{O}_X$-modules of finite rank. Then the locus where the rank of g is less than n is a Poisson subscheme of X for any n.*

Proof. Using the duality and the exterior power operations on $\mathcal{D}_+$-Poisson modules, we reduce the problem to showing that the vanishing locus of a morphism $g : M \to \mathcal{O}_X$ (a Poisson module structure on $\mathcal{O}_X$ being the canonical one) is a Poisson subscheme. It remains to apply the previous lemma to $\operatorname{coker}(g)$. □

Corollary 2.3. *The degeneration loci of the structural morphism $H : \Omega_X \to T_X$ of a Poisson variety X are Poisson subschemes of X.*

Proof. The action (7) of the Lie derivative along the vector field $H(df)$ preserves H, hence, it can be considered as a morphism of $\mathcal{D}_+$-modules. □

Remark. One can show that any hamiltonian vector field preserves the degeneration loci of H, which is a stronger property than just being a Poisson subscheme.

Even when X is singular, the first definition of a $\mathcal{D}_+$-Poisson module still works, and we have a $\mathcal{D}_+$-Poisson module structure on Ω_X which is uniquely (and correctly) defined by the condition $\{f, dg\} = d\{f, g\}$. Applying Lemma 2.1 above to this case, we get the following corollary.

Corollary 2.4. *The singular locus of a Poisson scheme is a Poisson subscheme.*

Remark. The Lie algebroid structure on $\mathcal{P}_2$ induces an $\mathcal{O}_X$-linear Lie algebra structure on $S^2\Omega_X$ as on the kernel of the natural projection $\mathcal{P}_2 \to \mathcal{P}_1 = \Omega_X$. If $x \in X$ is a closed point, then the stalk of $S^2\Omega_X$ at x is isomorphic to m_x^2/m_x^3, where m_x is the maximal ideal corresponding to x, and the Lie algebra structure on this space is the one induced by the Poisson bracket (note that $\{m_x^2, m_x^2\} \subset m_x^2$). The original Poisson structure can be recovered from this family of Lie algebras, as one can easily see from the formula $\{f^2, g^2\} = 4\{f, g\}fg$.

Let X be a Poisson scheme, $Y \subset X$ be a Poisson subscheme with the defining ideal J. Then there is a natural $\mathcal{O}_Y$-linear Lie bracket on J/J^2. If $Y = y$ is a point (that is, the Poisson bracket vanishes at y), the corresponding Lie algebra J/J^2 is called classically the *linearization* of the bracket at Y (or the *cotangent Lie algebra*).

The following lemma gives a useful criterion of smoothness of the vanishing locus of a Poisson structure.

Lemma 2.5. *Let $Z \subset X$ be a vanishing locus of a Poisson structure on a smooth variety X, $\mathfrak{g} = \mathfrak{m}_x/\mathfrak{m}_x^2$ be a conormal Lie algebra of a closed point $x \in Z$, where $\mathfrak{m}_x \subset \mathcal{O}_X$ is the corresponding maximal ideal. Then there is a natural isomorphism of $k(x)$-vector spaces*

$$T_x Z \simeq (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*,$$

where T_xZ is a Zariski tangent space to Z at x. In particular, Z is smooth at x if and only if

$$\dim_{k(x)}(\mathbf{g}/[\mathbf{g},\mathbf{g}]) = \dim_x Z,$$

where $\dim_x Z$ is the local dimension of Z at x.

Proof. By definition, $(T_xZ)^* \simeq \mathbf{m}_x/(J_Z + \mathbf{m}_x^2)$, where $J_Z \in \mathcal{O}_X$ in the ideal of Z. Let (x_i) be local coordinates at x. Then J_Z is locally generated by the functions $\{x_i, x_j\}$, while the Lie algebra structure on $\mathbf{m}_x/\mathbf{m}_x^2$ is given by $[x_i, x_j] = \{x_i, x_j\} \mod (\mathbf{m}_x^2)$, so the assertion follows. □

For any Poisson module F on X, the restriction $F|_Y$ becomes naturally a module over this Lie algebra. More generally, for any Lie algebroid L on X, one can consider L-ideals in $\mathcal{O}_X$. If J is such an ideal, and $Y \subset X$ is the corresponding subscheme, then $L|_Y$ becomes a Lie algebroid on Y. This follows from the fact that the image of the structural morphism $L \to T_X$ is contained in the subsheaf of derivations preserving J. An L-module F on X such that $JF = 0$ is the same as an $L|_Y$-module and the functor $F \mapsto F/JF$ from the category of L-modules to the category of $L|_Y$-modules is left adjoint to the inclusion functor.

In the case of the Poisson–Lie algebroid $L = \Omega_X$ and a Poisson ideal $J \subset \mathcal{O}_X$, we obtain a Lie algebroid structure on $\Omega_X|_Y$. Furthermore, the canonical morphism $r : \Omega_X|_Y \to \Omega_Y$ is a morphism of Lie algebroids. Now assume that X and Y are smooth. Then there is an exact sequence

$$0 \to J/J^2 \to \Omega_X|_Y \to \Omega_Y \to 0, \tag{8}$$

where J/J^2 is a Lie ideal in $\Omega_X|_Y$. One can check that the induced Lie bracket on J/J^2 is the natural one. When this bracket is zero the sequence (8) defines a Poisson module structure on J/J^2.

Thus, if $Y \subset X$ is a Poisson subscheme of a Poisson scheme and F is a Poisson module over X, then there is an $\Omega_X|_Y$-module structure on $F|_Y$ extending the $\mathcal{O}_Y$-linear Lie action of J/J^2 on $F|_Y$. When the latter action is zero, we obtain a Poisson module structure on $F|_Y$. For example, if $\mathcal{L}$ is a Poisson line bundle on X such that the corresponding Poisson connection $\mathcal{L} \to T_X \otimes \mathcal{L}$ is defined locally by a hamiltonian vector field v preserving Y, i.e., $v(J) \subset J$, then there is a natural Poisson module structure on $\mathcal{L}|_Y$. This condition is always satisfied when $Y \subset X$ is some degeneration locus of the Poisson structure.

3. Batalin–Vilkovisky Structures and the Koszul Operator

Let A be a (sheaf of) (associative) (super)commutative graded algebra(s), where the parentheses contain the words we will omit further.

Let $\mathcal{D} = \mathcal{D}(A)$ be the algebra of (super)differential operators on A. By definition, it contains $\mathcal{D}_0 = A$ as a subalgebra of the left multiplication operators. It is also endowed with the natural increasing filtration $\mathcal{D}_k \subset \mathcal{D}$, where the elements of $\mathcal{D}_k$ are called operators of order $\leq k$ and are characterized by the property $[D, f] \in \mathcal{D}_{k-1}$ for any $f \in \mathcal{D}_0$ (the commutator of two operators is $[D_1, D_2] = D_1D_2 - (-1)^{|D_1||D_2|}D_2D_1$). Algebra $\mathcal{D}$ also has the natural grading $\mathcal{D} = \bigoplus \mathcal{D}^i$ by the degree of operators.

Lemma 3.1. *Assume that A is generated by A_1 as an algebra over A_0. Then for any nonzero differential operator $D \in \mathcal{D}(A)$ one has*

$$\deg D + \operatorname{order} D \geq 0.$$

Proof. Induction in order shows that if the inequality does not hold for some D of degree < 0, then for any $a_1 \in A_1$ one has $[D, a_1] = 0$. Now since $D(A_0) = 0$ and A is generated by A_1 over A_0, this implies that $D(A) = 0$. □

Following E. Getzler, we call a commutative graded algebra A endowed with a Lie bracket of degree 1 (which is a graded Lie bracket on $A[1]$) a *braid algebra* if the following identity is satisfied:

$$[u, vw] = [u, v]w + (-1)^{|v||w|}[u, w]v.$$

In other words, $\mathrm{ad}(u)$ is a derivation of degree $|u|-1$. Note that since the bracket has degree 1, the usual sign rule does not apply in general. Let us also define the notion of an anti-braid algebra in the same manner, with the only difference being that the Lie bracket should be of degree -1. For a graded algebra A, denote by A^- the same algebra with the opposite grading: $A^-_i = A_{-i}$. Then A is a braid algebra if and only if A^- is an anti-braid algebra.

Definition. Let $\sigma : L \to T_X$ be a Lie algebroid. Then the Schouten bracket is the unique braid algebra structure on the exterior algebra $\bigwedge(L)$ such that the bracket on L is the old one and $[l, f] = \sigma(l)(f)$ for $l \in L$, $f \in \mathcal{O}_X$.

Conversely, any braid structure on the exterior algebra $\bigwedge(L)$ corresponds to some Lie algebroid structure on L.

Proposition 3.2 ([10]). *Let $\Delta \in \mathcal{D}_2^{-1}$ be a d.o. on A of degree -1 and order 2 such that $\Delta^2 = 0$. Then the following bracket gives a braid algebra structure on A:*

$$[u, v] = [\varepsilon(u), [\Delta, \varepsilon(v)]](1), \tag{9}$$

where $u, v \in A$, $\varepsilon(x) \in \mathcal{D}_0$ is the left multiplication by x, $1 \in A_0$ is the unit.

A commutative graded algebra with such an operator Δ is called a *Batalin–Vilkovisky algebra* (or a *B–V algebra*).

This construction can be generalized as follows: let $\mathcal{B} \in \mathcal{D}$ be a graded commutative subalgebra of a graded associative algebra $\mathcal{D}$. Then it defines an increasing filtration on $\mathcal{D}$: consider $\mathcal{B}$ as the set of elements of order 0; for $k > 0$, an element $x \in \mathcal{D}$ has order $\leq k$ with respect to $\mathcal{B}$ if and only if the supercommutator $[x, b]$ has order $\leq k-1$ for any homogeneous element $b \in \mathcal{B}$. Now an element $\Delta \in \mathcal{D}^{-1}$ of order 2 with respect to $\mathcal{B}$ such that $\Delta^2 = 0$ defines a braid algebra structure on $\mathcal{B}$ by the same formula: $[b_1, b_2] = [b_1, [\Delta, b_2]] \in \mathcal{B}$. Similarly, if $\Delta \in \mathcal{D}^1$ is an element with the same properties, it defines an anti-braid structure on $\mathcal{B}$.

In the above situation, $\mathcal{D} = \mathcal{D}(A)$, $\mathcal{B} = \mathcal{D}_0$. Here are some other examples where such a scheme applies.

Examples. 1. Let $\mathcal{D} = \mathcal{D}(A)$ and assume that $\mathcal{D}_k^{-k-1} = 0$ for any $k \geq 0$. According to Lemma 3.1 this is so if A is generated by A_1 as an algebra over A_0. Put $\mathcal{B} = \bigoplus_{k \geq 0} \mathcal{D}_k^{-k}$; this is a supercommutative subalgebra of $\mathcal{D}$. Now any $\Delta \in \mathcal{D}_1^1$ is of order 2 with respect to $\mathcal{B}$. Indeed the degree of $[[\Delta, b_1], b_2]$ is equal to $|b_1| + |b_2| + 1$ while its order does not exceed $\mathrm{order}(b_1) + \mathrm{order}(b_2) - 1 \leq -|b_1| - |b_1| - 1$, as required. Therefore, any element $\Delta \in \mathcal{D}_1^1$ such that $\Delta^2 = 0$ defines an anti-braid algebra structure on $\mathcal{B}$.

2. This is a particular case of the previous example. Let X be smooth, $A = \Omega_X^\bullet$ be the De Rham complex. Then $\mathcal{B} \simeq (\bigwedge^\bullet T_X)^-$ (polyvector fields act by contractions on A) and the braid algebra structure on $\bigwedge^\bullet T_X$ induced by the De Rham differential d is given by the usual Schouten bracket. To prove this, it is sufficient to check the following two simple identities: $[i(v), [d, f]](1) = v(f)$, $[i(v), [d, i(w)]](df) = [v, w](f)$, where $v, w \in T_X$, f is a function, $i(\cdot)$ is the contraction operator.

3. More generally, for any Lie algebroid L which is a locally free $\mathcal{O}_X$-module, the Schouten bracket on $\bigwedge^\bullet L$ is obtained by the same construction from the Koszul differential on $\bigwedge^\bullet L^\vee$ (see Sec. 1). Thus, in this case we have the classical identity (see [14])

$$[i(x), [d, i(y)]] = i([x, y]). \tag{10}$$

Lemma 3.3. *Let $A = \bigwedge^\bullet(L)$ be the exterior algebra of a locally free $\mathcal{O}_X$-module L of finite rank, $\mathcal{B} \in \mathcal{D} = \mathcal{D}(A)$ be the subalgebra consisting of operators of contraction with elements of $\bigwedge^\bullet(L^\vee)$. Then an operator $\Delta \in \mathcal{D}$ has order $\leq k$ with respect to $\mathcal{B}$ if and only if it belongs to $\bigoplus_i \mathcal{D}_{k-i}^i$.*

Proof. Clearly, the elements of $\mathcal{D}_{k-i}^i$ have order $\leq k$ with respect to $\mathcal{B}$. To prove another inclusion, we use induction in k. Let Δ be an operator of degree i and order $\leq k$ with respect to $\mathcal{B}$. Then by the induction

hypothesis this is equivalent to the following two conditions: $[\Delta, f] \in \mathcal{D}^i_{k-i-1}$ for any $f \in \mathcal{O}_X$ and $[\Delta, i(\varphi)] \in \mathcal{D}^{i-1}_{k-i}$ for any $\varphi \in L^\vee$. Applying the same criterion for $k-1$ instead of k, one can check that for any $l \in L$ the operator $[\Delta, \varepsilon(l)]$ has order $\leq k-1$ with respect to $\mathcal{B}$. Therefore, $[\Delta, \varepsilon(l)] \in \mathcal{D}^{i+1}_{k-i-1}$ for any $l \in L$, which implies that $\Delta \in \mathcal{D}^i_{k-i}$. □

Now we are going to find all B–V structures on the exterior algebra. Consider first the case, where the line bundle $\bigwedge^n(L)$ is trivial. Let $\phi \in \bigwedge^n(L)$ be a nonvanishing section, $P = P_\phi : \bigwedge^i L^\vee \to \bigwedge^{n-i} L$ be the corresponding $\mathcal{O}_X$-linear isomorphism defined by the relation

$$\langle P_\phi(x), y\rangle \phi = x \wedge y$$

for any $x \in \bigwedge^i L^\vee$, $y \in \bigwedge^{n-i} L^\vee$. Note that P induces an isomorphism of graded algebras

$$P_D : \mathcal{D}(\bigwedge^\bullet L^\vee) \simeq \mathcal{D}(\bigwedge^\bullet L)^-$$

given by $D \mapsto P \circ D \circ P^{-1}$.

For $x \in \bigwedge^i L^\vee$, $v \in \bigwedge^i L$, the following identities hold:

$$P_D(i(v)) = \varepsilon(v), \qquad P_D(\varepsilon(x)) = i(x),$$

where $\varepsilon(\cdot)$ denotes the (left) exterior product operator.

Proposition 3.4. *An operator*

$$\Delta_\phi = P \circ d \circ P^{-1},$$

where d is the Koszul differential on $\bigwedge^\bullet L^\vee$, defines a B-V structure on $\bigwedge^\bullet L$ extending the usual braid algebra structure on it. The Schouten bracket of two tensors v, w is given by the following formula:

$$[v, w] = [\varepsilon(v), [\Delta_\phi, \varepsilon(w)]](1). \tag{11}$$

Proof. The fact that Δ_ϕ is a differential operator of the second order follows from Lemma 3.3. The remaining statements follow from comparison of the formulas (9) and (10), since P_D interchanges $\varepsilon(v)$ and $i(v)$. □

Applying the formula (11) to $L = T_X$ for a smooth variety X we obtain an operator Δ_ϕ on polyvector fields, which was introduced by J.-L. Koszul in [14]. Later we will need the following property of this operator.

Lemma 3.5. *Let $L = T_X$, $\psi = f\phi$, where f is an invertible function. Then $\Delta_\psi = \Delta_\phi + i(dln(f))$.*

Proof. Indeed $\Delta_\psi - \Delta_\phi$ is a derivation of $\bigwedge^\bullet T_X$, so it is sufficient to check this equality on elements of T_X. Now it follows from the formula $\Delta_\phi(v) = \mathrm{Lie}_v\, \phi$, which is straightforward. □

Theorem 3.6. *Let L be a locally free $\mathcal{O}_X$-module of rank n. Then the following data are equivalent:*

(1) *the B–V structure on the exterior algebra $\bigwedge^\bullet(L)$;*
(2) *the Lie algebroid structure on L together with an L-module structure on $\bigwedge^n L$;*
(3) *the Lie algebroid structure on L and an operator δ of degree -1 on $\bigwedge^\bullet(L)$ such that $\delta^2 = 0$ and $[\delta, i(x)] = i(dx)$ for any $x \in \bigwedge^\bullet(L^\vee)$, where d is the Koszul differential.*

Proof. Let us show that (1) is equivalent to (3). We have already seen that a braid algebra structure on $\bigwedge^\bullet(L)$ is the same as a Lie algebroid structure on L. Let $\Delta : \bigwedge^\bullet(L) \to \bigwedge^\bullet(L)$ be an operator of degree -1 which is a B–V structure. Consider its action on $\bigwedge^n L$:

$$\Delta : \bigwedge^n L \to \bigwedge^{n-1} L \simeq \mathcal{H}om(L, \bigwedge^n L).$$

We claim that $\{l, x\} = \Delta(x)(l)$ is an L-module structure on $\bigwedge^n L$. Conversely, given an L-module structure on $\bigwedge^n L$, we have the complex $K_L(\bigwedge^n L)$ which induces an operator of degree -1 on $\bigwedge^\bullet(L)$, and the latter is claimed to give a B–V structure. Both these statements are local, so we can assume that $\bigwedge^n(L) \simeq \mathcal{O}_X$. If we fix such an isomorphism, then the assertion reduces to the fact that an operator Q of degree 1 on $\bigwedge^\bullet(L^\vee)$ such that $Q^2 = 0$ is a Koszul operator for some L-module structure on $\mathcal{O}_X$ if and only if it is dual to some B–V

operator compatible with the Lie algebroid structure. But both these conditions mean that $Q = d + \varepsilon(\alpha)$, where d is the Koszul differential, $\alpha \in L^\vee$ — this follows from the fact that d induces the Schouten bracket on $\bigwedge^\bullet(L)$, so the operators of such a form are exactly those inducing the same bracket on $\bigwedge^\bullet(L)$.

It remains to prove that the third data is equivalent to the first two. Note that δ has order 1 with respect to $i(\bigwedge^\bullet(L^\vee))$, which means that its usual order is ≤ 2 by Lemma 3.3. Moreover, one can see that the condition $[\delta, i(x)] = i(dx)$ means that the corresponding operator $P_D(\delta)$ acting on $\bigwedge^\bullet(L^\vee)$ has the form $d + \varepsilon(\alpha)$ for some $\alpha \in L^\vee$. □

According to the theorem above, a B–V structure extending the usual Schouten bracket on polyvector fields is the same as a flat connection on the canonical bundle ω_X. We conclude this section by the corresponding invariant statement (not depending on the choice of connection).

Proposition 3.7. *Let $\mathcal{D}_{\leq 1}(\omega_X)$ be the bundle of differential operators of the first order on ω_X. Then for any i there is a canonical splitting*

$$\bigwedge^i(\mathcal{D}_{\leq 1}(\omega_X)) \simeq \bigwedge^i T_X \oplus \bigwedge^{i-1} T_X$$

such that for any flat connection $T_X \to \mathcal{D}_{\leq 1}(\omega_X)$ the corresponding map $\bigwedge^i T_X \to \bigwedge^i(\mathcal{D}_{\leq 1}(\omega_X))$ decomposes as $v \mapsto (v, \Delta(v))$, where $\Delta : \bigwedge^i T_X \to \bigwedge^{i-1} T_X$ is the Koszul operator associated with this connection.

Proof. Apply Theorem 3.6 to $L = \mathcal{D}_{\leq 1}(\omega_X)$. The canonical L-module structure on L induces a B–V operator $\Delta : \bigwedge^{i+1} L \to \bigwedge^i L$. Note that for every i there is a canonical exact sequence

$$0 \to \bigwedge^{i-1} T_X \to \bigwedge^i L \to \bigwedge^i T_X \to 0.$$

Moreover, the composition

$$\bigwedge^{i-1} T_X \to \bigwedge^i L \xrightarrow{\Delta} \bigwedge^{i-1} L \to \bigwedge^{i-1} T_X$$

is the identity map. Thus, we obtain the required splitting. A flat connection on ω_X induces a morphism of B–V algebras $\bigwedge^\bullet(T_X) \to \bigwedge^\bullet(L)$, from which the last assertion follows. □

4. Batalin–Vilkovisky Structure on the De Rham Complex of a Poisson Variety

The remarkable property of the Poisson–Lie algebroid on a smooth Poisson variety X is that it admits a canonical compatible B–V structure; in other words, there is a canonical Poisson module structure on ω_X.

Theorem 4.1. *Let G be the Poisson bivector field. The operator*

$$\delta = [i(G), d]$$

defines a B–V structure on $\Omega_X^\cdot$ compatible with the Poisson–Lie algebroid structure on Ω_X^1.

Lemma 4.2. *Let X be a smooth Poisson variety. The differential d_P on $K_{\Omega_X} = \bigwedge^\bullet T_X$ constructed in Sec. 1 is given by the formula $v \mapsto [G, v]$, where v is a polyvector field, G is a bivector field defining the Poisson structure.*

Proof. It is sufficient to check that $d_P(f) = [G, f]$ for any $f \in \mathcal{O}_X$ and $d_P(v) = [G, v]$ for any $v \in T_X$. The first equality is easy, so let us prove the second one. First note that for $f, g \in \mathcal{O}_X$ we have

$$\langle d_P(v), df \wedge dg \rangle = \{f, v(g)\} + \{v(f), g\} - v(\{f, g\}) = -\langle \lambda_v G, df \wedge \rangle,$$

i.e., $d_P(v) = -\lambda_v G$. On the other hand,

$$[i(G), [d, i(v)]] = [i(G), \lambda_v] = -i(\lambda_v G),$$

so that $[G, v] = -\lambda_v G$, and the assertion follows. □

Corollary 4.3. *For a smooth Poisson variety, one has the following formulas:*

$$[G, G] = 0,$$
$$[\Delta_\phi(G), G] = 0,$$
$$[\Delta_\phi, \varepsilon(G)] = -d_P + \varepsilon(\Delta_\phi(G)),$$

where in the last formula the brackets mean commutator.

Proof of Theorem 4.1. According to the previous section, it is sufficient to check the following two identities:

$$\delta^2 = 0,$$
$$[\delta, i(x)] = i(d_P x)$$

for any $x \in \bigwedge^i T_X$. The second is a reformulation of the lemma above because

$$[\delta, i(x)] = [[i(G), d], i(x)] = [i(G), [d, i(x)]] = i([G, x]).$$

The first equality follows from the Jacoby identity: first we apply it to check that $[\delta, d] = [\delta, i(G)] = 0$ and then to conclude that $\delta^2 = [\delta, \delta] = 0$. □

Now we want to describe the Poisson module structure on ω_X corresponding to the B–V operator δ. We claim that if we choose locally a nonvanishing form ϕ of the highest degree, then the corresponding Poisson connection on ω_X is given by the formula

$$\nabla(\phi) = -\Delta_\phi(G) \otimes \phi,$$

where $G \in \bigwedge^2 T_X$ is the Poisson structure tensor, Δ_ϕ is the operator on $\bigwedge^\bullet T_X$ defined in Sec. 3. In fact, this can be easily seen from the third formula of the corollary above, which shows that the dual operator to δ is $d_P - \varepsilon(\Delta_\phi(G))$.

Lemma 4.4. *For a smooth Poisson variety, the following identities hold:*

$$L_v(\phi) = \Delta_\phi(v)\phi,$$
$$L_{H(df)}(\phi) = -\Delta_\phi(G)(f)\phi,$$

where $v \in T_X$, $\phi \in \omega_X$, $f \in \mathcal{O}_X$, L is the Lie derivative.

Proof. The first identity is easy. The second follows from the first and the fact that $i(df)$ and Δ_ϕ anticommute with each other:

$$L_{H(df)}(\phi) = \Delta_\phi(H(df))\phi = \Delta_\phi \circ i(df)(G) = -i(df)(\Delta_\phi(G))\phi = -\Delta_\phi(G)(f)\phi. \quad \square$$

We will also need an explicit formula for the vector field $\Delta_\phi(G)$.

Lemma 4.5. *Let $y_1, \ldots, y_n$ be the local coordinates. Then for $\phi = dy_1 \wedge \ldots \wedge dy_n$ one has*

$$\Delta_\phi(G) = \sum_{i,j} \frac{\partial\{y_i, y_j\}}{\partial y_i} \cdot \frac{\partial}{\partial y_j}.$$

This is checked by a direct computation, which is left to the reader.

The operators d, δ, and $\Lambda = i(G)$ on the De Rham complex of a Poisson variety satisfy the following supercommutation relations:

$$[d, d] = [\delta, \delta] = [d, \delta] = 0, \tag{12}$$
$$[\Lambda, d] = \delta, [\Lambda, \delta] = 0, \tag{13}$$

where Λ is even, d and δ are odd.

Lemma 4.6. *Assume that Λ, d and δ satisfy the relations* (12), (13). *Then the following identities hold:*

$$\delta\Lambda^n = \frac{1}{n+1}(\Lambda^{n+1}d - d\Lambda^{n+1}),$$
$$d + \delta = \exp(\Lambda)d\exp(-\Lambda).$$

The proof is left to the reader.

Proposition 4.7. *Assume that some operator Λ acts on a complex of $\mathbf{Q}$-vector spaces (C, d) and satisfies the relation*

$$[\Lambda, [\Lambda, d]] = 0.$$

Assume that d is strictly compatible with some finite decreasing filtration F^p on C such that $\Lambda(F^p) \subset F^{p+1}$. Then $d + [\Lambda, d]$ is also strictly compatible with this filtration.

Proof. According to the previous lemma, $\exp(\Lambda)$ gives an isomorphism of filtered complexes (C, d) and $(C, d + [\Lambda, d])$, which implies the claim. □

For a smooth Poisson variety, consider the bicomplex $C^{i,j} = \Omega^{j-i}$ with the differentials $d : C^{i,j} \to C^{i,j+1}$ and $\delta : C^{i,j} \to C^{i+1,j}$ (see [5]).

Corollary 4.8. *The spectral sequence of the bicomplex C beginning with $d_0 = d$ degenerates in E_1.*

5. Poisson Structures on Line Bundles

Let $p : Y \to X$ be a line bundle over the Poisson manifold X. We want to describe all Poisson structures on Y such that p is a Poisson morphism. First we consider local situation. Namely, given a Poisson algebra A, we are interested in compatible Poisson structures on $A[[t]]$. Such a structure is determined uniquely by the derivation $v = \{t, \cdot\} : A \to A[[t]]$. We consider $v = v(t)$ as a vector field on $X = \operatorname{Spec} A$ depending on the parameter t. Then the Jacoby identity admits the following interpretation.

Proposition 5.1. *The Poisson structures on $A[[t]]$ compatible with the given one on A are in one-to-one correspondence with the formal families $v(t)$ of vector fields on X parametrized by t such that the following identity holds:*

$$L_{v(t)}G = \frac{\partial}{\partial t}v(t) \wedge v(t), \tag{14}$$

where G is a skew-symmetric bivector field defining the Poisson structure on A, L denotes the Lie derivative.

Proof. One can easily see that the bivector field defining a Poisson structure on $A[[t]]$ compatible with the given one on A has the form

$$\widetilde{G} = G + \frac{\partial}{\partial t} \wedge v.$$

Now one can easily compute that

$$\left[\widetilde{G}, \widetilde{G}\right] = 2L_vG \wedge \frac{\partial}{\partial t} - 2\left[\frac{\partial}{\partial t}, v\right] \wedge v \wedge \frac{\partial}{\partial t}.$$

The required assertion follows immediately because $L_vG - \frac{\partial}{\partial t}v \wedge v \in T_Y[[t]]$. □

In the particular case where $v(t) = f(t)v$, where v does not depend on t, the condition (14) reduces to $L_vG = 0$. Note that in this case G and $v(t) \wedge \frac{\partial}{\partial t}$ form a Hamiltonian pair (see [9]).

Example. In the case of a symplectic structure, Eq. (14) is equivalent to the following:

$$d_y\omega = \omega \wedge \frac{\partial}{\partial t}\omega,$$

where $\omega = \omega(t)$ is a 1-parameter family of 1-forms on X such that $v = H(\omega)$, d_y is the differential along X, so that the complete differential is

$$d\omega = d_y\omega + dt \wedge \left(\frac{\partial}{\partial t}\omega\right).$$

The proposition above can be globalized easily in one direction—to the case of the trivial line bundle over any base (one should only replace formal families of vector fields by polynomial ones). Now we want to generalize this to the case of a nontrivial line bundle $\mathcal{L}^{-1}$ over X. Then the sections of the sheaf of algebras $\mathrm{Sym}(\mathcal{L})$ over X can be considered as functions on the total space Y of $\mathcal{L}^{-1}$, so a Poisson bracket on Y compatible with the given one on X defines a map

$$\{\ ,\ \} : \mathcal{O}_X \times \mathrm{Sym}(\mathcal{L}) \to \mathrm{Sym}(\mathcal{L})$$

which induces a Poisson connection on $\mathcal{L}$ (not necessarily flat) and $\mathcal{O}_X$-linear homomorphisms $\phi_i : \mathcal{L} \to \mathcal{L}^i$ for $i \neq 1$. Note that the condition $\phi_i = 0$ for $i \neq 1$ means that the corresponding Poisson structure on Y is preserved by the canonical vector field $t\frac{\partial}{\partial t}$ on Y, where t is a (local) parameter along the fibers of the projection $Y \to X$. In this case, the Jacoby identity is equivalent to the condition that the connection on $\mathcal{L}$ is flat.

So we have arrived at the following statement.

Proposition 5.2. *There is a one-to-one correspondence between the flat Poisson connections on a line bundle $\mathcal{L}$ over X and the Poisson structures on the total space Y of the line bundle $\mathcal{L}^{-1}$ preserved by the fiberwise $\mathbf{C}^*$-action and compatible with the Poisson structure on X.*

Note also that a Poisson structure on the total space Y of the line bundle $\mathcal{L}^{-1}$ induces the Poisson bracket on the algebra of global regular functions $H^0(Y, \mathcal{O}_Y) = \bigoplus H^0(X, \mathcal{L}^n)$.

Corollary 5.3. *A flat Poisson connection $v : \mathcal{L} \to T_X \otimes \mathcal{L}$ induces the homogeneous Poisson bracket on the graded algebra $\bigoplus H^0(\mathcal{L}^n)$ given by the formula*

$$\{fs^n, gs^m\} = (\{f, g\} + mv_s(f)g - nv_s(g)f)s^{n+m}$$

for $s \in \mathcal{L}$, $f, g \in \mathcal{O}_X$, where $v(s) = v_s \otimes s$.

6. Poisson Structures on Projective Line Bundles

Now let $p : Y \to X$ be a projective line bundle. Then Poisson structures on Y compatible with a given Poisson structure on X can be treated in a similar way. Namely, let $Y = \mathbf{P}(E)$, where E is a rank-2 vector bundle on X. Then we claim the following.

Theorem 6.1. *Assume that there exists a Poisson module structure on $\det E = \bigwedge^2 E$. Then there is a natural one-to-one correspondence between Poisson structures on Y compatible with a given Poisson structure on X and Poisson module structures on E inducing the given one on $\det E$.*

Remark. Notice that Poisson module structures on a line bundle $\det E$ form an affine space $\mathrm{Pois}(\det E)$ over the vector space of global hamiltonian vector fields on X, where the action of the latter space on the former one is given by the tensor product of Poisson modules ($\mathcal{O}_X$ can be considered as a Poisson module via a hamiltonian vector field). On the other hand, the space of hamiltonian vector fields acts on the set $\mathrm{Pois}(E)$ of Poisson module structures on E, such that the action of a vector field v is compatible with the action of $2v$ on $\mathrm{Pois}(\det E)$. Hence, the choice of the Poisson module structure on $\det E$ in the theorem above is inessential.

Proof of Theorem 6.1. Assume that E has the structure of a Poisson module. Then for any local function f on X and local sections s_0, s_1 of $E^\vee$, we can define $\{f, \frac{s_1}{s_0}\}$ using the Leibnitz rule and the induced Poisson

module structure on $E^\vee$, which gives a Poisson structure on Y. If the sections s_0 and s_1 give local trivialization for $E^\vee$, then we have

$$\begin{aligned} \{f, s_0\} &= v_{00}(f)s_0 + v_{01}(f)s_1, \\ \{f, s_1\} &= v_{10}(f)s_0 + v_{11}(f)s_1 \end{aligned} \tag{15}$$

for some 2×2-matrix v of vector fields on X. Then

$$\{f, t\} = v_{10}(f) + (v_{11}(f) - v_{00}(f))t - v_{01}(f)t^2,$$

where $t = \frac{s_1}{s_0}$. Note also that the formulas (15) define (locally) a Poisson module structure on $E^\vee$ if and only if the identities below hold

$$\begin{aligned} L_{v_{00}} G &= v_{10} \wedge v_{01}, \\ L_{v_{01}} G &= v_{01} \wedge (v_{00} - v_{11}), \end{aligned}$$

as well as two others with 0 and 1 permuted. The induced Poisson module structure on $\det(E^\vee)$ is given by the hamiltonian vector field $v_{00} + v_{11}$.

On the other hand, Y is locally isomorphic to $X \times \mathbf{P}^1$, and if $t = \frac{s_1}{s_0}$ is a parameter on the projective line, then we can write

$$\{f, t\} = u_0(f) + u_1(f)t + u_2(f)t^2 + \dots,$$

where f is a function on X, and u_i are some vector fields. Changing t to t^{-1}, we see that $u_i = 0$ for $i > 0$. Equation (14) is equivalent in this case to the following system:

$$\begin{aligned} L_{u_0} G &= u_0 \wedge u_1, \\ L_{u_1} G &= 2u_0 \wedge u_2, \\ L_{u_2} G &= u_1 \wedge u_2. \end{aligned} \tag{16}$$

The solution corresponding to a Poisson module structure on $E^\vee$ has the form

$$u_0 = v_{10}, u_1 = v_{11} - v_{00}, u_2 = -v_{01}.$$

Therefore, a Poisson module structure on $E^\vee$ is determined uniquely by the corresponding Poisson bracket on Y and by the induced Poisson module structure on $\det E^\vee$ (which is given locally by the hamiltonian vector field $v_{00} + v_{11}$).

Thus, we have constructed a morphism from the sheaf of Poisson module structures on $E^\vee$ inducing the given one on $\det E^\vee$ to the sheaf of Poisson structures on Y compatible with the given one on X, which is a local isomorphism, hence a global isomorphism. □

Examples. 1. Let X be a smooth projective curve. Then a Poisson structure on X is zero, and a Poisson module structure on E inducing the zero one on $\det E$ is just an $\mathrm{End}_0(E)$-valued vector field, i.e., a traceless morphism $f : E \to E \otimes K^{-1}$, where K is the canonical class. If the genus of X is greater than 0, then for any such a pair (E, f) with $f \neq 0$, either one has $E = L \oplus M$, where L and M are line bundles, or this pair is obtained by tensoring with a line bundle from the following one: E is the unique nontrivial extension of $\mathcal{O}$ by K; f is the composition

$$E \to \mathcal{O} \to E \otimes K^{-1}.$$

2. If X is a symplectic variety, then the compatible Poisson structures on $Y = \mathbf{P}(E)$ are in bijections with flat connections on E inducing the given one on $\det E$.

Proposition 6.2. *Let E be a rank-2 Poisson bundle on a Poisson variety X, $Y = \mathbf{P}(E)$ be the corresponding Poisson variety. Then Poisson sections $\sigma : X \to Y$ of the projection $p : Y \to X$ are in bijective correspondence with surjective morphisms of Poisson modules $E^\vee \to L$, where L is a Poisson line bundle on X.*

Proof. All sections σ of p are in one-to-one correspondence with $\mathcal{O}_X$-linear surjections $g : E^\vee \to L$, where $L = \sigma^* \mathcal{O}_Y(1)$. Let us choose local trivializing sections s_0, s_1 of E such that s_1 generates the kernel of g, and

let $t = \frac{s_1}{s_0}$ be the corresponding fiber parameter. Then σ is a Poisson morphism if and only if $\{f, t\}$ is divisible by t. This is equivalent to the requirement that s_1 generate a Poisson submodule in E, that is, the kernel of g is a Poisson submodule. □

Another interesting property of projective line bundles in a Poisson category is that the image of the vanishing locus upstairs is precisely the vanishing locus downstairs.

Proposition 6.3. *Let E be a Poisson rank-2 bundle on a Poisson variety X, $p : Y = \mathbf{P}(E) \to X$ be the corresponding Poisson morphism. Assume that the Poisson structure on X vanishes at a point $x \in X$. Then there exists a point $y \in p^{-1}(x)$ such that the Poisson structure on Y vanishes at y.*

Proof. If the Poisson structure on X vanishes at $x \in X$, then the Poisson connection $E^\vee \to T_X \otimes E^\vee$ corresponding to the Poisson structure on Y induces a family of commuting operators on E_x^* parametrized by T_X^*, so that if we denote by A_f the operator corresponding to df_x for a local function f, then the value of the Poisson bracket on Y at a point $y \in p^{-1}(x) = \mathbf{P}(E_x)$ can be computed as follows:

$$\left\{f, \frac{s_1}{s_0}\right\}(y) = \frac{A_f(s_1)s_0 - A_f(s_0)s_1}{s_0^2}(y),$$

where s_0 and s_1 are local sections of $E^\vee$ (on the right-hand side, we consider s_i as homogeneous coordinates on $\mathbf{P}(E_x)$). Now if we take a common eigenvector of operators A_f, it defines a line in E_x, and the above formula implies that the Poisson structure on Y vanishes at the corresponding point $y \in p^{-1}(x)$. □

7. Poisson Divisors

In this section, we study an interplay of the standard correspondence between line bundles and divisors with a Poisson structure. We will see that in this context Poisson connections arise naturally.

Let X be a smooth (connected) Poisson variety X. Denote the Cartier divisor group of X by $\mathrm{Div} = H^0(X, K^*/\mathcal{O}^*)$, where K^* is the constant sheaf of invertible rational functions on X. Put $K_1^* = \{f \in K^* | \{\mathcal{O}, f\} \subset \mathcal{O}f\}$. This is a subsheaf of K^* containing $\mathcal{O}^*$. There is a natural homomorphism of sheaves $Hd\log : K_1^* \to T_h$, where T_h is a sheaf of hamiltonian vector fields. Namely, any function $f \in K_1^*$ defines the vector field $Hd\log(f)(g) = f^{-1}\{f, g\}$.

Put $\mathrm{PDiv} = H^0(X, K_1^*/\mathcal{O}^*) \subset \mathrm{Div}$. The intersection of PDiv with the subgroup of principal divisors consists of rational functions f such that $\{\mathcal{O}, f\} \subset \mathcal{O}f$ (considered up to multiplication with global invertible functions). Denote the corresponding divisor class group by $\mathrm{PCl} \subset \mathrm{Pic}\, X$. It is easy to see that the group PDiv can also be described in terms of Weyl divisors; one has to consider formal linear combinations of irreducible divisors which are Poisson subvarieties of X.

Proposition 7.1. *The group* PDiv *is isomorphic to the group of isomorphism classes of triples (L, ∇, s), where L is a line bundle, ∇ is a flat Poisson connection on L, and $s \neq 0$ is a rational section of L which is horizontal with respect to ∇.*

Proof. Consider a Cech representative for an element of PDiv, i.e., the collection of functions $f_i \in K_1^*(U_i)$ for some open covering U_i such that $g_{ij} = f_i f_j^{-1} \in \mathcal{O}^*$. The corresponding line bundle L is trivialized over U_i, that is, for each i there is a nowhere vanishing section $s_i \in L(U_i)$ such that $s_i = g_{ij}s_j$ over the intersection. Now define the connection on L by the formula $\nabla(s_i) = -Hd\log(f_i) \otimes s_i$. Then the formula $s|_{U_i} = f_i^{-1}s_i$ gives a well-defined rational horizontal section of L.

Now assume that we have a flat Poisson connection ∇ on L and a horizontal rational section s. Then trivializing L over an open covering as above, we can write $s = f_i^{-1}s_i$ for some rational functions f_i. Now the condition $\nabla(s) = 0$ implies the equality $\nabla(s_i) = -Hd\log(f_i) \otimes s_i$. It follows that $f_i \in K_1^*(U_i)$. □

Let L be a line bundle with a flat Poisson connection ∇. The natural question is whether there exists a nonzero rational section of L horizontal with respect to ∇. In fact, there is an obstruction which is a characteristic class of (L, ∇) with values in some kind of Poisson cohomology.

Note that just as in the case of usual connections, the group of isomorphism classes of pairs (line bundle, flat Poisson connection on it) is $H^1(\mathcal{O}^* \to T_h)$, where the homomorphism $\mathcal{O}^* \to T_h$ is $Hd\log$.

Definition. For a pair (L, ∇) as above, $c_1(L, \nabla) \in H^1(K_1^* \to T_h)$ is the image of the isomorphism class of this pair under the natural homomorphism $H^1(\mathcal{O}^* \to T_h) \to H^1(K_1^* \to T_h)$.

Proposition 7.2. *For a pair (L, ∇) as above, there exists a nonzero rational horizontal section if and only if $c_1(L, \nabla) = 0$.*

Proof. This follows from the exact sequence

$$H^0(K_1^*/\mathcal{O}^*) \to H^1(\mathcal{O}^* \to T_h) \to H^1(K_1^* \to T_h) \to 0.$$

Indeed, it is easy to see that under the identification of the previous proposition, the first arrow corresponds to the forgetting map: $(L, \nabla, s) \mapsto (L, \nabla)$. □

Remark. There is an exact sequence

$$0 \to H^1(K_0^*) \to H^1(K_1^* \to T_h) \to H^0(H_h^1),$$

where $K_0^* = \{f \in K^* | \ H_f = 0\}$ is the sheaf of nonzero rational Casimir functions, and H_h^1 is the first cohomology sheaf of the complex $K_1^* \to T_h$. Note that if the Poisson structure is symplectic at the general point, then $K_0^* = \mathbf{C}^*$ is a constant sheaf.

Definition. We will say that two Poisson divisors are strongly linear equivalent if they define the same pair (L, ∇) (in particular, they are linear equivalent in the usual sense).

Corollary 7.3. *For a given (L, ∇), the corresponding set of strongly linear equivalent effective Poisson divisors is in bijection with the set of points of the projective space $\mathbf{P}(H^0(L)^\nabla)$ — the projectivization of the space of ∇-horizontal global sections of L.*

Remark. The kernel of the natural map

$$\delta : \mathrm{PDiv} \to H^1(\mathcal{O}^* \to T_h)$$

defined above is isomorphic to $H^0(K_0^*)/H^0(\mathcal{O}^*)$. So in the nondegenerate (even-dimensional) case, $H^0(L)^\nabla$ is either zero or 1-dimensional.

Recall (see Sec. 4) that for any smooth Poisson variety X there is a canonical Poisson module structure on the canonical line bundle ω_X. In particular, for any Poisson structure H on a smooth variety X, the image of $c_1(\omega_X) \in H^1(\Omega_X^1)$ under the natural map $H^1(\Omega_X^1) \to H^1(T_X)$ induced by H is zero.

Proposition 7.4. *The divisor of degeneration of a nondegenerate Poisson structure (defined by the Pfaffian) is a Poisson divisor. The connection on ω_X^{-1} that it defines is the canonical one.*

Proof. We only have to check that the Pfaffian is a horizontal section of ω_X^{-1} with respect to the canonical connection. Now Lemma 4.4 shows that the canonical Poisson module structure on ω_X is given by the Lie derivative along the hamiltonian vector fields. Since these fields preserve the Pfaffian, the assertion follows. □

8. Blow-up in a Poisson Category

Definition. A Lie algebra L (sheaf of Lie algebras) is called degenerate if the following canonical map $\bigwedge^3 L \to S^2 L$ is zero:

$$x \wedge y \wedge z \mapsto [x,y]z + [y,z]x + [z,x]y.$$

For example, an abelian Lie algebra or a 2-dimensional Lie algebra over a field is degenerate.

Proposition 8.1. *A degenerate Lie algebra L of dimension n over a field k is either abelian or is isomorphic to the algebra with basis $(e_i, 0 \le i \le n-1; f)$ such that $[e_i, e_j] = 0$, $[f, e_i] = e_i$.*

Proof. The condition of degeneracy implies that for any $x, y \in L$ the commutator $[x,y]$ is a linear combination of x and y. In particular, any linear subspace of L is a degenerate Lie algebra itself. Choose a hyperplane $L_1 \subset L$. By induction, we may assume that either $[L_1, L_1] = 0$ or $[L_1, L_1] = L_2$ has codimension 1 in L_1 and $L_1 = L_2 \oplus k \cdot f$, where $\mathrm{ad}(f)$ acts as identity on L_2. In the latter case, let $g \in L \setminus L_1$. Then for any $e \in L_2$ we have $[g,e] = \lambda g + \mu e$ for some constants $\lambda, \mu \in k$. Now the commutator $[g+f, e] = \lambda g + (\mu+1)e$ is a linear combination of $g+f$ and e, which implies that $\lambda = 0$, i.e., $[g,e]$ is proportional to e for any $e \in L_2$. Therefore, there exists a constant $\mu \in k$ such that $[g,e] = \mu e$ for any $e \in L_2$. Hence, $[g - \mu f, e] = 0$, and replacing L_1 by the hyperplane spanned by L_2 and $g - \mu f$, we may assume that $[L_1, L_1] = 0$. Now one can easily see that L_1 is an ideal in L (otherwise, there is a decomposition $L_1 = L_2 \oplus k \cdot f$ such that L_2 is an ideal in L and we can apply the same argument as above), and the operator $\mathrm{ad}(g)$ for $g \in L \setminus L_1$ acts as a scalar on L_1. □

Definition. Let X be a Poisson scheme. A Poisson ideal sheaf $J \subset \mathcal{O}_X$ is called degenerate if for any $x, y, z \in J$ one has

$$\{x,y\}z + \{y,z\}x + \{z,x\}y \in J^3.$$

For example, J is degenerate if the Lie algebra sheaf J/J^2 is degenerate.

Theorem 8.2. *Let X be a scheme with a Poisson structure H, $Y \subset X$ be a Poisson subscheme such that J_Y is degenerate, $p : \widetilde{X} \to X$ be the blow-up of X along Y. Then there is a unique Poisson structure on $\widetilde{X}$ such that p is a Poisson morphism. If the linearization of H at Y is abelian (that is, $\{J_Y, J_Y\} \subset J_Y^2$), then the exceptional divisor E is a Poisson subvariety of $\widetilde{X}$.*

Proof. By definition, $\widetilde{X}$ is the projectivization of the following sheaf of algebras on X: $\mathcal{O}_X \oplus J \oplus J^2 \oplus \ldots$, where $J = J_Y$. So we have to check that for any $f, g, h \in J$ there exists a number n such that

$$\left\{\frac{f}{h}, \frac{g}{h}\right\} \subset h^{-n} \cdot J^n.$$

In fact, $h^3 \cdot \left\{\frac{f}{h}, \frac{g}{h}\right\} = \{f,g\} \cdot h + \{g,h\} \cdot f + \{h,f\} \cdot g$ belongs to J^3 by assumption, and so we are done.

If $\{J, J\} \in J^2$, then for any $f, g \in J$ we have $\left\{f, \frac{g}{f}\right\} \in f^{-1} \cdot J^2$, which means that the exceptional divisor is Poisson. □

Here is a partial inversion of this theorem.

Proposition 8.3. *Assume that $Y \subset X$, X and Y are smooth, and there exist compatible Poisson structures on X and $\widetilde{X}$. Then Y is a Poisson subvariety of X and the Lie algebra J_Y/J_Y^2 is degenerate. Furthermore, if the exceptional divisor $E \subset \widetilde{X}$ is Poisson, then $\{J_Y, J_Y\} \subset J_Y^2$.*

Proof. Let $x_1, \ldots, x_k$ be a local regular generating system for J_Y. Then the existence of the compatible Poisson structure on $\widetilde{X}$ implies that for any $f \in \mathcal{O}_X$ one has

$$\{x_i, f\}x_j - \{x_j, f\}x_i \in J_Y^2.$$

By regularity we obtain $\{x_i, f\} \in J_Y$, which proves that J_Y is a Poisson ideal. The remaining statements are simpler and are left to the reader. □

Example. A homogeneous Poisson structure on a vector space V has zero of the second order at $0 \in V$. Therefore, it induces a Poisson structure on the corresponding blow-up which is isomorphic to the total space T of the tautological line bundle $\mathcal{O}(-1)$ over the projective space $\mathbf{P}(V)$. This structure is compatible with the induced Poisson structure on $\mathbf{P}(V)$ and is preserved by the $\mathbf{C}^*$-action along the fibers of the projection $T \to \mathbf{P}(V)$ because the original Poisson structure was stable under the action of $\mathbf{C}^*$. Thus, a homogeneous Poisson structure on V can be described by the Poisson structure on the projective space together with a Poisson module structure on $\mathcal{O}(1)$ (see Sec. 5).

The blow-down is much easier, as the following general result shows.

Proposition 8.4. *Let $f : X \to Y$ be a morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then a Poisson structure on X induces canonically a Poisson structure on Y such that f is a Poisson morphism. Furthermore, if $\mathcal{F}$ is a Poisson module on X, then $f_*(\mathcal{F})$ is a Poisson module on Y.*

Proof. Let $U \subset Y$ be an open subset. Then for any functions $\phi, \psi \in \mathcal{O}_Y(U)$ the bracket $\{f^{-1}(\phi), f^{-1}(\psi)\}$ is a section of $f_*\mathcal{O}_X$ over U, hence, a function on U. Clearly, this defines a Poisson bracket on Y. The proof of the second statement is similar. □

This applies in particular to a proper morphism with connected fibers and to an open embedding having the complement of codimension 2 in a normal variety.

9. Degeneration Loci of Poisson Structures

In [2], A. Bondal conjectured that given a Poisson structure on a smooth projective variety X with ample anticanonical class (a so-called *Fano variety*) the locus where the rank of the structure map $H : \Omega^1_X \to T_X$ is $\leq 2k$ has a component of dimension $\geq 2k+1$. We are going to give some evidence in favor of this conjecture. Namely, we will consider only the maximal degeneration locus consisting of points where the rank of H is less than at the general point. We prove the required estimate for the dimension of such a locus in the following two cases: when X is a projective space and when the Poisson structure is nondegenerate (has maximal possible rank at the general point). Note that in the latter case we may assume that the dimension of X is odd, otherwise the assertion is obvious.

Let X be a smooth variety of odd dimension $n = 2k+1$. A Poisson structure on X is nondegenerate if the corresponding morphism $H : \Omega_X \to T_X$ has rank $2k$ at the general point. In other words, if $G \in \bigwedge^2 T_X$ is the structural tensor of the Poisson structure, then the product $g = \underbrace{G \wedge G \wedge \ldots \wedge G}_{k} \in \bigwedge^{2k} T_X \simeq \Omega^1_X \otimes \omega_X^{-1}$ is nonzero, hence g induces an embedding $i : \omega_X \to \Omega^1_X$. At the general point the image $\mathrm{im}(i) \subset \Omega^1_X$ coincides with the annihilator of the Lie subsheaf $\mathrm{im}(H) \subset T_X$, hence, it defines a corank-1 foliation on X, which means that for any local section $\nu \in \omega_X$ the 1-form $\omega = i(\nu)$ satisfies the Pfaff equation

$$\omega \wedge d\omega = 0. \tag{17}$$

Now the set of points in X where the rank of H drops coincides with the vanishing locus of i, so we may apply the following general result.

Theorem 9.1. *Let $i : L \to \Omega^1_X$ be an embedding of a line bundle defining a corank-1 foliation on a smooth variety X. Let $c_1(L) \in H^2(X, \mathbf{C})$ be a first Chern class of L. Assume that either $c_1(L)^2 \neq 0$, or $c_1(L) \neq 0$ and $H^1(X, L) = 0$. Then the vanishing locus of i has a component of codimension ≤ 2.*

Proof. Let S be a vanishing locus of i, $U = X \setminus S$. Then over U we have an integrable corank-1 subbundle $\ker(i^\vee) \subset T_U$, hence, by Bott's theorem (see [4]) we get $c_1(L|_U)^2 = 0$. Thus, if $\mathrm{codim}\, S > 2$ we conclude

that $c_1(L)^2 = 0$, which proves the first part of the statement. To prove the second part, note that the sheaf $\Omega^1_U/L|_U$ is locally free and we have a morphism $\nabla : L|_U \to \Omega^1_U/L|_U \otimes L|_U$ which fits in the commutative diagram

$$\begin{array}{ccc} L|_U & \xrightarrow{i|_U} & \Omega^1_U \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle d} \\ \Omega^1_U/L|_U \otimes L|_U & \longrightarrow & \Omega^2_U, \end{array} \tag{18}$$

where the lower horizontal arrow is induced by the wedge product and the embedding i. Thus, we have a class $e \in H^1(U, \mathcal{O}^* \to \Omega^1/L)$ which goes to the class representing $L|_U$ under the natural map $H^1(U, \mathcal{O}^* \to \Omega^1/L) \to H^1(U, \mathcal{O}^*)$. There is an exact sequence

$$H^1(U, \mathcal{O}^* \to \Omega^1) \to H^1(U, \mathcal{O}^* \to \Omega^1/L) \to H^1(U, L).$$

Let $j : U \to X$ be the embedding morphism. Assume that $\operatorname{codim} S > 2$. Then one can easily see that $R^1 j_*(\omega_U) = 0$, $j_*(\omega_U) = \omega_X$, so that $H^1(U, L) = H^1(X, L) = 0$. Hence, e comes from some element of $H^1(U, \mathcal{O}^* \to \Omega^1)$ which represents a connection on $L|_U$. Furthermore, since S has codimension > 1 in X, this connection extends to a connection over X which implies the triviality of $c_1(L) \in H^2(X, \mathbf{C})$. □

Corollary 9.2. *The rank of a nondegenerate Poisson structure on a Fano variety of odd dimension drops along the subset of codimension* ≤ 2.

Now we turn to the case $X = \mathbf{P}^n$. The proof of the next result follows closely the argument of J. P. Jouanolou (see [13, Proposition 2.7]).

Theorem 9.3. *Let $H : \Omega^1_{\mathbf{P}^n} \to T_{\mathbf{P}^n}$ be a Poisson structure on $\mathbf{P}^n$ such that the rank of H at the general point is equal to $2k$. Then the locus $S \subset \mathbf{P}^n$ where the rank of H drops has dimension $\geq 2k - 1$.*

Proof. Let $U = \mathbf{P}^n \setminus S$; then $\operatorname{im}(H)|_U$ is an integrable subbundle in T_U, hence, by the Bott theorem $c_1(\operatorname{coker}(H)|_U)^{n-2k+1} = 0$. If $\operatorname{codim} S \leq n - 2k + 1$, this implies that $c_1(\operatorname{coker}(H))^{n-2k+1} = 0$, i.e., $c_1(\operatorname{coker}(H)) = 0$. On the other hand, the tangent bundle $T_{\mathbf{P}^n}$ is stable (see [16]), hence, $\deg(Q) > 0$ for any quotient Q of $T_{\mathbf{P}^n}$ and we get a contradiction. □

10. The Differential Complex of a Nondegenerate Even-Dimensional Poisson Variety

Let $H : \Omega_X \to T_X$ be a nondegenerate Poisson structure on a smooth algebraic variety X of even dimension, Z be the degeneration locus defined by the Pfaffian form of H, and $U = X \setminus Z$ be the symplectic open part. Consider the complex of multivectors $\bigwedge^\bullet T_X$ with the standard Poisson differential (see Sec. 1). Let $H_\bullet : \Omega^\bullet_X \to \bigwedge^\bullet T_X$ be the morphism of dg-algebras induced by the Poisson structure: $H|_{\Omega^i_X} = (-1)^i \bigwedge^i(H)$. Then $H_\bullet|_U$ is an isomorphism, hence, there is a natural morphism of complexes $r : \bigwedge^\bullet T_X \to j_*\Omega^\bullet_U$, where $j : U \hookrightarrow X$ is an embedding, $\Omega^\bullet_U$ is the De Rham complex on U. Thus, we can identify $\bigwedge^\bullet T_X$ with the subcomplex $\mathcal{A}^\bullet \subset j_*\Omega^\bullet_U$ consisting of differential forms ω on U such that $H_\bullet(\omega)$ extends to a regular multivector on X.

Lemma 10.1. *Let ω be an i-form regular over U. Then*

(1) *If $H_i(\omega)$ is regular over X, then $f^2\omega$ is regular, where f is a local equation of Z.*
(2) *If $H_{i+1}(d\omega)$ and $f^k\omega$ are regular over X for some $k \geq 2$, then $f^{k-1}df \wedge \omega$ is regular.*

Proof. Locally there exists an operator $\widetilde{H}_\bullet$ such that $\widetilde{H}_\bullet H_\bullet = f^2 \operatorname{id}$. Namely, a nonvanishing top-degree form defines an isomorphism $\Omega^i_X \simeq \bigwedge^{n-i} T_X$, and with this identification we have $\widetilde{H}_i = H^*_{n-i}$, where H_j acts on j-forms. This implies the first assertion immediately. To prove the second statement, denote $\omega_1 = f^k\omega$,

$v = H_{i+1}(d\omega)$. Then

$$H_{i+1}(d\omega_1) = kf^{k-1}H_{i+1}(df \wedge \omega) + f^k \cdot v.$$

Applying $\widetilde{H}_{i+1}$ we obtain

$$d\omega_1 = kf^{k-1}df \wedge \omega + f^{k-2}\widetilde{H}_{i+1}v,$$

from which the assertion follows. □

Corollary 10.2. *If ω is an i-form on U such that $H_i(\omega)$ extends to a regular form over X, then the forms $f^2\omega$ and $fdf \wedge \omega$ are regular over X, where f is a local equation of Z.*

Now assume that $Z = \bigcup Z_i$ is a union of smooth divisors with normal crossings (in particular, Z is reduced) and let $\Omega^\bullet_X(\log Z) \subset j_*\Omega^\bullet_U$ be the subcomplex of forms with logarithmic singularities along Z (see [6, II.3]). We want to describe the subcomplex $\mathcal{A}^\cdot \subset j_*\Omega^\cdot_U$ in this case. Since Z_i are Poisson divisors, we have the inclusion

$$\Omega^\bullet_X(\log Z) \subset \mathcal{A}^\bullet.$$

Indeed, $\Omega^\bullet_X(\log Z)$ is generated locally as an $\Omega^\bullet_X$-algebra by the forms $d\log f_i = f_i^{-1}df_i$, where f_i is a local equation of f_i. Now the condition that Z_i is a Poisson divisor means that $H(df_i)$ is divisible by f_i, i.e., $H(d\log f_i)$ is regular along Z, as required. To proceed further we need a simple lemma on logarithmic singularities.

Lemma 10.3. *Let $\alpha \in j_*\Omega^k_U$ be a k-form regular outside Z. Then the following conditions are equivalent:*

(1) $f\alpha \in \Omega^k_X$, $f \cdot d\log(f_i) \wedge \alpha \in \Omega^{k+1}_X$;

(2) $f\alpha \in \Omega^k_X$, $df \wedge \alpha \in \Omega^{k+1}_X$;

(3) *α has logarithmic singularities along Z,*

where f (resp. f_i) is a local equation of Z (resp. Z_i).

Proof. It is easy to see that (3) implies (1) and (1) implies (2). Let us prove that (2) implies (3). According to [6, II.3.2(i)], we have to check that $f \cdot d\alpha \in \Omega^{k+1}_X$. But this follows from (2) since $fd\alpha = d(f\alpha) - df \wedge \alpha$. □

Lemma 10.4. *With the above assumptions we have $J_Z\mathcal{A}^\bullet \subset \Omega^\bullet_X(\log Z)$, where $J_Z = \mathcal{O}_X(-Z) \subset \mathcal{O}_X$ is an ideal sheaf of Z. Moreover, if we put*

$$\mathcal{F} = J_Z\mathcal{A}^1/J_Z\Omega^1_X(\log Z) \subset \Omega^1_X(\log Z)|_Z,$$

then $\mathcal{F} \wedge \mathcal{F} = 0$ and $d\mathcal{F} \wedge \mathcal{F} = 0$ (the product and the differential are in $\Omega^\bullet_X(\log Z)|_Z$). In particular, $\mathcal{F}$ considered as a sheaf on Z has rank at most 1.

Proof. Let $f \in \mathcal{O}_X$ be a local equation of Z. Then according to Corollary 10.2 we have the following inclusions: $f^2\mathcal{A}^\bullet \subset \Omega^\bullet_X$, $fdf \wedge \mathcal{A}^\bullet \subset \Omega^\bullet_X$. Hence, $f\mathcal{A}^\bullet \subset \Omega^\bullet_X(\log Z)$ by Lemma 10.3. It follows that $f^2\mathcal{A}^\bullet \subset f\Omega^\bullet_X(\log Z)$ and $fdf \wedge \mathcal{A}^\bullet \subset df \wedge \Omega^\bullet_X(\log Z) \subset f\Omega^\bullet_X(\log Z)$, which implies that $\mathcal{F} \wedge \mathcal{F} = d\mathcal{F} \wedge \mathcal{F} = 0$. □

As we have seen in Sec. 7, the degeneration locus $Z \subset X$ is a Poisson divisor, so that there is an induced Poisson structure on Z which is given by some map $\Omega_Z \to T_Z$, where Ω_Z is the sheaf of differentials on Z, T_Z is the sheaf of derivations of $\mathcal{O}_Z$. When $Z = \bigcup Z_i$ is the union of smooth components with normal crossings, we can also consider the following dg-algebra:

$$\underline{\Omega}^\bullet_Z = \ker \bigoplus \Omega^\bullet_{Z_i} \to \bigoplus \Omega^\bullet_{Z_{ij}},$$

where $Z_{ij} = Z_i \cap Z_j$, the map is induced by differences of restrictions. We need some easy facts about these sheaves on Z.

Lemma 10.5. *With the above assumptions, we have:*

(1) $\underline{\Omega}^\bullet_Z \simeq \Omega^\bullet_X/(J_Z\Omega^\bullet_X(\log Z))$;

(2) $T_Z \simeq T_{X,Z}/(J_Z T_X)$, *where* $T_{X,Z} \subset T_X$ *is the subsheaf of vector fields preserving* J_Z;
(3) *there are natural exact sequences*

$$\bigoplus_i \mathcal{O}_{Z_i}(-Z) \to \Omega^1_Z \to \underline{\Omega}^1_Z \to 0, \tag{19}$$

$$0 \to T_Z \to T_X|_Z \to \bigoplus \mathcal{O}_{Z_i}(Z_i) \to 0. \tag{20}$$

Proof. (2) is well-known. We have the natural restriction morphism $\Omega^\bullet_X \to \underline{\Omega}^\bullet_Z$. An easy local computation shows that it induces the isomorphism of (1). The morphism $\mathcal{O}_{Z_1}(-Z) \to \Omega^1_Z$ in (**19**) is induced by the morphism $d_1 : \mathcal{O}_{Z_1}(-Z) \to \Omega^1_X|_Z$ which is given locally by the element $f_2 \dots f_n df_1 \in \Omega^1_X|_Z$ annihilated by f_1 (where f_i are local equations of Z_i). The morphism $T_X|_Z \to \mathcal{O}_{Z_1}(Z_1)$ is obtained from d_1 by duality. The exactness is checked by a simple local computation. □

Remark. For any Poisson structure on Z, the components Z_i are Poisson subschemes, i.e., $\{f_i, h\} \in f_i\mathcal{O}_Z$ for any $h \in \mathcal{O}_Z$. In particular, the morphism $h : \Omega^1_Z \to T_Z$ induced by a Poisson bracket vanishes on the element $f_2 \dots f_n df_1$. Thus, the exact sequence (19) shows that h factors through a morphism $H : \underline{\Omega}^1_Z \to T_Z$.

Theorem 10.6. *Assume that a degeneration locus* Z *of a nondegenerate Poisson structure* $H : \Omega^1_X \to T_X$ *on a smooth variety* X *of even dimension is the union of smooth components* Z_i *with normal crossings. Then*

(1) $J_Z\mathcal{A}^1 \subset \Omega^1_X$;
(2) $T_Z \simeq \Omega^1_X(\log Z)/(J_Z\mathcal{A}^1)$;
(3) *for an induced Poisson structure* $H_Z : \underline{\Omega}^1_Z \to T_Z$ *on* Z, *one has the following isomorphisms:*

$$\operatorname{coker}(H_Z) \simeq \bigoplus_i \mathcal{O}_{Z_i},$$

$$\bigoplus_i \mathcal{O}_{Z_i}(Z_i - Z) \simeq \ker(H_Z) = J_Z\mathcal{A}^1/J_Z\Omega^1(\log Z) \subset \underline{\Omega}^1_Z;$$

(4) *there is an exact sequence of sheaves on* Z

$$0 \to \bigoplus_i \mathcal{O}_{Z_i} \to \operatorname{coker}(H|_Z) \to \bigoplus_i \mathcal{O}_{Z_i}(Z_i) \to 0, \tag{21}$$

where $H|_Z : \Omega_X|_Z \to T_X|_Z$ *is the restriction of* H *to* Z.

Proof. (1). Let f be a local equation of Z. To prove that $f \cdot \mathcal{A}^1 \subset \Omega^1_X$ consider the canonical exact sequence

$$0 \to \Omega^1_X/(f\Omega^1_X(\log Z)) \to \Omega^1_X(\log Z)/(f\Omega^1_X(\log Z)) \to \bigoplus_i \mathcal{O}_{Z_i} \to 0.$$

We have a subsheaf $f\mathcal{A}^1/(f\Omega^1_X(\log Z))$ in the middle term, and we have to prove that it goes to zero under the map induced by the Poincaré residue. It is enough to prove this at the general point of each component Z_i, so we may assume that Z is smooth. As we have seen above, $\mathcal{A}^1/\Omega^1_X(\log Z)$ has rank at most 1. So at the general point of Z we can write $\mathcal{A}^1 = \Omega^1_X + \mathcal{O}_X(f^{-1}df) + \mathcal{O}_X(\omega)$, where $\omega = a \cdot f^{-2}df + f^{-1}\alpha$, $a \in \mathcal{O}_X$, $\alpha \in \Omega^1_X$. Equivalently,

$$T_X = H\big(\Omega^1_X + \mathcal{O}_X(f^{-1}df) + \mathcal{O}_X(\omega)\big). \tag{22}$$

Therefore,

$$fT_X + H(\Omega^1_X) = H(\Omega^1_X + \mathcal{O}_X(a \cdot f^{-1}df)). \tag{23}$$

It follows from (22) and (23) that $\operatorname{coker} H|_Z = T_X/(fT_X + H(\Omega^1_X))$ is generated by the images of $f^{-1}df$ and ω. Moreover if a is invertible at the general point of Z, then this cokernel is generated by the image of ω, which is impossible because H is skew-symmetric, so it has even rank. Hence, a is divisible by f and $f\mathcal{A}^1 \in \Omega^1_X$ as required.

(2). As we have seen in Lemma 10.5, $T_Z \simeq T_{X,Z}/(J_Z T_X) \subset T_X|_Z$. Under the isomorphism $r : T_X \tilde{\to} \mathcal{A}^1$ the subsheaf $T_{X,Z} \subset T_X$ goes to the subsheaf $\mathcal{B} \subset \mathcal{A}^1$ consisting of 1-forms $\alpha \in \mathcal{A}^1$ such that $\langle H(x), df_i \rangle \in f_i \cdot T_X$ for any i, or equivalently, the function $\langle \alpha, H(dln(f_i)) \rangle$ is regular along Z for any i (where f_i are local equations of Z_i). Thus, we have an isomorphism $\mathcal{B}/(J_Z \mathcal{A}^1) \simeq T_Z$, so that the morphism

$$H_Z : \underline{\Omega}^1_Z \simeq \Omega^1_X/(J_Z \Omega^1_X(\log Z)) \to \mathcal{B}/(J_Z \mathcal{A}^1) \simeq T_Z \tag{24}$$

is induced by the embedding $\Omega^1_X \subset \mathcal{B}$. In particular, we get an isomorphism

$$\operatorname{coker}(H_Z) \simeq \mathcal{B}/\Omega^1_X \tag{25}$$

(this follows from the inclusion $J_Z \mathcal{A}^1 \subset \Omega^1_X$ proved above). Hence, $\mathcal{B}/\Omega^1_X$ has rank 1 at the general point of each component Z_i. On the other hand, $\Omega^1_X(\log Z) \subset \mathcal{B}$ because $\langle df_i, H(df_j) \rangle = \{f_i, f_j\}$ is divisible by $f_i f_j$. The quotient $\mathcal{B}/\Omega^1_X(\log Z)$ is a subsheaf of $\mathcal{A}_1/\Omega^1_X(\log Z) \subset \underline{\Omega}^1_Z(Z)$ of rank zero over each component Z_i, hence, it is zero. Thus, we get

$$\mathcal{B} = \Omega^1_X(\log Z) \tag{26}$$

and $T_Z \simeq \Omega^1_X(\log Z)/J_Z \mathcal{A}^1$.

(3). Combining (25) and (26), we get

$$\operatorname{coker}(H_Z) \simeq \Omega^1_X(\log Z)/\Omega^1_X \simeq \bigoplus_i \mathcal{O}_{Z_i}.$$

Also, from (24) we deduce immediately that $\ker(H_Z) \simeq J_Z \mathcal{A}^1/J_Z \Omega^1_X(\log Z)$. Now the exact sequence (20) tells us that the cokernel of the embedding $T_Z \simeq \Omega^1_X(\log Z)/J_Z \mathcal{A}^1 \subset \mathcal{A}^1/J_Z \mathcal{A}^1 \simeq T_X|_Z$ is isomorphic to $\bigoplus_i \mathcal{O}_{Z_i}(Z_i)$. Hence,

$$\mathcal{A}^1/\Omega^1_X(\log Z) \simeq \bigoplus_i \mathcal{O}_{Z_i}(Z_i),$$

as required.

(4). The morphism $H|_Z$ factors through H_Z. Hence, $\operatorname{coker}(H|_Z)$ is an extension of $(T_X|_Z)/T_Z \simeq \mathcal{A}^1/\Omega^1_X(\log Z)$ by $\operatorname{coker}(H_Z)$. □

Corollary 10.7. *Under the assumptions of Theorem* 10.6 *and the additional assumption that X is projective, the rank of H is constant over every connected component of the set $Z^{(k)}$ consisting of those points in X where exactly k irreducible components of Z meet. Furthermore, the following inequalities hold: $k \leq \dim(X) - \operatorname{rk}(H|_{Z^{(k)}}) \leq 2k$.*

Proof. Let $Y \subset Z^{(k)}$ be a connected component. Then there are exactly k values of i such that $Y \subset Z_i$ and for the other i's one has $Y \cap Z_i = \emptyset$. We may assume that $Y \subset Z_i$ for $1 \leq i \leq k$ and $Y \cap Z_i = \emptyset$ for $i > k$. Then the closure $\overline{Y}$ of Y in X is a connected component of $Z_1 \cap \ldots \cap Z_k$. Hence, restricting the exact sequence (21) to $\overline{Y}$, we get a long exact sequence

$$\ldots \to \mathcal{T}or^1\left(\bigoplus_i \mathcal{O}_{Z_i}(Z_i), \mathcal{O}_{\overline{Y}}\right) \xrightarrow{h} \mathcal{O}^k_{\overline{Y}} \oplus \left(\bigoplus_{i>k} \mathcal{O}_{\overline{Y} \cap Z_i}\right) \to$$

$$\to \operatorname{coker}(H)|_{\overline{Y}} \to \left(\bigoplus_{i \leq k} \mathcal{O}_{\overline{Y}}(Z_i)\right) \oplus \left(\bigoplus_{i>k} \mathcal{O}_{\overline{Y} \cap Z_i}(Z_i)\right) \to 0.$$

This gives immediately the required estimate of the rank of H over Y. Now we claim that

$$\mathcal{T}or^1\left(\bigoplus_i \mathcal{O}_{Z_i}(Z_i), \mathcal{O}_{\overline{Y}}\right) \simeq \mathcal{O}^k_{\overline{Y}},$$

which implies that the rank of the map h above is constant over Y. Indeed, if $i > k$, then $\overline{Y}$ meets Z_i transversally, hence, $\mathcal{T}or^1(\mathcal{O}_{Z_i}(Z_i), \mathcal{O}_{\overline{Y}}) = 0$. Otherwise, $\overline{Y} \subset Z_i$ and $\mathcal{T}or^1(\mathcal{O}_{Z_i}(Z_i), \mathcal{O}_{\overline{Y}}) \simeq \mathcal{O}_{\overline{Y}}$. □

Corollary 10.8. *Assume that Z is smooth. Then for the induced Poisson structure $H_Z : \Omega^1_Z \to T_Z$ there are canonical isomorphisms:* $\ker(H_Z) \simeq \operatorname{coker}(H_Z) \simeq \mathcal{O}_Z$. *Locally $\mathcal{A}^\bullet$ is generated as an algebra over $\Omega^\bullet_X(\log Z)$ by the 1-form $f^{-1}\alpha$, where f is a local equation of Z, and α is a regular 1-form on X.*

11. Poisson Structures on the Projective Space

We start with one of a few general statements one can make about Poisson structures on the projective space. It shows, in particular, that the degeneration locus of a nondegenerate Poisson structure on $\mathbf{P}^{2r}$ is singular for $r > 1$.

Theorem 11.1. *Assume that for a nonzero Poisson structure on $\mathbf{P}^n$, there is a Poisson divisor $Z \subset \mathbf{P}^n$ of degree $n+1$ which is a union of k smooth components with normal crossings. Then $k \geq n-1$.*

Lemma 11.2. *Let $X \subset \mathbf{P}^n$ be a smooth complete intersection of k smooth hypersurfaces of degrees $d_1, \ldots, d_k$, such that $\sum_i d_i \leq n+1$. Then $H^q(X, \Omega^p_X(l)) = 0$ if $p+q < n-k$ and either $l < 0$, or $l = 0$ and $q < p$.*

The proof follows easily by induction from the standard vanishings for $H^q(\mathbf{P}^n, \Omega^p_{\mathbf{P}^n}(l))$ (see, e.g., [3]) and is left to reader.

Proof of Theorem 11.1. Assume first that $Z = Z_1 \cup \ldots \cup Z_k$, where $k \leq n-3$. Let $Y = Z_1 \cap \ldots \cap Z_k$. Then $\omega_Y \simeq \mathcal{O}_Y$ and $\dim Y = n-k \geq 3$. The induced Poisson structure on Y is given by a tensor in $H^0\left(Y, \bigwedge^2 T_Y\right) \simeq H^0\left(Y, \Omega_Y^{n-k-2}\right)$, hence, it is zero by the previous lemma. Now the induced Poisson structure on $Y' = Z_1 \cap \ldots \cap Z_{k-1}$ vanishes on the anticanonical divisor $Y' \cap Z_k \subset Y'$. Therefore, it corresponds to an element in $H^0\left(Y', \bigwedge^2 T_{Y'} \otimes \omega_{Y'}\right) \simeq H^0\left(Y', \Omega_{Y'}^{n-k-1}\right)$, and again using the previous lemma we conclude that it is zero. Similarly, the Poisson structure vanishes over all $(k-1)$-fold intersections of branches. Proceeding further by induction, we obtain that the Poisson structure on $\mathbf{P}^n$ is zero, which is a contradiction.

Now consider the case where $k = n-2$. Then $Y = Z_1 \cap \ldots \cap Z_k$ is a surface with trivial canonical class, hence, the induced Poisson structure on Y is either zero or a symplectic one. However, the latter case is impossible because the induced Poisson structure on the Fano threefold $Z_1 \cap \ldots \cap Z_{k-1}$ should vanish at least along a curve by 9.2. Therefore, the Poisson structure vanishes over Y, and we can proceed as above to get a contradiction. □

Now we want to characterize a Poisson structure on the projective space $\mathbf{P}^n$ by its restriction to the open affine subset U_m where $x_m \neq 0$. Put

$$P_{ij} = \left\{\frac{x_i}{x_m}, \frac{x_j}{x_m}\right\}.$$

Then we have

$$\left\{\frac{x_j}{x_i}, \frac{x_k}{x_i}\right\} = \frac{x_m^3}{x_i^3}\left(\frac{x_i}{x_m}P_{jk} + \frac{x_j}{x_m}P_{ki} + \frac{x_k}{x_m}P_{ij}\right), \tag{27}$$

$$\left\{\frac{x_j}{x_i}, \frac{x_m}{x_i}\right\} = \frac{x_m^3}{x_i^3}P_{ij}. \tag{28}$$

Hence, the regularity at infinity of the Poisson structure given by P_{ij} is equivalent to the following two conditions:

1. $x_m^3 \cdot P_{ij} \in S$ for all i, j;
2. $x_i \cdot P_{jk} + x_j \cdot P_{ki} + x_k \cdot P_{ij} \in x_m S$ for all i, j, k,

where S is the polynomial algebra in variables x_i. If we consider P_{ij} as polynomials in the variables $y_i = \frac{x_i}{x_m}$, then these conditions are equivalent to the following:

1. $\deg P_{ij} \leq 3$;
2. $\deg(y_i \cdot P_{jk} + y_j \cdot P_{ki} + y_k \cdot P_{ij}) \leq 3$.

In other words, if we write $P_{ij} = C_{ij} + R_{ij}$, where $\deg R_{ij} \leq 2$ and C_{ij} are homogeneous cubic polynomials, then we should have

$$y_i \cdot C_{jk} + y_j \cdot C_{ki} + y_k \cdot C_{ij} = 0.$$

The obtained bracket on U_m has a zero cubic part if and only if the hyperplane $x_m = 0$ is a Poisson subvariety of $\mathbf{P}^n$ (this follows from the formula (28)). Thus, we can extend any (nonhomogeneous) quadratic Poisson structure on the affine space to the Poisson structure on the projective space of the same dimension such that the complementary hyperplane is a Poisson subvariety.

12. Quadratic Poisson Structures

As we have noted in 8.2, there is a bijection between the set of quadratic (=homogeneous) Poisson structures on a vector space V and the following set of pairs: a Poisson structure on the projective space $\mathbf{P}(V)$ and a Poisson module structure on the line bundle $\mathcal{O}(1)$. On the other hand, we know (see Secs. 2, 4, and 7) that for any Poisson variety X there is a canonical Poisson structure on the canonical line bundle ω_X. For the projective space $\mathbf{P}^n$ this means that for any Poisson structure on it there is a canonical Poisson module structure on $\mathcal{O}(-n-1)$, hence on $\mathcal{O}(1)$. Since two flat Poisson connections on the same line bundle differ by a hamiltonian vector field, we arrive at the following statement, which was first proven by Bondal [2].

Theorem 12.1. *There is a bijection between the set of quadratic Poisson structures on a vector space V and the following set of pairs: a Poisson structure on the projective space $\mathbf{P}(V)$ and a global hamiltonian vector field with respect to it.*

The passage from a quadratic Poisson structure on V to a Poisson structure on $\mathbf{P}(V)$ is clear: one just uses the Leibnitz identity to define the Poisson bracket between the rational functions of degree zero which are local functions on $\mathbf{P}(V)$. The rest of this section is devoted to making the passage in the opposite direction more explicit.

According to Sec. 5, for any Poisson algebra A and any derivation v preserving Poisson structure, we can define a Poisson structure on $A[t]$ by the formula

$$\{a, t\} = t \cdot v(a),$$

where $a \in A$. Then the following identity holds:

$$\{ta, tb\} = t^2(\{a, b\} - a \cdot v(b) + b \cdot v(a)) \tag{29}$$

for any $a, b \in A$. Now if A has an increasing filtration (A_i) compatible with multiplication, we can consider the following subalgebra of $A[t]$:

$$\widetilde{A} = \oplus A_i \cdot t^i.$$

In our case, A is a polynomial algebra in the variables $\frac{x_j}{x_m}$ with the filtration by total degree, $t = x_m$, and $\widetilde{A}$ is a polynomial algebra in the variables x_j. Now $\widetilde{A}$ inherits a Poisson structure if and only if the bracket

$$\{a, b\}_1 = \{a, b\} - a \cdot v(b) + b \cdot v(a)$$

is compatible with filtration in the following sense: $\{A_1, A_1\}_1 \subset A_2$ (note that $\{\cdot,\cdot\}_1$ is not a Poisson bracket in general). In our case, the Poisson bracket on A is cubic (see the previous section), and if we choose v to be quadratic, then the compatibility will mean that the homogeneous cubic part of the Poisson bracket on A is equal to $\varepsilon \wedge v_2$, where ε is the Euler vector field, v_2 is the homogeneous quadratic part of the vector field v. To get the canonical Poisson bracket associated with a Poisson bracket on the projective space, we have

to apply this to $v = v_{\mathrm{can}} = \frac{1}{n+1}\Delta_\phi(G)$, where $\phi = dy_1 \wedge \ldots \wedge dy_n$ (see Sec. 4). Thus, we get the following explicit formula for the Poisson bracket on the polynomial algebra associated with a Poisson structure on the projective space.

Theorem 12.2. *Let a Poisson structure on* $\mathbf{P}^n$ *be given by the bracket*

$$\left\{\frac{x_i}{x_0}, \frac{x_j}{x_0}\right\} = \frac{p_{ij}}{x_0^3},$$

where $x_0, \ldots, x_n$ *are homogeneous coordinates,* p_{ij} *are homogeneous polynomials in* x_i *of degree* 3. *Then the associated quadratic Poisson bracket in variables* x_i *is given by the formula*

$$\{x_i, x_j\} = x_0^{-1}\left(p_{ij} - \frac{x_i}{n+1}\sum_{k=1}^{n}\frac{\partial p_{kj}}{\partial x_k} + \frac{x_j}{n+1}\sum_{k=1}^{n}\frac{\partial p_{ki}}{\partial x_k}\right).$$

Example. Let $n = 2$; then a Poisson structure on $\mathbf{P}^2$ has the form

$$\left\{\frac{x_1}{x_0}, \frac{x_2}{x_0}\right\} = \frac{f}{x_0^3},$$

where f is a cubic form. The corresponding quadratic Poisson structure is $\{x_1, x_2\} = \frac{1}{3}\frac{\partial f}{\partial x_0}$, ... (the other brackets are obtained by cyclic permutation of x_0, x_1, x_2).

13. Poisson Structures on 3-Dimensional Varieties

In this section, X is always a 3-dimensional smooth variety. A (nonzero) Poisson structure on X is the same as an embedding $i : \omega_X \to \Omega^1_X$ defining a corank-1 foliation on X. As we have seen in Sec. 9, if X is a Fano 3-fold, then the vanishing locus Z of i has a component of dimension ≥ 1. We are particularly interested in the case where there is a smooth connected component of Z of dimension 1.

Theorem 13.1. *Let* $C \subset X$ *be a smooth curve which is an irreducible component of the vanishing locus of a Poisson structure (equipped with the reduced scheme structure) on a 3-dimensional smooth variety* X. *Then the conormal Lie sheaf of* C *is abelian, i.e.,* $\{J_C, J_C\} \subset J_C^2$, *where* $J_C \subset \mathcal{O}_X$ *is an ideal sheaf of* C.

Proof. Consider the situation in the formal neighborhood of a point of C (or analytically). Then we can choose a coordinate system x_1, x_2, x_3 on X such that C is defined by the ideal $J_C = (x_1, x_2)$. Let the Poisson structure be given by

$$\{x_1, x_2\} = f_3,\ \{x_2, x_3\} = f_1,\ \{x_3, x_1\} = f_2.$$

By assumption, J_C is an associated prime ideal of the ideal (f_1, f_2, f_3). Now consider the hamiltonian vector field

$$\Delta_\phi(G) = \left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right)\frac{\partial}{\partial x_1} + \ldots$$

defined in Sec. 4: we use the formula of Lemma 4.5; the skipped terms are obtained by cyclic permutation of indices. We know by Lemma 1.1 that $\Delta_\phi(G)$ preserves $J_C = (x_1, x_2)$. This means that $\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \in J_C$ and $\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \in J_C$. Note that $f_i \in (x_1, x_2)$, hence, $\frac{\partial f_i}{\partial x_3} \in (x_1, x_2)$. Therefore, $\frac{\partial f_3}{\partial x_1}, \frac{\partial f_3}{\partial x_2} \in (x_1, x_2)$. Since $f_3 \in (x_1, x_2)$ we can write $f_3 = x_1 g_1 + x_2 g_2$, so that $\frac{\partial f_3}{\partial x_1} \equiv g_1 \pmod{x_1, x_2}$, $\frac{\partial f_3}{\partial x_2} \equiv g_2 \pmod{x_1, x_2}$, which implies that $g_1, g_2 \in (x_1, x_2)$ and $f_3 \in (x_1, x_2)^2 = J_C^2$, as required. □

Corollary 13.2. *Let* $C \subset X$ *be a smooth curve which is a connected component of the vanishing locus of a Poisson structure on* X *(with its natural scheme structure). Then*

1. $\omega_C^{\otimes 2} \simeq \mathcal{O}_C$;

2. *the induced Poisson structure on the projectivization* $\mathbf{P}(N)$ *of the normal bundle of the embedding* $C \subset X$ *vanishes on the fiber over a point* $x \in C$ *if and only if the cotangent Lie algebra at* x *is degenerate.*

Proof. The cotangent Lie algebra at any point $x \in C$ contains a 2-dimensional abelian subalgebra, namely, the stalk of J_C/J_C^2 at x. By Lemma 2.5, this implies that the adjoint action of ω_C on J_C/J_C^2 is everywhere nondegenerate, i.e., we have an isomorphism $\omega_C \otimes J_C/J_C^2 \tilde{\to} J_C/J_C^2$. Considering the determinants, we get $\omega_C^{\otimes 2} \simeq \mathcal{O}_C$. For the proof of the second statement, note that the cotangent Lie algebra is degenerate at x if and only if the corresponding operator $(J_C/J_C^2)_x \to (J_C/J_C^2)_x$ is scalar (see 8.1). Now the proof of Theorem 6.1 shows that this is equivalent to the vanishing of the Poisson bracket on $\mathbf{P}(N)$ over x. □

The main source of Poisson structures on 3-dimensional varieties is given by the following construction. Let $f : X \to Y$ be a morphism, where X and Y are smooth of dimensions 3 and 1 respectively. Let F_i be the multiple fibers of f and m_i be their multiplicities; then there is a pull-back morphism on 1-forms

$$i_f : f^*\omega_Y(\sum_i (m_i - 1)F_i) \to \Omega^1_X,$$

which defines generically an integrable subbundle. Now if D is a divisor in the linear system

$$\left| f^*\omega_Y \left(\sum_i (m_i - 1)F_i \right) \otimes \omega_X^{-1} \right|,$$

then we have a Poisson structure (which is defined up to a scalar)

$$i_{f,D} : \omega_X \simeq f^*\omega_Y(-D) \hookrightarrow f^*\omega_Y \to \Omega^1_X.$$

When $Y \simeq \mathbf{P}^1$, we say that this structure is associated with the corresponding pencil and D. Note that the same construction works for a birational morphism since it is defined in codimension 1. Notice also that the fibers of f are Poisson divisors with respect to such a Poisson structure.

Lemma 13.3. *Let* $i : \omega_X \to \Omega^1_X$ *be a Poisson structure. Then a smooth divisor* $D \in X$ *is Poisson with respect to* i *if and only if the composition*

$$\omega_X|_D \xrightarrow{i|_D} (\Omega^1_X)|_D \longrightarrow \Omega^1_D$$

is zero.

Proof. Locally we can choose a volume form $\eta \in \omega_X$ so that the Poisson bracket is given by the formula

$$i(\eta) \wedge df \wedge dg = \{f, g\}\eta.$$

Let f be a local equation of D. Then D is a Poisson divisor if and only if $\{f, g\}$ is divisible by f for any $g \in \mathcal{O}_X$, or equivalently, $i(\eta) \wedge df \wedge dg \in \mathcal{O}_X f\eta$ for any g. The latter condition means that $i(\eta) \in \mathcal{O}_X df$ mod $(\mathcal{O}_X f\Omega^1_X)$, that is, $i(\eta)|_D \in \Omega^1_D$ is zero. □

Lemma 13.4. *Let* E *be a locally free sheaf on a smooth variety,* $L_1 \hookrightarrow E$, $L_2 \hookrightarrow E$ *be a pair of morphisms of sheaves, where* L_i *are line bundles. Assume that* $L_1|_U \subset L_2|_U \subset E$ *for some dense open subset* U *and that* $L_2 \hookrightarrow E$ *is a subbundle outside some closed subset of codimension* 2. *Then* $L_1 \subset L_2$.

Proof. Let $\widehat{L_2} \subset E$ be a maximal normal extension of L_2 (see [16, Chapter II, 1.1]). Since $\widehat{L_2}$ is normal and has no torsion, it is reflexive (see loc. cit., 1.1.12). But a reflexive sheaf of rank 1 is a line bundle (loc. cit., 1.1.15), hence, $\widehat{L_2}$ is a line bundle. Since the morphism $L_2 \to E$ does not vanish on any divisor, it follows that $L_2 = \widehat{L_2}$, that is, E/L_2 has no torsion. Now the canonical morphism $L_1 \to E/L_2$ should be zero since it vanishes at the general point and the target has no torsion. □

Theorem 13.5. *Let $f : X \to Y$ be a morphism, where $\dim X = 3$, $\dim Y = 1$, $i : \omega_X \to \Omega^1_X$ be a Poisson structure on X such that a general fiber of f is a Poisson divisor with respect to i. Then $i = i_{f,D}$ for some divisor $D \in H^0\left(f^*\omega_Y\left(\sum_i (m_i - 1)D_i\right) \otimes \omega_X^{-1}\right)$.*

Proof. Over an open subset where i and i_f are the subbundles, their images coincide. Indeed, Lemma 13.3 implies that locally over such a subset the 1-form corresponding to i is proportional to df. It remains to apply Lemma 13.4 to i and i_f. □

Examples. 1. A pencil of quadrics on $\mathbf{P}^3$ induces a Poisson bracket on $\mathbf{P}^3$, since for the corresponding birational morphism $f : \mathbf{P}^3 \to \mathbf{P}^1$ we have $f^*\mathcal{O}(-2) \simeq \mathcal{O}(-4)$.

2. A pencil of cubics on $\mathbf{P}^3$ containing a triple hyperplane $3L$ also induces a Poisson bracket, since in this case we have $f^*\mathcal{O}(-2)(2L) \simeq \mathcal{O}(-4)$.

14. Poisson Structures on $\mathbf{P}^3$

We need two lemmas concerning the geometry of an elliptic curve of degree 5 in $\mathbf{P}^3$.

Lemma 14.1. *Let $C \subset \mathbf{P}^3$ be an elliptic curve of degree 5 which is cut out scheme-theoretically by cubics. Then there exists a smooth cubic S containing C. Furthermore, S can be represented as a blow-up of $\mathbf{P}^2$ in six points in such a way that $C \in \left|\mathcal{O}_S\left(5l - 2\sum_{i=1}^{5} e_i\right)\right|$, where e_i, $i = 1, \ldots, 6$, are the exceptional divisors, $\mathcal{O}_S(l)$ is the pull-back of $\mathcal{O}_{\mathbf{P}^2}(1)$. In particular, C is not contained in any quadric.*

Proof. Let $J_C \subset \mathcal{O}_{\mathbf{P}^3}$ be the ideal sheaf of C. Then, by assumption, $J_C(3)$ is generated by global sections. In particular, if $\pi : X \to \mathbf{P}^3$ is the blow-up of $\mathbf{P}^3$ along C, then the linear series $|3H - E|$ on X is base-point free (here H is the pull-back of the hyperplane class, E is the exceptional divisor). Hence, by the Bertini theorem, a general divisor in $|3H - E|$ is smooth. Let $f : X \to \mathbf{P}^n$ be a morphism defined by $|3H - E|$, so that $n = h^0(J_C(3)) - 1 \geq 4$. Then f maps each projective line $\pi^{-1}(x)$, where $x \in C$, isomorphically onto a line in $\mathbf{P}^n$. Since this family of lines is one-dimensional, an easy dimension count shows that a general hyperplane in $\mathbf{P}^n$ does not contain $f(\pi^{-1}(x))$ for any $x \in C$. Therefore, a general divisor $D \in |3H - E|$ is smooth and intersects each fiber $\pi^{-1}(x)$, where $x \in C$, by a simple point. It follows that the projection π maps D isomorphically onto a smooth cubic hypersurface $S \subset \mathbf{P}^3$ containing C. It is well known that S is isomorphic to the blow-up of $\mathbf{P}^2$ at 6 points in such a way that $\mathcal{O}_S(1) \simeq \mathcal{O}\left(3l - \sum_{i=1}^{6} e_i\right)$ where e_i are the exceptional divisors, $\mathcal{O}(l)$ is the pull-back of $\mathcal{O}_{\mathbf{P}^2}(1)$. Assume that $\mathcal{O}_S(C) \simeq \mathcal{O}\left(al - \sum_{i=1}^{6} b_i e_i\right)$. Then $a \geq 2$, $b_i \geq 0$ for any i. The condition that C is an elliptic curve of degree 5 implies the following two equations on a, b_i:

$$3a - \sum_{i=1}^{6} b_i = 5, \tag{30}$$

$$a^2 - \sum_{i=1}^{6} b_i^2 = 5. \tag{31}$$

Without loss of generality, we may assume that $b_i \geq 1$ for $i = 1, \ldots, k$, $b_i = 0$ for $i > k$ for some $1 \leq k \leq 6$. Now, using the inequality $\sum_{i=1}^{k} b_i^2 \geq \frac{1}{k}\left(\sum_{i=1}^{k} b_i\right)^2$ and denoting $\sum_i b_i$ by x, we get the following inequality:

$$\left(\frac{x+5}{3}\right)^2 \geq \frac{x^2}{k} + 5, \tag{32}$$

which implies that either $k = 5$ and $x \leq 10$, or $k = 6$ and $x < 18$. Also, it follows from Eq. (30) that $x \equiv 1 \pmod 3$. If $k = 5$, this implies that either $x = 7$ or $x = 10$. One can easily see that the former case is impossible, and in the latter case the only solution (up to renumeration of b_i) is $a = 5$, $b_1 = \ldots = b_5 = 2$, $b_6 = 0$. Now let $k = 6$; then it follows from Eq. (31) that there exists i such that b_i is divisible by 3, say $b_6 \equiv 0 \pmod 3$. In particular, $x \geq 8$ and the only possible values of x are 10, 13, and 16. One can easily see that the case $x = 10$ is impossible; in the case $x = 13$, the only solution is $a = 6$, $b_1 = 1$, $b_2 = b_3 = b_4 = 2$, $b_5 = b_6 = 3$, and in the case $x = 16$ the only solution is $a = 7$, $b_1 = b_2 = 2$, $b_i = 3$ for $i \geq 3$. Note that in the latter case C is not scheme-theoretically cut out by cubics. Indeed, consider following exact sequence of sheaves on $\mathbf{P}^3$:

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-3) \to J_C \to \mathcal{O}_S(-C) \to 0. \tag{33}$$

It shows that if $J_C(3)$ is generated by global sections, then so is $\mathcal{O}_S(3)(-C)$. However, if $\mathcal{O}(C) \simeq \mathcal{O}\left(7l - 2e_1 - 2e_2 - 3\sum_{i=3}^{6} e_i\right)$, then $\mathcal{O}_S(3)(-C) \simeq \mathcal{O}(2l - e_1 - e_2)$, which is not globally generated. Now assume that $\mathcal{O}_S(C) \simeq \mathcal{O}_S(6l - e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6)$. Consider the following lines on S: $e_1' = l - e_1 - e_5$, $e_2' = e_2$, $e_3' = e_3$, $e_4' = e_4$, $e_5' = l - e_1 - e_6$, $e_6' = l - e_5 - e_6$. These lines are mutually disjoint, so they define a blow-down of S to $\mathbf{P}^2$. Let l' be the pull-back of $\mathcal{O}_{\mathbf{P}^2}(1)$ under this morphism; then we have

$$3l' - \sum_{i=1}^{6} e_i' = 3l - \sum_{i=1}^{6} e_i,$$

hence $l' = 2l - e_1 - e_5 - e_6$. Now we see that

$$\mathcal{O}_S(6l - e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6) \simeq \mathcal{O}_S\left(5l' - 2\sum_{i=1}^{5} e_i'\right),$$

as required.

Now the fact that C is not contained in any quadric follows immediately because $H^0(S, \mathcal{O}_S(2)(-C)) = 0$. Indeed, we have $\mathcal{O}(C) \simeq \mathcal{O}_S\left(5l - 2\sum_{i=1}^{5} e_i\right)$; therefore, $H^0(S, \mathcal{O}_S(2)(-C)) = H^0(S, \mathcal{O}_S(l - 2e_6)) = 0$. □

Lemma 14.2. *Let $C \subset \mathbf{P}^3$ be an elliptic curve of degree 5 which is cut out scheme theoretically by cubics. Then $h^0(\mathbf{P}^3, J_C(3)) = 5$. Let X be a blow-up of $\mathbf{P}^3$ along C, $f : X \to \mathbf{P}^4$ be a morphism defined by the linear system $|3H - E|$, where H is the hyperplane class, $E \subset X$ is an exceptional divisor. Then f maps X onto a smooth quadric hypersurface $Y \subset \mathbf{P}^4$ contracting the irreducible divisor $Q \subset X$ which is the only effective divisor in the linear series $|5H - 2E|$ onto a curve $f(Q)$ of degree 5 in $\mathbf{P}^4$. Moreover, f induces an isomorphism $X \setminus Q \to Y \setminus f(Q)$ and factors through a morphism $\tilde{f} : X \to \tilde{Y}$, where $\tilde{Y} \to Y$ is a blow-up of Y along $f(Q)$, which is an isomorphism in codimension 1.*

Proof. According to the previous lemma, we can find a smooth cubic $S \subset \mathbf{P}^3$ containing C. Then using the explicit form of $\mathcal{O}_S(C)$ obtained in the previous lemma, one can easily show that $H^1(S, \mathcal{O}_S(3)(-C)) = 0$. Indeed, as $\omega_S \simeq \mathcal{O}_S(-1)$ by Kodaira vanishing it is sufficient to prove that $\mathcal{O}_S(4)(-C)$ is ample, which can be checked using [11, V, 4.13]. Therefore, using (33) we get $H^1(\mathbf{P}^3, J_C(3)) = 0$, and from the exact sequence

$$0 \to J_C(3) \to \mathcal{O}_{\mathbf{P}^3}(3) \to \mathcal{O}_C(3) \to 0,$$

we conclude that $h^0(\mathbf{P}^3, J_C(3)) = 5$.

Now let us check that $h^0(X, \mathcal{O}_X(5H - 2E)) = 1$. Note that a smooth cubic S containing C can be embedded canonically in X so that $\mathcal{O}_X(S) \simeq \mathcal{O}(3H - E)$ and $S \cap E = C \subset S$. Consider the following exact sequence:

$$0 \to \mathcal{O}_X(2H - E) \to \mathcal{O}_X(5H - 2E) \to \mathcal{O}_S(5)(-2C) \to 0.$$

By Lemma 14.1, we have $H^0(X, \mathcal{O}_X(2H - E)) = H^0(\mathbf{P}^3, J_C(2)) = 0$. Also, we claim that $H^1(X, \mathcal{O}_X(2H - E)) = H^1(\mathbf{P}^3, J_C(2))$. Indeed, this follows from the exact sequence

$$0 \to H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) \to H^1(J_C(2)) \to 0,$$

since $h^0(\mathcal{O}_{\mathbf{P}^3}(2)) = h^0(\mathcal{O}_C(2)) = 10$. Therefore, $h^0(X, \mathcal{O}_X(5H - 2E)) = h^0(S, \mathcal{O}_S(5)(-2C))$. Now as $\mathcal{O}_S(C) \simeq \mathcal{O}_S\left(5l - 2\sum_{i=1}^{5} e_i\right)$, we get $h^0(\mathcal{O}_S(5)(-2C)) = h^0\left(\mathcal{O}_S\left(\sum_{i=1}^{5}(l - e_i - e_6)\right)\right) = 1$. Thus, there is a unique effective divisor in $|5H - 2E|$ which is easily seen to be irreducible.

One can compute easily the intersection number $(3H - E)^3 = 2$, which means that f is a birational morphism onto a quadric hypersurface $Y \subset \mathbf{P}^4$ (f cannot be a double covering of a hyperplane in $\mathbf{P}^4$ because it is given by a complete linear system). Consider the restriction of f to a general divisor $S \in H^0(X, \mathcal{O}_X(3H - E))$ corresponding to a smooth cubic in $\mathbf{P}^3$ containing C. One can easily see that if we represent S as a blow-up of $\mathbf{P}^2$ in 6 points so that $\mathcal{O}_S(C) \simeq \mathcal{O}_S\left(5l - 2\sum_{i=1}^{5}\right)$, then the restriction $f|_S$ is the blow-down of 5 lines $l - e_i - e_6$ ($i = 1, \dots, 5$) onto a smooth 3-dimensional quadric. Hence, Y is smooth.

Note that a line in $\mathbf{P}^3$ maps to a point under f if and only if it is a trisecant of C, that is, intersects C in 3 points. A general chord of C is not a trisecant (e.g. the exceptional line $e_1 \subset S$), hence, there is at most a 1-dimensional family of trisecants of C. It follows that a general plane $L \subset \mathbf{P}^3$ does not contain any trisecant of C, so it intersects C by 5 points in general linear position. Hence, the proper preimage $\widetilde{L}$ of L under π is a blow-up of $\mathbf{P}^2$ at 5 points in general position and the restriction of f to $\widetilde{L}$ induces an isomorphism of $\widetilde{L}$ with an intersection of two quadrics in $\mathbf{P}^4$. Thus, the divisor class of $f(\widetilde{L}) \subset Y$ is $\mathcal{O}_Y(2)$, so that $f^{-1}(f(\widetilde{L})) \in |6H - 2E|$ and $f^{-1}(f(\widetilde{L})) - \widetilde{L} \in |5H - 2E|$. As we have seen above, the latter linear series contains the only effective divisor $Q \subset X$ which therefore is contracted by f to an irreducible curve. Indeed, the restriction of f to $Q \cap \widetilde{L}$ is an immersion, hence, $f(Q)$ is a curve. To prove that f factors through a morphism $\widetilde{f} : X \to \widetilde{Y}$ it is sufficient to show that the subscheme $f^{-1}(f(Q)) \subset X$ coincides with Q (where $f(Q)$ is equipped with the reduced scheme structure). However, this follows easily from the fact that Q is the scheme-theoretic intersection of divisors $f^{-1}(f(\widetilde{L}))$, where L runs through all planes in $\mathbf{P}^3$ while $f(Q)$ is the intersection of divisors $f(\widetilde{L})$. Now let $Z \subset Y$ be the exceptional locus of f. We claim that Z is a union of $f(Q)$ and a finite number of points. Indeed, let $Y_1 \subset Y$ be a general hyperplane section of Y. Then $f^{-1}(Y_1)$ is a proper preimage of the smooth cubic containing C. The restriction of f to the cubic $S = f^{-1}(Y_1)$ is the blow-down of 5 lines $l - e_i - e_6$ ($i = 1, \dots, 5$) which constitute the intersection $S \cap Q$. Hence, $Y_1 \cap Z$ is contained in $f(Q)$, which implies our claim. This argument also shows that $\widetilde{f}$ is an isomorphism over a general point of $f(Q)$, which finishes the proof. □

Theorem 14.3. *Let $\omega \in H^0(\Omega^1_{\mathbf{P}^3}(4))$ be a Poisson structure on $\mathbf{P}^3$ such that the vanishing locus of ω has a connected component which is a smooth curve C. Then C is an elliptic curve of degree 3 or 4.*

Proof. One can compute easily that $c_3(\Omega^1_{\mathbf{P}^3}(4)) = 20$. Thus, there is a 0-cycle of degree 20 (*intersection product*) on the vanishing locus of ω (see [8]). According to [8, Proposition 9.1.1], the contribution of the part supported on C is equal to

$$\deg\left(\Omega^1_{\mathbf{P}^3}(4)|_C\right) - \deg\left(T_{\mathbf{P}^3}|_C\right) = 4 \deg C.$$

Together with 13.1, this implies that $C \subset \mathbf{P}^3$ is an elliptic curve of degree ≤ 5. Let us prove that the case $\deg C = 5$ is impossible. Indeed, then the vanishing locus of ω coincides with C, i.e., we have a surjection $\omega^\vee : T_{\mathbf{P}^3} \to J_C(4)$. In particular, C is cut out scheme-theoretically by cubics. The blow-up X of $\mathbf{P}^3$ along

C has a compatible Poisson structure such that the exceptional divisor $E \subset X$ is a Poisson divisor. First of all, we claim that the induced Poisson structure on E is zero. Indeed, we have a morphism $\widetilde{f} : X \to \widetilde{Y}$ constructed in Lemma 14.2, which is an isomorphism in codimension 1. According to Sec. 8.4, there is a compatible Poisson structure on $\widetilde{Y}$ such that $\widetilde{f}$ is a Poisson morphism. It follows that there is a compatible Poisson structure on Y vanishing over $f(Q)$. If the Poisson structure on E is not zero, then the vanishing locus of the Poisson structure on X has dimension 1 (note that the vanishing locus of the original Poisson structure has dimension 1 because otherwise a component of dimension 2 should intersect C). Hence, $f(Q)$ is an irreducible component of the vanishing locus of the induced Poisson structure on Y. By Theorem 13.1, this implies that the exceptional divisor $\widetilde{E}$ in the blow-up $\widetilde{Y}$ of Y along $f(Q)$ is a Poisson divisor. Furthermore, a morphism $\widetilde{f} : X \to \widetilde{Y}$ is an isomorphism at the general point of Q and $\widetilde{f}^{-1}(\widetilde{E}) = Q$, hence Q is a Poisson divisor. Therefore, the Poisson structure on E vanishes over the curve $Q \cap E$. However, $Q \cap E$ is a divisor in the linear system $\mathcal{O}(5H - 2E)|_E \simeq \mathcal{O}_E(2) \otimes p^*\mathcal{O}_C(5)$, where $p : E \to C$ is the projection. Thus, the Poisson structure on E is given by an element in $H^0(E, \omega_E^{-1}(-Q \cap E)) \simeq H^0(E, p^*\mathcal{O}_C(-1)) = 0$, hence, it is zero. Now, as follows from Lemma 14.2, the restriction of f to E is birational; therefore $f(E)$ is a surface of degree $(3H - E) \cdot (3H - E) \cdot E = 10$. Thus, $f(E)$ is a divisor in the linear system $|\mathcal{O}_Y(5)|$, hence the induced Poisson structure on Y is given by an element in $H^0(\Omega^1_Y \otimes \omega_Y^{-1}(-5)) \simeq H^0(\Omega^1_Y(-2)) = 0$, which is a contradiction. □

We treat two cases of the conclusion of the previous theorem separately in the following two propositions. In these propositions (but not in the subsequent theorem), the words "vanishing over an elliptic curve" mean that the curve is a connected component of the vanishing locus with the reduced scheme structure.

Proposition 14.4. *Let $\omega \in H^0(\Omega^1_{\mathbf{P}^3}(4))$ be a Poisson structure on $\mathbf{P}^3$ vanishing over an elliptic curve $C \subset \mathbf{P}^3$ of degree 4 and at a finite number of points. Then ω is associated with the pencil of quadrics containing C.*

Proof. Let $p : \widetilde{\mathbf{P}}^3 \to \mathbf{P}^3$ be the blow-up along C. Then there is a morphism $f : \widetilde{\mathbf{P}}^3 \to \mathbf{P}^1$ given by the pencil $|p^*\mathcal{O}(2)(-E)|$, where E is the exceptional divisor, so that the fibers of f are isomorphic to quadrics in $\mathbf{P}^3$ passing through C. The Poisson structure on $\widetilde{\mathbf{P}}^3$ is given by some foliation $\widetilde{\omega} : p^*\mathcal{O}(-4)(E) \to \Omega^1$. We claim that $\widetilde{\omega}$ vanishes over E. Indeed, according to Theorems 13.1 and 8.2, E is a Poisson divisor. Note that there is an isomorphism $E \simeq C \times \mathbf{P}^1$ such that $f|_E$ is the natural projection to $\mathbf{P}^1$. A nonzero Poisson structure on E vanishes along the divisor in the linear system $(f|_E)^*\mathcal{O}(2)$, hence, if $\widetilde{\omega}$ does not vanish over E, then it defines a subbundle everywhere on $\widetilde{\mathbf{P}^3}$ except for a finite number of fibers of f. But this contradicts the Bott vanishing theorem (see [4]) because $c_1^2(f^{-1}(U)) \neq 0$ for any nonempty open subset $U \subset \mathbf{P}^1$. Therefore, $\widetilde{\omega}$ vanishes over E, in other words, it factors through a morphism $\omega' : p^*\mathcal{O}(-4)(2E) \simeq f^*\mathcal{O}(-2) \to \Omega^1$. Note that for any smooth fiber $D = f^{-1}(x)$ the restriction of ω' as a 1-form to D is a section of $\Omega^1_D \otimes p^*\mathcal{O}(4)(-2E)|_D \simeq \Omega^1_D$. Since D is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, this section should be zero, hence, by Lemma 13.3 D is a Poisson divisor. Now our assertion follows from Theorem 13.5. □

Proposition 14.5. *Let $\omega \in H^0(\Omega^1_{\mathbf{P}^3}(4))$ be a Poisson structure on $\mathbf{P}^3$ vanishing over an elliptic curve C of degree 3 and over at least one point outside C. Then either ω vanishes on the plane L containing C or ω is associated with the pencil of cubics spanned by $3L$ and some cubic $S \subset \mathbf{P}^3$ such that $C = L \cap S$.*

Proof. Let us restrict ω to the 3-dimensional vector space $V = \mathbf{P}^3 \setminus L$. We can assume that ω vanishes at $0 \in V$. Hence, we can write $\omega = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, where $\deg p_i \leq 2$, $p_i(0) = 0$. Let $p_i = q_i + l_i$, where q_i (resp. l_i) are quadratic (resp. linear) forms. Then $q_1 dx_1 + q_2 dx_1 + q_3 dx_3$ is also a Poisson structure and $3f = x_1 q_1 + x_2 q_2 + x_3 q_3 = 0$ is the equation of C where x_i are considered as homogeneous coordinates on L. Since C is smooth, there is a unique quadratic Poisson structure on V inducing the given one on L —this follows, e.g., from Theorem 12.1 and Proposition 15.1. Therefore, we have $q_i = \frac{\partial f}{\partial x_i}$, for $i = 1, 2, 3$, that is,

$$\omega = l_1 dx_1 + l_2 dx_2 + l_3 dx_3 + df.$$

Now the Pfaff equation (17) takes the form

$$d(l_1dx_1 + l_2dx_2 + l_3dx_3) \wedge df = 0,$$

which is equivalent to the equation $\left(\frac{\partial l_2}{\partial x_1} - \frac{\partial l_1}{\partial x_2}\right)\frac{\partial f}{\partial x_3} + \ldots = 0$, where the other terms are obtained by cyclic permutation. Since $\frac{\partial f}{\partial x_i}$, $i = 1, 2, 3$, are linearly independent, we obtain that $d\omega = 0$, i.e., $\omega = d(\widetilde{f})$, where $\deg \widetilde{f} \leq 3$. It is easy to see that this is equivalent to the conclusion of the proposition. □

Thus, we arrive at the following theorem.

Theorem 14.6. *Let ω be a Poisson structure on $\mathbf{P}^3$ such that the vanishing locus of ω has a connected component C which is a smooth curve. Then ω is either associated with the pencil of quadrics containing an elliptic curve of degree 4 or with the pencil of cubics spanned by a triple plane $3L$ and a cubic S such that $L \cap S$ is smooth.*

Proof. According to Theorem 14.3, C is an elliptic curve of degree 3 or 4. In the former case, the contribution to the degree of the 0-cycle corresponding to ω (which is equal to $c_3(\Omega^1_{\mathbf{P}^3}(4)) = 20$) of the part concentrated on C is equal to $4 \deg C = 12$, hence ω vanishes at some point outside C and we may apply Proposition 14.5. If $\deg C = 4$, then the contribution of the part concentrated on C is equal to 16. Assume that there is another connected component Z of the vanishing locus of ω which has dimension 1. Applying Theorem 12.2 of [8] in our situation, we get that the contribution of the part concentrated on Z is at least $2 \deg Z$ (since $\Omega^1_{\mathbf{P}^3}(2)$ is generated by global sections). Therefore, $\deg Z \leq 2$, which implies that Z is either a plane conic or a line. In both cases, one can compute using Proposition 9.1.1 of [8] that the contribution of the part concentrated on Z is bigger than 4, which is a contradiction. Hence, we can apply Proposition 14.4 to finish the proof. □

15. Hamiltonian Vector Fields

The problem of finding of all global vector fields on $\mathbf{P}^n$ preserving a given nondegenerate Poisson structure H boils down to determining the set of closed algebraic 1-forms ω with singularities along Z, the degeneration locus of H, for which $H(\omega)$ is a regular vector field on $\mathbf{P}^n$. Assume that Z is irreducible and reduced. Then we claim that $H^1(\mathbf{P}^n \setminus Z, \mathbf{C}) = 0$, hence, all such forms are exact (note that $U = \mathbf{P}^n \setminus Z$ is smooth and affine, so the cohomology of the algebraic De Rham complex coincides with the usual one). To see this we note that $\operatorname{Pic} U$ is a torsion group and global invertible functions on U are constant. So the Kummer sequence implies that $H^1_{\text{et}}(U, \mathbf{Z}/p\mathbf{Z}) = 0$ for almost all primes p. It follows that $\operatorname{rk} H^1(U, \mathbf{Z}) = 0$, as required. Thus, any vector field on U preserving H has the form $v = H(dg)$, where $g \in H^0(U, \mathcal{O}_U)$.

Proposition 15.1. *Assume that the Pfaffian form of a nondegenerate Poisson structure H on $\mathbf{P}^n$ is irreducible. Then there are no global vector fields on $\mathbf{P}^n$ preserving H.*

Proof. By the discussion above, we should consider an equality $v = H(dg)$, where v is a global vector field on $X = \mathbf{P}^n$, g is a global function on U. Now the lemma below implies that g is regular everywhere on $\mathbf{P}^n$, hence, constant. □

Lemma 15.2. *Let X be a normal nondegenerate (even-dimensional) Poisson variety such that the degeneration divisor Z is reduced at the general point of each component. Let g be a rational function on X which is regular on $X \setminus Z$, such that $H(dg)$ extends to a regular vector field on X. Then g is regular everywhere on X.*

Proof. Note that X is regular in codimension 1, so we can speak about the degeneration divisor of a Poisson structure on X. At the general point of a component of Z we can write $g = \frac{p}{f^m}$, where f is a local equation of Z, p is not divisible by f. By Corollary 10.8, we have $d(\frac{p}{f^m}) \in \Omega^1(\log Z) + \mathcal{O} \cdot f^{-1}\alpha$ for some regular 1-form

α, which is impossible for $m > 0$. Therefore, g is regular at the general point of every component of Z, hence, it is regular everywhere on X (since X is normal). $\square$

The proof of this proposition can be generalized to the case of hamiltonian vector fields with values in $\mathcal{O}(i)$. Here by a hamiltonian $\mathcal{O}(i)$-valued vector field, we mean an element $v \in T(i)$ such that $d(v) = 0$ for the Koszul differential $d : T(i) \to \bigwedge^2 T(i)$ associated with the canonical Poisson module structure on $\mathcal{O}(i)$.

Theorem 15.3. *Let X be a Fano variety such that* $\operatorname{Pic} X = \mathbf{Z}$, *and H be a nondegenerate Poisson structure on X. Assume that the degeneration locus Z of H is irreducible and reduced at the general point. Assume also that $\omega_X \simeq \mathcal{O}(-n)$, where $\mathcal{O}(1)$ is an ample generator of* $\operatorname{Pic} X$ *and $n|p^k i$ for some integer i and some prime number p. Then any $\mathcal{O}(i)$-valued hamiltonian vector field has the form $d(s)$ for some global section $s \in H^0(X, \mathcal{O}(i))$, where $d : \mathcal{O}(i) \to \mathcal{O}(i) \otimes T_X$ is the canonical differential.*

Proof. Let $U = X \setminus Z$, $\mathcal{A}$ be a local system on U associated with the canonical flat connection on $\mathcal{O}(i)|_U$. Then using Lemma 15.2 one can see that it is sufficient to prove that $H^1(U, \mathcal{A}) = 0$. As $\operatorname{Pic} U \simeq \mathbf{Z}/n\mathbf{Z}$, the Kummer sequence shows that the abelianization of the fundamental group π of U is $\mathbf{Z}/n\mathbf{Z}$. Consider the Galois covering $g : \widetilde{U} \to U$ associated with the normal subgroup $K \subset \pi$ such that $\pi/K \simeq \mathbf{Z}/q\mathbf{Z}$, where $q = p^l$ is the maximal power of p which divides n. Then $g^*(\mathcal{O}(i)|_U)$ is trivial, hence, it is sufficient to prove that $H^1(\widetilde{U}, \mathbf{C}) = 0$. So we have to show that the abelianization K^{ab} of K is a torsion group. Note that the commutant $[K, K] \subset K$ is a normal subgroup of π, so we can consider the group $G = \pi/[K, K]$ which is an extension of $\mathbf{Z}/q\mathbf{Z}$ by K^{ab}. The generator of $\mathbf{Z}/q\mathbf{Z}$ acts by conjugation as some automorphism σ of K^{ab}. Clearly $[G, G] = (\sigma - 1)K^{ab} \subset K^{ab}$. Therefore, $G^{ab} \simeq \pi^{ab} \simeq \mathbf{Z}/n\mathbf{Z}$ is an extension of $\mathbf{Z}/q\mathbf{Z}$ by $K^{ab}/(\sigma - 1)K^{ab}$. In particular, the order of the latter group divides n/q. Now assume that K^{ab} has nonzero rank r. Let $K^{ab}_{tors} \subset K^{ab}$ be the torsion subgroup; then σ preserves K^{ab}_{tors} and induces the automorphism σ_0 of the quotient $K^{ab}/K^{ab}_{tors} \simeq \mathbf{Z}^r$. The cokernel of $\sigma - 1$ surjects onto the cokernel of $\sigma_0 - 1$; therefore, $\mathbf{Z}^r/(\sigma_0 - 1)\mathbf{Z}^r$ is finite of order $\det(\sigma_0 - 1)$ prime to p. But this is impossible because $\sigma_0^q = 1$. Indeed, let $P(t) = \det(t - \sigma_0)$ be the characteristic polynomial of σ_0, $Q(t)$ be its minimal polynomial. Then $Q|P$ and $Q|((t^q - 1)/(t - 1))$. The latter implies that $Q(1)$ is divisible by p, hence, $P(1) = \det(1 - \sigma_0)$ is divisible by p, which is a contradiction. $\square$

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