FIELD GUIDE TO A_∞ SIGN CONVENTIONS

1. Signs in the definition of A_∞-algebra

We are concerned with the following "standard" definition. An A_∞-*algebra* is a $\mathbb{Z}$-graded vector space A equipped with linear maps m_n of degree $2-n$, for $n \geq 1$, satsifying for each n the identity

$$\sum_{k+l=n+1} \sum_{k=1}^{k} \epsilon \cdot (-1)^{l(\widetilde{a_1}+\ldots+\widetilde{a_{j-1}})} \cdot m_k(a_1, \ldots, a_{j-1}, m_l(a_j, \ldots, a_{j+l-1}), a_{j+l}, \ldots, a_n),$$

where $\widetilde{a}$ is the degree of a homogeneous element $a \in A$, $\epsilon = \epsilon(k,l,j)$ is a certain sign. The second sign in the definition is obtained from the Koszul sign rule, since m_l, that has degree $2-l$, is exchanged with the elements $a_1, \ldots, a_{j-1}$. Note that Seidel and Fukaya use another format of the definition where the Koszul sign is not inserted (see below).

The problem is that the sign $\epsilon(k,l,j)$ in different sources is written differently. The works [2] and [4] use

$$\epsilon_1(k,l,j) = (-1)^{r+st},$$

where $r = j-1$, $s = l$ and $t = n-j-l+1$. The original work where A_∞-algebra was introduced, [10], uses

$$\epsilon_2(k,l,j) = (-1)^{j(l+1)+ln}.$$

[6] and [3] (and following them, [7]) use

$$\epsilon_3(k,l,j) = (-1)^{(j-1)(l-1)+(k-1)l} = (-1)^{j(l+1)+kl+1}.$$

[1] (and following it, [8]) uses

$$\epsilon_4(k,l,j) = (-1)^{j(l+1)}.$$

As one can easily check, the connection between the first three signs is the following:

$$\epsilon_1 = \epsilon_3 = -\epsilon_2.$$

In particular, these signs all give the same definition of an A_∞-algebra. This definition corresponds to associating to (m_n) a coderivation D of the free coalgebra $T(A[1])$ cogenerated by $A[1]$, and setting $D^2 = 0$.

The relation with ϵ_4 is more complicated:

$$\epsilon_4 = (-1)^{\binom{n+1}{2}+1+\binom{k}{2}+\binom{l}{2}} \epsilon_1.$$

This means that in order to match the ϵ_4-definition of an A_∞-algebra with the ϵ_1-definition one has to change the m_n as follows:

$$m'_n = (-1)^{\binom{n}{2}} m_n.$$

Seidel uses very different sign conventions. In [9] the A_∞-identity looks as follows:

$$\sum_{k+l=n+1}\sum_{k=1}^{k}(-1)^{\widetilde{a_1}+\ldots+\widetilde{a_{j-1}}+j-1}\cdot m_k(a_1,\ldots,a_{j-1},m_l(a_j,\ldots,a_{j+l-1}),a_{j+l},\ldots,a_n),$$

which for example does not correspond to the usual associativity if there is only m_2. To connect this to the definition corresponding to ϵ_1 one has to make the following change of sign:

$$m'_n(a_1,\ldots,a_n)=(-1)^{(n-1)\widetilde{a_1}+(n-2)\widetilde{a_2}+\ldots+\widetilde{a_{n-1}}}m_n(a_1,\ldots,a_n).$$

2. Sign in the definition of the Gerstenhaber bracket

Interpreting a Hochschild cochain $f\in\mathrm{Hom}(A^{\otimes n},A)$ as giving rise to a coderivation D_f of the free coalgebra $T(A[1])$, we obtain the definition of Gerstenhaber bracket $[f,g]$ of Hochschild cochains, so that

$$D_{[f,g]}=[D_f,D_g],$$

where on the right we have a supercommutator. This leads to the following formula for $f\in\mathrm{Hom}(A^{\otimes m},A)$, $g\in\mathrm{Hom}(A^{\otimes n},A)$

$$[f,g]=f\bar{\circ}g-(-1)^{|f||g|}g\bar{\circ}f,\quad\text{where}\quad f\bar{\circ}g(a_1,\ldots,a_{m+n-1})=$$

$$\sum_{i=1}^{m}(-1)^{(\widetilde{a_1}+\ldots+\widetilde{a_{i-1}}+m-1)\deg(g)+(i-1)(n-1)}f(a_1,\ldots,a_{i-1},g(a_i,\ldots,a_{i+n-1}),a_{i+n},\ldots,a_{m+n-1}),$$

where for a cochain $f\in\mathrm{Hom}(A^{\otimes m},A)$, homogeneous of degree $\deg(f)$, we set $|f|=\deg(f)+m-1$. Note that in the case when A sits in degree 0 this is compatible with the usual definition, say, in [5, E.1.5.2].

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