Ex. Configurations of linear subspaces

$W_1, \dots, W_m \subset V$, parametrized by $X = \prod_{i=1}^{m} G(r_i, V)$

where $r_i = \dim W_i$

$SL(V) \curvearrowright X$, $\mathcal{O}(\kappa_1, \dots, \kappa_m)$ on X has $SL(V)$-equiv. str-re

$\mathcal{O}(1)$ on $G(r, V)$ corresp. to Plücker embedding

$G(r, V) \hookrightarrow \mathbb{P}(\Lambda^r V)$, so fiber of $\mathcal{O}(1)$ at W

$W \longmapsto \Lambda^r W \subset \Lambda^r V$ is $\Lambda^r W^*$.

Assume $\kappa_i > 0$ $\forall i$.

Thm. $(W_1, \dots, W_m)$ is $\mathcal{O}(\kappa_1, \dots, \kappa_m)$-semist. (stable) $\iff$

$$\forall\, 0 \neq W \underset{\neq}{\subset} V: \quad \frac{\sum \kappa_i \cdot \dim W_i \cap W}{\dim W} \underset{(<)}{\leq} \frac{\sum \kappa_i \dim W_i}{\dim V}$$

Ex. All $\kappa_i = 1$, $\dim W_i = r$ $\quad \frac{1}{m} \sum \frac{\dim W_i \cap W}{\dim W} \leq \frac{r}{\dim V}$

In terms of $Q_i = W/_{W_i \cap W} = \mathrm{im}(W \to V/_{W_i})$: $\quad \frac{1}{m} \sum \frac{\dim Q_i}{\dim W} \geqslant \frac{q}{\dim V}$, $q = \dim V - r = \dim(V/_{W_i})$

Pf: Apply HM-criterion. $e_1, \dots, e_n$ basis of V

$\nu(\lambda) = \mathrm{diag}(\lambda^{q_1}, \dots, \lambda^{q_n})$, $q_1 \geqslant \dots \geqslant q_n$, $\Sigma q_i = 0$

$V_{\leq j} = \langle e_1, \dots, e_j \rangle$ look at jumps in $0 \leq \dim(W_i \cap V_{\leq 1}) \leq \dim(W_i \cap V_{\leq 2}) \leq \dots$

$\rightsquigarrow 1 \leq \nu_1^{(i)} < \dots < \nu_{r_i}^{(i)} \leq n$, $\dim W_i = r_i$

$\dim W_i \cap V_{\leq \nu_j^{(i)}} = j$, $\dim W_i \cap V_{\leq \nu_j^{(i)} - 1} = j-1$

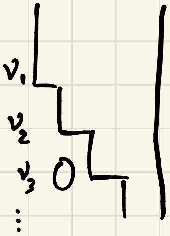
Pl. coord-s $P_I(W_i) = \det(W_i \to V \to V_I)$

$|I| = r_i \qquad \mathrm{wt}_\nu(P_{j_1 \cdots j_{r_i}}) = q_{j_1} + \dots + q_{j_{r_i}}$

$$W = \mathrm{span}\begin{bmatrix} \vdots & \vdots & \\ a_{\nu_1, 1} \neq 0 & \vdots & \\ 0 & a_{\nu_2, 2} \neq 0 & \\ \vdots & 0 & \cdots \\ 0 & \vdots & \end{bmatrix}$$

Note: $P_{\nu_1^{(i)}, \dots, \nu_{r_i}^{(i)}}(W_i) \neq 0$

Claim: $P_{j_1 < \dots < j_{r_i}}(W_i) = 0$ if $j_s > \nu_s^{(i)}$ for some s



$j_1 > \nu_1 \Rightarrow$ 1st col. (minor) = 0 zero $r_i \times 1$ -subm.

$j_2 > \nu_2 \Rightarrow$ 1st two col: $\begin{matrix} * & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix}$ zero $(r_i-1)\times 2$-subm. $\Rightarrow \det = 0$

$j_3 > \nu_3 \Rightarrow$ 1st three col.: $\begin{matrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{matrix}$ zero $(r_i-2)\times 3$ -subm. $\Rightarrow \det=0$

Thus $p_{\nu_1^{(i)}\cdots\nu_{r_i}^{(i)}}(W_i)$ is the $\neq 0$ coord. of min. ν-weight

so $$-\mu_\nu(W_1,\dots,W_m) = \sum_{i=1}^{m} k_i\left(\sum_{j=1}^{r_i} q_{\nu_j^{(i)}}\right)$$

$\dim(W_i\cap V_{\le s}) - \dim(W_i \cap V_{\le s-1}) = 0$ unless $s = \nu_j^{(i)}$ for some j

so $$-\mu = \sum_i k_i \left(\sum_{j=1}^{n} q_j \left(\dim(W_i\cap V_{\le j}) - \dim(W_i\cap V_{\le j-1})\right)\right) =$$

$$= \sum_i k_i\left[r_i q_n + \sum_{j=1}^{n-1}(q_j-q_{j+1})\dim(W_i\cap V_{\le j})\right]$$

$$= q_n\cdot\sum k_i r_i + \sum_{j=1}^{n-1}(q_j-q_{j+1})\cdot\sum_{i=1}^{m} k_i\cdot\dim(W_i\cap V_{\le j})$$

ν = pos. lin. comb. of ν_s, $1\le s\le n$: $q_1=\dots=q_s=n-s$, $q_{s+1}=\dots=q_n=-s$

for $\nu=\nu_s$ get

$$-\mu = -s\cdot\sum k_i r_i + n\sum_{i=1}^{m} k_i \dim(W_i\cap V_{\le s}) \le 0$$

$$\Leftrightarrow \quad \frac{\sum k_i \dim(W_i\cap V_{\le s})}{\underset{=\dim V_{\le s}}{s}} \le \frac{\sum k_i r_i}{n}$$

<u>Cor</u>. If $\gcd(\sum k_i r_i, n) = 1$ then semistable = stable

Stable / semistable bundles on curves.

C smooth proj. curve

$\forall$ coh. sheaf $\mathcal{F}$, $T(\mathcal{F}) \subset \mathcal{F}$ torsion subsheaf

$\mathcal{F}/_{T(\mathcal{F})}$ torsion free $\Rightarrow$ loc. free, i.e. vec. bundle

The sequence $0 \to T(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}/_{T(\mathcal{F})} \to 0$ splits

Riem.-Roch: $\underset{\overset{\|}{h^0(V)-h^1(V)}}{\chi(V)} = \deg(V) - \mathrm{rk}\,V\cdot(g-1)$

$\boxed{\mu = \frac{\deg}{\mathrm{rk}}}$

where $\deg(V) = \deg(\underset{\text{line bundle}}{\Lambda^{top} V})$

Note: $\deg(V\otimes \underset{\text{line bundle}}{L}) = \deg V + \mathrm{rk}\,V\cdot \deg L$

ω_C can. line bundle

$H^1(V) \simeq H^0(V^\vee \otimes \omega_C)^\vee$ - Serre duality

$\mathcal{F} \hookrightarrow V$ subsheaf of a vec. bundle $\Rightarrow$

$\mathcal{F}$ loc. free but not nec. subbundle

$\mathcal{F} \overset{\text{saturation}}{\hookrightarrow} \overline{\mathcal{F}} = \ker\left(V \to {(V/\mathcal{F})}/_{T(V/\mathcal{F})}\right) \hookrightarrow \underset{\text{subbundle}}{V}$

s.t. $\overline{\mathcal{F}}/_{\mathcal{F}}$ torsion

Lemma. $\forall$ $\overset{\neq 0}{\text{vec.}}$ bundle V $\exists$ $\overset{\text{line}}{\text{subbundle}}$ $L \hookrightarrow V$.

Pf: Let L be ample l.b. Then $H^0(V\otimes L^n)$ generates $V \otimes L^n$

$\Rightarrow \exists\ \mathcal{O} \xrightarrow{\neq 0} V\otimes L^n \Rightarrow$ injection $L^{-n} \hookrightarrow V$

$\Rightarrow$ saturation of the image is a line subbundle.

Thus, any vec. bundle is an iterated ext-n of line bundles.

Lem. Suppose V vec. bundle s.t. $H^0(V)$ generates V generically. Then $h^0(V) \leq \deg V + \operatorname{rk} V$.

Pf: induction on rk.

Let $\operatorname{rk} V = 1$, $0 \to \mathcal{O} \to V \to T \to 0$, T torsion, $h^0(T) = \ell(T) = \deg V$

$\Rightarrow h^0(V) \leq 1 + \deg V$

$\operatorname{rk} V > 1$. Take $s \in H^0(V) \neq 0$,

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[rr, bend right, "s"'] & L \arrow[r, hook] & V \arrow[r] & V/L \end{tikzcd}$$

vec. bun.

$H^0(V) \otimes \mathcal{O} \to V$ gener. surj. $\Rightarrow H^0(V/L) \otimes \mathcal{O} \to V/L$ gener. surj.

$\Rightarrow h^0(V) \leq h^0(L) + h^0(V/L) \leq (\deg L + 1) + (\deg V/L + \operatorname{rk} V/L)$

Def. V is stable (semist.) if $\forall$ subbundle $0 \neq \mathcal{F} \subsetneq V$

$$\mu(\mathcal{F}) < \mu(V) \quad (\leq)$$

$\Leftrightarrow \forall$ quot. bundle $V \underset{\neq 0}{\twoheadrightarrow} G \neq 0$: $\mu(G) > \mu(V)$ $(\geq)$

(since deg & rk additive)

Lem. 1) V_1, V_2 semistable, $\mu(V_1) > \mu(V_2) \Rightarrow \operatorname{Hom}(V_1, V_2) = 0$

2) V_1, V_2 stable, $\mu(V_1) = \mu(V_2) \Rightarrow \underset{0}{\forall}{}^{\!\!*} f: V_1 \to V_2$ is an isom-sm

Pf: Let $\underset{\neq 0}{f}: V_1 \to V_2$. Consider $\operatorname{im}(f) = \mathcal{F} \subset V_2$

V_1, V_2 semist. $\Rightarrow \mu(\mathcal{F}) \geq \mu(V_1) > \mu(V_2)$

$\mu(\mathcal{F}) \leq \mu(V_2)$ contrad.

Now V_1, V_2 stable, $\mu(V_1) = \mu(V_2)$.

$\mathcal{F} \neq 0 \Rightarrow \mu(\mathcal{F}) < \mu(V_2)$ (consider case $\operatorname{rk}\mathcal{F} = \operatorname{rk} V_2$ separately)

also $\mu(\mathcal{F}) \geq \mu(V_1)$ contradiction.

Rem. $(\deg V, \operatorname{rk} V) = 1 \Rightarrow$ (stability $\Leftrightarrow$ semistability)

Cor. (over $\mathbb{C}$) V stable $\Rightarrow$ $\mathrm{Hom}(V,V)=\mathbb{C}$.

$f: V \to V$, let λ = eigenvalue of $f|_{V_p}$

then $f-\lambda\cdot id$ not isom. $\Rightarrow$ $f-\lambda\cdot id = 0$

Cor. V semist., $\deg V > (2g-2)\,\mathrm{rk}\,V \Rightarrow H^1(V)=0$

Pf: By Serre duality, need $\mathrm{Hom}(V,\omega)=0$

But $\mu(V) > 2g-2 = \mu(\omega)$.

Cor. V semist., $\deg V > (2g-1)\,\mathrm{rk}\,V \Longrightarrow$ V gener. by global sect-s.

Pf: consider long ex. seq. of coh. ass. with

$$0 \to V(-p) \to V \to V_p \to 0 \qquad \forall\, p \in C$$

need $H^1(V(-p))=0$, $\deg V(-p) = \deg V - \mathrm{rk}\,V > (2g-2)\,\mathrm{rk}\,V$

Ex. $\mathrm{rk}\,V = 2$ semistability $\Leftrightarrow$ $\forall$ line subbundle $L \subset V$ $\deg L \le \frac{\deg V}{2}$

Ex. V stable/ss $\Rightarrow$ so is $L \otimes V$, L line bundle.

Ex dual of stable bundle is stable

Ex. $0 \to \mathcal{O} \to V \to L \to 0$, $\deg L = 1$ $\Rightarrow$ V stable.
nontrivial ext-n

$\nwarrow$ M destabilizing

$\deg M \geq 1 \Rightarrow \mathrm{Hom}(M,\mathcal{O})=0 \Rightarrow M \hookrightarrow L$ isom-sm $\Rightarrow$ splitting.

Ex. $g=1$, $\deg V=1$, $\mathrm{rk}\,V=2$, V stable

$\Rightarrow h^1(V)=0$, $h^0(V)=1$, $\mathcal{O} \hookrightarrow V$ cannot factor thru $L \hookrightarrow V$ of $\deg > 0$

$\Rightarrow 0 \to \mathcal{O} \to V \to L \to 0$ $h^1(L^{-1})=1$ $\Rightarrow$ unique such ext-n.

Similarly, one can classify

all stable bundles on elliptic curve (Atiyah),
then all bundles.

Ex. $g=2$, $rk=2$, consider ext-ns $0\to\mathcal{O}\to V\to L\to 0$

$\leftrightarrow \mathbb{P}\mathrm{Ext}^1(L,\mathcal{O}) \simeq \mathbb{P}H^1(L^{-1}) \simeq \mathbb{P}H^0(L\otimes\omega)^* \simeq \mathbb{P}^3$ $\quad \deg L = 3$, $h^0(L\omega) = 4$

$C \underset{\phi}{\overset{|L\omega|}{\hookrightarrow}} \mathbb{P}H^0(L\otimes\omega)$: $p \mapsto \{L\omega|_p^* \hookrightarrow H^0(L\omega)^*\}$

$$\begin{tikzcd}[column sep=small] = \ker\big(H^0(L\omega)^* \arrow[r] \arrow[d, phantom, "\wr"] & H^0(L\omega(-p))^*\big) \arrow[d, phantom, "\wr"] \\ \mathrm{Ext}^1(L,\omega) \arrow[r] & \mathrm{Ext}^1(L(-p)) \end{tikzcd}$$

so $\phi(C) \leftrightarrow$ extensions

$$\begin{tikzcd} \mathcal{O} \arrow[r] & V \arrow[r] & L \\ & & L(-p) \arrow[ul, dotted] \arrow[u, hook] \end{tikzcd}$$

that split over $L(-p)$ for some p

V unstable $\iff \exists\, M \hookrightarrow V$ $\deg M \geq 2$ $\iff$ $V \to L$ splits over $L(-p)$

so all V corr. to $\mathbb{P}^3 - C$ are stable

All remaining $^{rk=2}$ stable bundles w. $\Lambda^2 V \simeq L$ are ext.s $\mathcal{O}(p)\to V\to L(-p)$

Lemma. E vector bundle, $V \subset H^0(E)$ subspace s.t. $V\otimes\mathcal{O}\to E$ gener. surjective.

$$\Rightarrow \#\{p\in C \mid \mathrm{coker}(V\to E|_p)\neq 0\} \leq \deg E$$

Proof: induction on $rk\,E$.

$rk\,E = 1$, pick $s\in V-0$, $0\to\mathcal{O}\xrightarrow{s}E\to T\to 0$, T torsion

our number $\leq \#\,\mathrm{supp}(T) \leq \ell(T) = \deg E$.

$rk\,E > 1$. Pick $s\in V-0$,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & E \arrow[r] & E/L \arrow[r] & 0 \\ & \mathcal{O} \arrow[u] \arrow[ur, "s"] \arrow[r] & V\cdot\mathcal{O} \arrow[u] & & \end{tikzcd}$$

By induction assumption, $S = \{p\in C : \mathrm{coker}(V\to E/L|_p)\neq 0\}$ has card $\leq \deg E/L$

$\Rightarrow$ away from $\mathrm{supp}(L/\mathcal{O})$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & V\cdot\mathcal{O} \arrow[r] \arrow[d] & V/\langle s\rangle\cdot\mathcal{O} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & E \arrow[r] & E/L \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ middle arrow is surjective

Relation to semistability.

Given E vec. bundle generated by global sections, $p_1, \dots, p_N \in C$

$$E \longmapsto \left(H^0(E) \to E|_{p_1}, \dots, H^0(E) \to E|_{p_N}\right)$$

point of $Gr(r, H^0(E)^*)^N$.

Let $r = \mathrm{rk}\, E$, $d = \deg E$

Theorem. Assume $d > r(2g-2) + r(r-1)g$

Then $\exists\, N_0 = N_0(d, r, g)$ s.t. for $N \geqslant N_0$ the point in Gr^N associated with E is GIT-semistable (stable) $\iff$ E is semistable (stable

w.r.to $L = \mathcal{O}(1, \dots, 1)$ on Gr^N

Lemma. Assume E is stable, d as above

$\forall$ subbundle $0 \neq F \subsetneq E$ s.t. $H^0(F)\mathcal{O} \to F$ is generically surjective, one has

$$\frac{h^0(F)}{\mathrm{rk}\, F} < \frac{h^0(E)}{\mathrm{rk}\, E}.$$

Pf: know: $h^1(E) = 0$, so $h^0(E) = \deg E - \mathrm{rk}\, E\, (g-1)$

if $h^1(F) = 0$ then $h^0(F) = \deg F - \mathrm{rk}\, F\, (g-1)$ $\Rightarrow$ get required inequality from $\mu(F) < \mu(E)$.

Assume $H^1(F) \neq 0 \Rightarrow \exists \neq 0$ map $\phi: F \to \omega$

$\Rightarrow$ saturation of $\ker(\phi)$ is a subbundle $F' \subset F$ of $\mathrm{rk} = \mathrm{rk}\, F - 1$ & degree $= \deg F - (2g-2)$

stability $\Rightarrow$ $r \cdot (\deg F - (2g-2)) \leq (\mathrm{rk}\, F - 1) \cdot d$

(also true if $\mathrm{rk}\, F = 1$ & $F' = 0$)

so $\deg F \le 2g-2 + (\mathrm{rk} F - 1)\frac{d}{r}$

Also know: $h^0(F) \le \deg F + \mathrm{rk} F$

$$\Rightarrow \frac{h^0(F)}{\mathrm{rk} F} \le \frac{d}{r} + 1 + \frac{(2g-2) - \frac{d}{r}}{\mathrm{rk} F}$$

while $\frac{h^0(E)}{\mathrm{rk} E} = \frac{d}{r} - (g-1)$

So need $1 + \frac{2g-2-\frac{d}{r}}{\mathrm{rk} F} < -(g-1)$

$$\frac{d}{r} - (2g-2) > g\,\mathrm{rk} F \Longleftarrow \frac{d}{r} > 2g-2 + (r-1)g$$

<u>Proof of Thm:</u> $(\Leftarrow)$

only give proof for stable E.

Set $V = H^0(E)$, $n := \dim V$ $(r = \mathrm{rk} E)$

Given $0 \ne W \subsetneq V$, let $W_j = \mathrm{im}(W \to E|_{p_j})$

<u>Want to prove:</u> $\frac{1}{N}\sum_{j=1}^{N} \frac{\dim W_j}{\dim W} \geqslant \frac{r}{n}$

Consider composed map $W \otimes \mathcal{O} \hookrightarrow V \otimes \mathcal{O} \to E$

Let $F \subset E$ be saturation of the image so

$0 \ne F$ subbundle & $W \otimes \mathcal{O} \to F$ generically surjective

Assume first $\mathrm{rk} F < r$.

Then By Lemma, $\frac{h^0(F)}{\mathrm{rk} F} < \frac{n}{r}$

$$\Rightarrow \frac{\mathrm{rk} F}{h^0(F)} > \frac{r}{n}$$

$$\Rightarrow \frac{\mathrm{rk} F}{h^0(F)} - \frac{r}{n} \geqslant \frac{1}{n^2}$$

By another Lemma, $\#\{p \mid W \not\twoheadrightarrow F|_p\} \leq \deg F$

(use the fact that $W \otimes \mathcal{O} \to F$ is gener. surj.)

$\Rightarrow$ for $\geq N - \deg F$ indices j : $W_j = F|_{p_j}$, so $\dim W_j = \operatorname{rk} F$

$$\Rightarrow \frac{1}{N} \sum \frac{\dim W_j}{\dim W} - \frac{r}{n} \geq$$

$$\frac{1}{N} \sum \left(\frac{\dim W_j}{\dim W} - \frac{r}{n} \right) \geq \frac{1}{N} \left((N - \deg F)\left(\frac{\operatorname{rk} W}{\dim W} - \frac{r}{n} \right) - \deg F \cdot \frac{r}{n} \right)$$

$$\geq \frac{1}{N} \cdot \left(\frac{N - \deg F}{n^2} - \deg F \cdot \frac{r}{n} \right) = \frac{N - \deg F (1 + rn)}{N \; n^2}$$

enough to have

$$N > \deg F \cdot (1 + rn)$$

By stability $\deg F < \operatorname{rk} F \cdot \frac{d}{r} \leq (r-1) \frac{d}{r}$.

So enough to have $N \geq (r-1)\frac{d}{r} + (r-1) d n$

Now assume $\operatorname{rk} F = r$, i.e. $F = E$

Then for $\geq N - d$ indices j: $W_j = E|_{p_j}$ so $\dim W_j = r$

so $\frac{1}{N} \sum \left(\frac{\dim W_j}{\dim W} - \frac{r}{n} \right) \geq \frac{1}{N} \left((N-d)\left(\frac{r}{n-1} - \frac{r}{n} \right) - d \cdot \frac{r}{n} \right) =$

$= \frac{r}{Nn} \frac{N - nd}{n-1}$, so enough to take

$$N > nd$$

$\Rightarrow$) Assume E is unstable.

Let $F \subset E$ be a subbundle of min. rk s.t. $\mu(F) > \mu(E)$

$\Rightarrow$ F is stable.

$\mu(F) > \mu(E) \geqslant 2g-1 \Rightarrow H^0(F) \otimes \mathcal{O} \twoheadrightarrow F$, $h^1(F) = 0$

$\Rightarrow$ For $W := H^0(F) \subset V = H^0(E)$, $\dim W_j = \mathrm{rk}\, F$

Claim: $\dfrac{\mathrm{rk}\, F}{h^0(F)} < \dfrac{r}{h^0(E)}$ so the point in Gr^N is unstable

$$\Updownarrow$$

$$\frac{h^0(F)}{\mathrm{rk}\, F} > \frac{h^0(E)}{\mathrm{rk}\, E}$$

$\Updownarrow$ R.R since $h^1(F) = h^1(E) = 0$

$$\frac{\deg F}{\mathrm{rk}\, F} > \frac{\deg E}{\mathrm{rk}\, E}$$

holds by assumption.

Harder - Narasimhan filtration.

$\forall$ vec. bun. E $\exists !\ 0 \subsetneq F_1 \subsetneq F_2 \subset \dots \subset F_n = E$ s.t. F_i/F_{i-1} ss. of sl. μ_i, $\mu_1 > \mu_2 > \dots > \mu_n$

Pf: $\exists$ max. subb. $\overset{F_1}{\text{of}}$ max. slope, induction on dim. (replace E by E/F_1)

=> can define

$$\boxed{\mu_1 =: \phi_+(E) \geq \mu_n =: \phi_-(E)}$$

Lem. $\phi_-(E) > \phi_+(F) \Rightarrow \mathrm{Hom}(E,F)$

Pf: use filtrations

Lem. $V \otimes \mathcal{O} \twoheadrightarrow E \Rightarrow 0 \leq \phi_-(E) \leq \phi_+(E) \leq \deg E$

Pf: $V\otimes\mathcal{O} \twoheadrightarrow E \twoheadrightarrow F_n/F_{n-1} \Rightarrow \mu_n \geq 0$

$r_i = \mathrm{rk}(F_i/F_{i-1})$, $r_1\mu_1 + \dots + r_n\mu_n = \deg E$

$\Rightarrow r_1\mu_1 \leq \deg E \Rightarrow \mu_1 \leq \deg E$

Recovering $V\otimes\mathcal{O} \to E$ from $(V \to E|_{p_i})_{i=1,\dots,N}$

Consider

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[rrd, "f"] & V\otimes\mathcal{O} \arrow[r] \arrow[d, equal] & E \arrow[r] & 0 \\ & & V\otimes\mathcal{O} \arrow[r] & E' & \end{tikzcd}$$

$f: K \to E'$ vanishes at $p_1, \dots, p_N$ => factors thru $K \to E'(-p_1 - \dots - p_N)$

By Lemma applied to $V^{*}\cdot\mathcal{O} \to K^{\vee}$,

$$\phi_-(K) \geq \deg K = -\deg E.$$

Similarly,

$$\phi_+(E'(-p_1 - \dots - p_N)) = \phi_+(E') - N \leq \deg E'$$

so if $-\deg E > \deg(E') - N$, i.e., $N > 2\deg E$

then $\mathrm{Hom}(K, E'(-p_1 - \dots - p_N)) = 0$

$$\begin{tikzcd} & V\otimes\mathcal{O} \arrow[r] \arrow[d, equal] & E \arrow[d, dashed] \\ \Rightarrow & V\otimes\mathcal{O} \arrow[r] & E' \end{tikzcd}$$

$\Rightarrow$ same quotients.

__Quot scheme__: quotients of $V\otimes\mathcal{O}_C$ of given Hilbert pol. h parametrized by a projective scheme $\mathrm{Quot}_h(V\otimes\mathcal{O})$

More precisely, it represents the functor

$$S \mapsto \left\{ V\otimes\mathcal{O}_{S\times C} \twoheadrightarrow \mathcal{F} \text{ - flat}/S,\ \mathcal{F}_s \text{ has H.p. } h \right\}$$

Idea of constr: closed subsch. of Grassmannian

we are interested in the open subset of $V\otimes\mathcal{O} \twoheadrightarrow Q$ s.t. Q loc. free, $H^1(Q)=0$, $V \xrightarrow{\sim} H^0(Q)$

why open?

__Lem.__ Given $\begin{matrix}\mathcal{F}\\ \downarrow \\ S\times X\end{matrix}$, flat$/S$, 1) $\mathcal{F}|_{s\times X}$ loc. free at x $\Rightarrow \mathcal{F}$ loc. free at (s,x)

2) X proper, $\mathcal{F}|_{s\times X}$ loc. free $\Rightarrow \exists\, U \ni s$: $\mathcal{F}|_{U\times X}$ loc. free

__Pf__: 1) $\underset{\substack{\cup \\ m}}{A} \to B$ local hom-sm of loc. Noeth. rings, M f.g. B-mod., flat$/A$, M/mM free over B/mB $\Rightarrow$ M free$/B$

Consider $f: B^n \to M$ s.t. $f \otimes_A A/m$ is an isom. $\Rightarrow f$ surj.

$0 \to K \to B^n \to M \to 0 \rightsquigarrow 0 = \mathrm{Tor}_1^A(M, A/m) \to K \otimes_A A/m \to 0$ $\Rightarrow K = 0$

2) $\exists$ fin. $U_i \subset S\times X$ s.t. $\mathcal{F}|_{U_i}$ loc. free & $s\times X \subset \cup U_i$

$U =$ complement to $\cap\, p_1(S\times X - U_i)$

$0 \to K \to V \otimes \mathcal{O}_{S\times C} \to \mathcal{F} \to 0$, K, $\mathcal{F}$ flat$/S$

By semicontinuity, $\overset{V \hookrightarrow H^0(\mathcal{F}_s) \Leftarrow}{H^0(K_s) = 0}$, $H^1(\mathcal{F}_s) = 0$ open conditions

Once $H^1(\mathcal{F}_s) = 0$, $p_{1*}\mathcal{F}$ is a vector bundle & its fiber at $s \simeq H^0(\mathcal{F}_s)$

$V \to H^0(\mathcal{F}_s)$ surj $\Rightarrow$ $V \otimes \mathcal{O}_S \to p_{1*}\mathcal{F}$ surj. near s

$\Rightarrow V \to H^0(\mathcal{F}_s)$ surj. near s.

What is achieved using GIT ?

Moduli (contravariant) functors: $S \mapsto \left\{ \begin{array}{l} \text{v. bun. of given } rk = r, deg = d \text{ on } S \times C \\ \text{s.t. } V_s \text{ semist. (st.)} \end{array} \right\} / V' \sim V \otimes p_1^* L$

$B \lesssim \begin{array}{c} Bun^{ss}_{r,d} \\ (Bun^{s}_{r,d}) \end{array}$ $S' \to S$ can form pullback

Fine mod. space : scheme representing moduli functor B

coarse mod. space : scheme X + nat. transform.

$$B(S) \to Mor(S, X)$$

s.t. $B(pt) \to X$ is bijection

Thm. $\forall (r,d)$ $\exists$ course mod. space $\mathcal{U}^s(r,d) \leftarrow$ qproj. var-ty for $Bun^s_{r,d}$

if $gcd(r,d) = 1$ then it is a fine mod. space

Idea of proof : 0) $\mathcal{U}^s(r,d) :=$ GIT quot.
1) can assume $\frac{d}{r} \gg 0$ by $V \mapsto V \otimes L$

2) Let $\begin{array}{c} V \\ \downarrow \\ S \times C \end{array}$ be family of vec. bundles, $p_{1*}V$ is a vector bundle$/S$ fiber at s : $H^0(V_s)$

Let $\tilde{S} \to S$ be a GL_n-torsor of bases in $H^0(V_s)$

Let $\begin{matrix}\tilde V\\ \downarrow\\ \tilde S\times C\end{matrix}$ be induced family. Then have trivializ-n of $H^0(\tilde V_s)$

$\rightsquigarrow \mathcal{O}^n \twoheadrightarrow \tilde V \rightsquigarrow$ get a morphism

$\tilde S \longrightarrow \mathrm{Quot}(\mathcal{O}^n)$ landing in our open locus

compose with map to $Gr^N \rightsquigarrow$ get

$\tilde S \longrightarrow (Gr^N)^s$ compatible with GL_n-action

$$\begin{tikzcd} \tilde S \arrow[r, bend left, "GL_n\text{-invariant}"] \arrow[r] \arrow[dr, "GL_n"'] & (Gr^N)^s /\!/_{L} PGL_n \\ & S \arrow[u, dashed] \end{tikzcd}$$

Consider the above; this gives nat. transform-n

Moduli of curves.

1. Hilbert scheme : Fix $n \geq 1$.

consider functor

$$\underset{\text{loc. Noeth.}}{S} \longmapsto \left\{ \begin{tikzcd} Y \arrow[r, phantom, "\overset{\text{closed}}{\subset}"] \arrow[dr, "\text{flat}"'] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd} \right\}$$

Grothendieck : representable by $\mathrm{Hilb}_{\mathbb{P}^n} = \bigsqcup \mathrm{Hilb}^h_{\mathbb{P}^n}$

h = Hilbert polyn. of $Y \subset \mathbb{P}^n$: $h(i) = \chi(Y, \mathcal{O}(i)|_Y)$

$\mathrm{Hilb}^h_{\mathbb{P}^n}$ is a projective scheme

2. Curves in multicanonical embedding.

C smooth (conn.) proj. curve, $g \geq 2$.

$m \geq 3 \Rightarrow i_m : C \xhookrightarrow{|\omega_C^m|} \mathbb{P}^N$

More generally, $i_L : C \xhookrightarrow{|L|} \mathbb{P}^r$, $h^0(L) = r+1$

$\deg L >> 0$

$$h_{i_L(C)}(i) = \chi(L^i) = i\cdot\deg L - g + 1$$

$C \hookrightarrow \mathbb{P}^r \rightsquigarrow$

point of $\mathrm{Hilb}_{\mathbb{P}^r} \hookrightarrow \mathrm{Gr}_k H^0(\mathbb{P}^r, \mathcal{O}(m))^*$

$$\mathcal{O}_{\mathbb{P}^r} \twoheadrightarrow \mathcal{O}_C \mapsto H^0(\mathbb{P}^r, \mathcal{O}(m)) \twoheadrightarrow H^0(C, \mathcal{O}(m)|_C)$$

<u>mth Hilbert point.</u>

<u>Def</u>. $C \subset \mathbb{P}^r$

not cont. in $\forall$ hyperplane

$\rightsquigarrow$ reduced degree $:= \frac{\deg C}{r}$

C linearly stable $\overset{\text{def}}{\iff}$ $\forall$ linear proj Γ of C from a lin. subsp. $\subset \mathbb{P}^r$

red. deg. $(C) <$ red. deg (Γ)

<u>caveat</u>: if $\underset{\text{lin. proj.}}{C \to p(L)}$ has degree d,

define Γ to be the cycle $d\cdot[p(L)]$

in other words, $\deg \Gamma = \deg$ (pull-back of $\mathcal{O}(1)$ to C)

<u>Thm</u>. Fix g, d, r, $\exists$ ∞-ly many $m > 0$ s.t.

$\forall$ $\underset{\text{genus } g}{C} \hookrightarrow \mathbb{P}H^0(C,L)^*$, C lin. stable $\Rightarrow$

mth Hilbert point of C is stable.

<u>Prop</u>. $\deg L > 2g$, $g > 0$ $\Rightarrow$

$C \hookrightarrow \mathbb{P}H^0(C,L)^*$

lin. stable

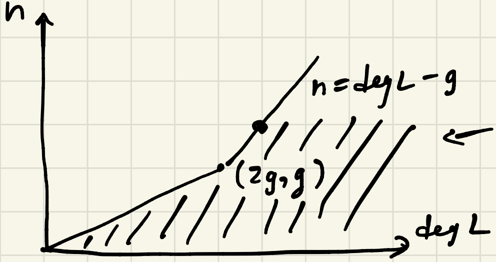
Pf of Prop. Need to know possible pairs $(\deg \underbrace{\varphi^*\mathcal{O}(1)}_{L}, n)$ for $\varphi : C \longrightarrow \mathbb{P}^n$ s.t. $\varphi(C) \not\subset$ hyperplane.

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \overset{\swarrow\!\nearrow}{\subset} H^0(C, L)$$

case 1. $H^1(L) = 0 \Rightarrow n \leq h^0(L) - 1 = \deg L - g$

case 2. $H^1(L) \neq 0 \Rightarrow$ by Clifford's thm

$$n \leq h^0(L) - 1 \leq \frac{\deg L}{2}$$



reduced slope $= \frac{\deg L}{n}$

region of possible pairs $(\deg L, n)$

start with a point $(\deg L, n = \deg L - g)$

lin. projection $\rightsquigarrow (\deg L', n')$

wher $\deg L' \leq \deg L$, $n' < n$

in the allowed range

$$\Rightarrow \frac{\deg L'}{n'} > \frac{\deg L}{n}.$$

Proof of Thm is based on the HM-criterion.

$C \overset{|L|}{\hookrightarrow} \mathbb{P}H^0(C, L)^*$, m th Hilb. pt $\leftrightarrow$

$\mu_m : S^m H^0(L) \twoheadrightarrow H^0(L^m)$ for large m

given a basis (x_i) of $H^0(L)$ & wts $r_i \in \mathbb{Z}$ s.t. $\sum r_i = 0$

$\rightsquigarrow$ wts of monomials in $S^m H^0(L)$

$\rightsquigarrow$ weight filtration $F_{\leq w} S^m H^0(L)$ (depends on (r_i))

Def. Let $s_1, \dots, s_N$ be a basis of $H^0(L^m)$. We say it is of *negative wt.* if $s_i \in \mu_m(F_{\le w_i} S^m H^0(L))$ with $\sum w_i < 0$.

key Lemma. mth Hilb. of $C \xrightarrow{|L|} \mathbb{P}H^0(L)^*$ is stable (assuming μ_m surjective) $\iff$ $\forall$ basis (x_i) of $H^0(L)$ & $\forall$ $\underset{\sum r_i = 0}{(r_i)^{\ne (0)}}$, $\exists$ a basis of negative wt. in $H^0(L^m)$.

Proof. $(x_i), \underset{\sum r_i = 0}{(r_i)} \longleftrightarrow$ 1 P.S. $\nu : \mathbb{G}_m \to SL(H^0(L))$

mth Hilb. pt $\longleftrightarrow$ $\Lambda^N \mu_m : \Lambda^N S^m H^0(L) \to \overset{1 \text{ dim.}}{\Lambda^N H^0(L^m)}$

Set $V = S^m H^0(L)$, $W = H^0(L^m)$. $N = \dim H^0(L^m)$

Given $\mu : \underset{\underset{n \in \mathbb{Z}}{\bigoplus} V_n}{\overset{\shortparallel}{V}} \twoheadrightarrow \underset{\dim W = N}{W}$, we say that $s_1, \dots, s_N$ is a basis of W of negative wt. if $s_i \in \mu(V_{\le w_i})$ with $\sum w_i < 0$

Consider $\Lambda^N \mu : \Lambda^N V \twoheadrightarrow \underset{1 \text{ dim}}{\Lambda^N W}$, where $\Lambda^N V$ has induced $\mathbb{Z}$-grading

Then $\exists$ basis of neg. wt in W $\iff$ $(\Lambda^N V)_{<0} \twoheadrightarrow \Lambda^N W$

Indeed, $wt(s_1 \wedge \dots \wedge s_N) = \sum wt(s_i)$

Given $\Lambda^N V = \tilde{V} = \bigoplus \tilde{V}_n \xrightarrow[\varphi]{} \mathbb{C}$, write $\varphi = \sum \varphi_n$, $\varphi_n \in \tilde{V}_n^*$

Then $\varphi|_{\tilde{V}_{<0}} \ne 0 \iff \exists n < 0 : \varphi_n \ne 0$

$\iff$ stability (by HM-criterion)

Ranks of polynomials. (j. w. Kazhdan, Wang)

Def. $\operatorname{srk}(f) = \min\{r : f \in (\ell_1, \dots, \ell_r)\}$

$= \min\{\operatorname{codim}_V L \mid L \subset (f=0)\}$.

Thm 1. (Kazhdan-P.) after Derksen

f hom. pol. deg $=d$ $/k$ - perfect field $\Rightarrow \operatorname{srk}_k(f) \leq d \cdot \operatorname{srk}_{\bar{k}}(f)$

Ex. $f = x^2 + y^2 / \mathbb{R}$ $\quad \operatorname{srk}_{\mathbb{R}} = 2$, $\operatorname{srk}_{\mathbb{C}} = 1$.

Idea of proof (similar to Derksen for tensors)

Introduce another notion of rk, **G-rank** s.t.

- $\operatorname{srk}_k(f) \leq r^G_k(f) \leq d \cdot \operatorname{srk}_k(f)$ $\quad \forall$ perfect k
- $r^G_k(f) = r^G_{\bar{k}}(f)$

Definition of G-rk: consider $G = GL(V)$ acting on $\underset{f}{\overset{\psi}{S^d V}}$

given $g(t) \in G(k[[t]])$, $\operatorname{val}_t(g(t)\cdot f) \in \mathbb{Z}_{\geq 0}$

Ex. $g = \operatorname{diag}(t, 1, \dots)$, $f = x_1^d$

$\Rightarrow \operatorname{val}_t(g(t)\cdot f) = d$

$$\mu(g(t), f) := d \cdot \frac{\operatorname{val}_t(\det g(t))}{\operatorname{val}_t(g(t)\cdot f)}$$

Ex. $\mu(g(t), x_1^d) = 1$

$$r^G(f) = \inf_{g(t)} \mu(g(t), f)$$

Ex. $r^G(x_1^d) = 1$

$\Big[$ Δ-le inequality: $r^G(f_1 + f_2) \leq r^G(f_1) + r^G(f_2)$ $\Big]$

Ex. $r^G(x_1^2 x_2) = \frac{3}{2}$

Proof of $r^G(f) \leq d \cdot \mathrm{srk}(f)$:

$f = l_1 f_1 + \dots + l_r f_r$ $\quad g(t) = \mathrm{diag}(\underbrace{t, \dots, t}, 1 \dots 1)$

$\mathrm{val}_t(\det g) = r, \quad \mathrm{val}_t(g(t) \cdot f) \geq 1$

Interpretation in terms of GIT-stability:

Lemma

$$r^G(f) \geq \frac{p}{q} \iff w = (f^{\otimes p}; e_1, \dots, e_n) \in W = (S^d V)^{\otimes p} \otimes \det^{-dq} \oplus V^n$$

is G-semistable $\qquad n = \dim V$

Hilbert-Mumford-Kempf criterion:

w unstable $\iff \exists$ 1-par. subgr. $\lambda : \mathbb{G}_m \to G /_k$ s.t.

$\lim_{t \to 0} \lambda(t) \cdot w = 0 \iff \mathrm{val}_t(\lambda(t) \cdot w) > 0$

Pf of Lem $\Rightarrow$) if w not st. find λ s.t. $\mathrm{val}_t(\lambda(t) \cdot w) > 0$

$\overset{''}{(f^{\otimes p}, e_1, \dots, e_n)}$

In particular, $\mathrm{val}_t(\lambda(t) \cdot (e_1, \dots, e_n)) > 0 \implies \lambda(t) \in G(k[t])$

$p\, \mathrm{val}_t(\lambda(t) \cdot f) - dq\, \mathrm{val}_t \det(\lambda) > 0$

$\iff \mu(\lambda, f) < \frac{p}{q}$

Conversely, assume $\exists\, g(t) \in G(k[[t]])$: $\mathrm{val}_t(g(t) \cdot f) > 0$ &

$\mu(g(t), f) < \frac{p}{q}$

$\implies \mathrm{val}_t(g(t) \cdot f^{\otimes p} \otimes 1) > 0$ $\quad$ (can assume $g \in G(k[t])$ truncating)

$\implies \lim_{t \to 0} g(t) \cdot w = (0, \underbrace{g(0) \cdot (e_1, \dots, e_n)}_{u}) \implies (0, u) \in \overline{G \cdot w}$

But $0 \in \overline{G \cdot (0, u)} \implies 0 \in \overline{G \cdot w}$, so w is unstable

Cor. $r_k^G(f) = r_{\bar{k}}^G(f)$, in def. of r^G enough to take

$g(t)$ 1 par. subgr. $/k$ (diag. in some basis)

Remains to prove: $\mathrm{srk}(f) \leq r^G(f)$

Assume $r^G(f) < r \Rightarrow \exists$ 1PS $g(t)$: $r \cdot v_t(g \cdot f) > d \cdot v_t(\det g)$

can assume $g = \mathrm{diag}(t^{a_1}, \dots, t^{a_n})$ for basis $(e_1, \dots, e_n)$, $a_i >$

Let $S := \left\{ i : a_i \geqslant \frac{v_t(g \cdot f)}{d} \right\}$

Then $v_t(\det g) \geqslant \sum_{i \in S} a_i \geqslant |S| \cdot \frac{v_t(g \cdot f)}{d}$

$$\Rightarrow |S| \leqslant \frac{d \cdot v_t(\det g)}{v_t(g \cdot f)} < r$$

<u>Claim</u>: $f \in \langle e_i \mid i \in S \rangle \cdot S^{d-1} V$ $\quad (\Rightarrow \mathrm{srk}(f) < r)$

Indeed, otherwise $\exists$ monomial M in f, s.t. M only involves e_j, $j \notin S$

$$\Rightarrow v_t(g \cdot f) \leq v_t(g \cdot M) < d \cdot \frac{v_t(g \cdot f)}{d}$$

contradiction

<u>About Kempf's</u> thm: let $\overset{\text{unstable}}{v \in V}$, $\overset{\text{1PS}}{\nu : \mathbb{G}_m \to G}$

Notion of length on 1PS, $\|\nu\|$ s.t.
$\|g \nu g^{-1}\| = \|\nu\|$ & $\|\nu\| = \sqrt{(\nu, \nu)}$ for $\nu \hookrightarrow T$

$B(v) := \sup_{\nu} \frac{m(v, \nu)}{\|\nu\|} < \infty$ (can view as continuous f-n on unit sphere of ν)

Key observation:

$$\left\{ \nu : \frac{m(v,\nu)}{\|\nu\|} = B(v) \right\} \cap \underset{\substack{\uparrow \\ \forall \text{ max. torus}}}{T} \text{ has 1 el-t}$$

use: $m(v, \nu) = \min_i \ell_i(\nu)$, $\ell_1, \dots, \ell_N$ linear functions