Torus actions:

$$a: \mathbb{G}_m \times X \to X$$

$\mathbb{G}_m \curvearrowright$ aff. var. X $\quad\leftrightarrow\quad$ $\mathbb{C}[X] \xrightarrow{a^*} \mathbb{C}[X] \otimes \mathbb{C}[T, T^{-1}]$

algebraic action $\qquad a^* f = \sum_{n \in \mathbb{Z}} \pi_n(f) \otimes T^n$

$$f = \sum \pi_n(f) \quad \Leftarrow \quad \lambda^* f = \sum \lambda^n \pi_n(f)$$

$$\mu^* \lambda^* f = \sum \lambda^n \mu^* \pi_n(f)$$

$$= \sum \lambda^n \mu^n \pi_n(f) \qquad \Rightarrow \quad \mu^* \pi_n(f) = \mu^n \pi_n(f)$$

So

$$\mathbb{C}[X] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}[X]_n,$$ compatible with product

So alg. $\mathbb{G}_m$-action $\longleftrightarrow$ $\mathbb{Z}$-grading of the alg. $\mathbb{C}[X]$.

Similarly, $T \simeq \mathbb{G}_m^n$ acts aff. var-ty X

$\leftrightarrow$ $X^\cdot(T)$ - grading of the alg. $\mathbb{C}[X]$ (comp. w. prod.)

$\mathrm{Hom}(T, \mathbb{G}_m) \ni \chi$, $\quad \mathbb{C}[X]_\chi = \{f : t^{-1*} f = \chi(t) f\}$

V vector space, $T \to GL(V)$ alg. repr.

$\leftrightarrow$ $V = \bigoplus_{\chi \in X^\cdot(T)} V_\chi$ $\qquad$ (i.e. diagonalizable)

$\Leftrightarrow$ $V = \bigoplus_{i=1}^{n} \mathbb{C} v_i$, $\quad t v_i = \chi_i(t) v_i$, $\quad \chi_1, \dots, \chi_n \in X^\cdot(T)$

$x_1, \dots, x_n \in V^*$ dual to (v_i), so $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$

$$t \cdot x_1^{\alpha_1} \dots x_n^{\alpha_n} = (\chi_1^{-\alpha_1} \dots \chi_n^{-\alpha_n})(t)\, x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

so $\quad \mathbb{C}[V]^T = \mathrm{span}\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \sum \alpha_i \chi_i = 0\}$

$M = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \mid \sum \alpha_i \chi_i = 0\} < \mathbb{Z}_{\geq 0}^n$ submonoid

$m_1, \dots, m_k$ gen-s of M $\Leftrightarrow$ (x^{m_i}) gen-s of $\mathbb{C}[V]^T$

Ex. $\mathbb{G}_m \curvearrowright \mathbb{C}^3 \quad \lambda \cdot (x, y, z) = (\lambda^2 x, y, \lambda^{-1} z)$

$$M = \{(\alpha_1, \alpha_2, \alpha_3) \mid 2\alpha_1 = \alpha_3\}$$

gen-s of M: $e_2, \; e_1 + 2e_3$

$$\mathbb{C}[V]^{\mathbb{G}_m} = \mathbb{C}[y, xz^2]$$

Orbit closures. $v = \sum_{\chi \in W} v_\chi$, $v_\chi \neq 0$, $\underset{\mathrm{supp}(v)}{W} \subset X(T)$

$\Rightarrow T \cdot v \subset \mathrm{span}(v_\chi)$, $\overline{T \cdot v} \subset \mathrm{span}(v_\chi)$

$\Rightarrow$ enough to study $\overline{T \cdot v}$ in the situation

$v = \sum_{i=1}^n v_i$, $\chi_1, \ldots, \chi_n$ distinct

$P :=$ convex hull$(W) \subset X(T) \otimes \mathbb{R}$

rat-l convex polytope

face $=$ inters. of $P \cap H$ s.t. P is on one side of H (includes P & $\emptyset$)

can take H rat-l

Def. Admissible face $:= P \cap H$, $H \ni 0$, P on one side

Ex.. $W = \{e_1, 2e_1, 3e_1, 3e_1 + 2e_2, e_1 + e_2, e_2\}$

admiss. faces: $\emptyset, \{e_2\}, [e_1, 3e_1], P$

Ex. Assume $0 \in P$. Then F admissible $\Leftrightarrow 0 \in F$.

$\emptyset$ admiss. $\Leftrightarrow 0 \notin P$

Set $F_0 = \begin{cases} \text{unique face cont. } 0 \text{ in rel. interior if } 0 \in P \\ \emptyset \quad \text{otherwise} \end{cases}$

For adm face F, set $v_F = \sum_{\chi \in F \cap W} v_\chi$

Thm. $\exists$ bijection $\{T\text{-orbits in } \overline{Tv}\} \leftrightarrow$ admissible faces

$$T v_F \longleftrightarrow F$$

$$\text{unique closed orbit} \longleftrightarrow F_0$$

$0 \in \overline{Tv} \iff F_0 = \emptyset$

$\overline{Tv} = Tv \iff 0 \in \text{rel. inter.}(P)$

Proof: need to prove (i) $Tv_F \subset \overline{Tv}$ $\forall$ admiss. F

(ii) $v_1 \in \overline{Tv} \Rightarrow \text{supp}(v_1) = W \cap F$, for unique admissible F

(iii) $Tv_1 = Tv_F$

Note: $\mathbb{A}^1 - 0 \xrightarrow{f} X$ alg. variety (separ.) Define $\lim_{t\to 0} f(t)$ as $f(0)$ if f extends to $\mathbb{A}^1$.
$\iff$ in classical topology

$\mathbb{G}_m \curvearrowright V = \oplus V_n$, $v = \sum v_i$, $\lim_{t \to 0} t \cdot v = \begin{cases} v_0, & v_i = 0 \ i<0 \\ \text{DNE} & \text{otherwise} \end{cases}$

Pf: (i) $\exists \nu \in \text{Hom}(\mathbb{G}_m, T) = X(T)^\vee : \nu|_P \geq 0, \ \nu|_F = 0$

$\Rightarrow \lim_{t \to 0} \nu(t) \cdot v = v_F$

(ii) write $v_1 = \sum_{\chi \in I} z_\chi \cdot v_\chi$, $z_\chi \neq 0$, $I \subset W$

$\tilde{I} := \{ \chi' \in W \mid (\chi' + \mathbb{Z}_{\geq 0}\langle W \rangle) \cap \mathbb{Z}\langle I \rangle \neq \emptyset \} \supset I$

Exer. $F = \text{conv}(\tilde{I})$ is an admissible face (consider first the case $I = \{0\}$)

Claim: $I = \tilde{I}$. $\chi' \in \tilde{I} \Rightarrow \sum_\chi n_\chi \chi = \sum_{\psi \in I} \alpha_\psi \psi$, $n_\chi \geq 0$, $n_{\chi'} > 0$, $\alpha_\psi \in \mathbb{Z}$

$\Rightarrow f = \prod x_\chi^{n_\chi} \cdot \prod_\psi x_\psi^{-\alpha_\psi} \in \mathbb{C}[V_I]^T$ where

$v \in V_I \subset V$ open $(x_i \neq 0, i \in I)$
$v_1 \in$

$v_1 \in \overline{Tv} \Rightarrow f(v_1) = f(v) \neq 0$

$\Rightarrow x_{\chi'}(v_1) \neq 0$

$\Rightarrow \chi' \in I$

(iii) $V(F) := \text{span}(v_\chi \mid \chi \in F \cap W)$, $V^\circ(F) \subset V(F)$ open where $x_\chi \neq 0 \ \forall \chi \in F \cap W$

Note: $v_1, v_F \in V^\circ(F)$

All T-orbits on $V^\circ(F)$ have $\dim = \dim F$

$\Rightarrow$ $\underset{\text{in } V^\circ(F)}{\text{closed}}$ $\Rightarrow$ sep. by invariants on $V^\circ(F)$

$V^\circ(F) \underset{\text{closed}}{\subset} V_{F\cap W}$ $\Rightarrow$ T-inv-s on $V^\circ(F)$ extend to T-inv-s on $V_{F\cap W}$

But $v_1, v_F \in \overline{Tv}$ so $f(v_1) = f(v_F) = f(v)$ $\forall$ T-inv. on $V_{F\cap W}$.

$\Rightarrow$ $Tv_1 = Tv_F$.

<u>Cor.</u> (of Thm + Pf of (i)). X affine, $T \curvearrowright X$, $x_0 \in \overline{Tx}$

$\Rightarrow \exists\, \nu : \mathbb{G}_m \to T$ s.t. $\lim\limits_{t\to 0} \nu(t)\cdot x \in Tx_0$

<u>Pf</u>: $X \underset{T\text{-equiv.}}{\hookrightarrow} V$, $Tx_0 = Tx_F$, use Pf of (i)

<u>Ex</u>. $\dim T = 1$, $T = \mathbb{G}_m$, $\underset{\text{supp}(v)}{\overset{\shortparallel}{W}} \subset \mathbb{Z}$, $P = [\min W, \max W]$

1) $0 \notin P$ $\Rightarrow$ $\emptyset$ & P admissible, so $\overline{Tv} = Tv \cup \{0\}$

2) $0 \in \operatorname{int} P$ $\Rightarrow$ only P admissible so $\overline{Tv} = Tv$

3) 0 is an endpoint of P $\Rightarrow$ 0 & P admissible $\overline{Tv} = Tv \cup \{v_0\}$

<u>Ex</u>. Cor. for $T = \mathbb{G}_m$: $\mathbb{G}_m \curvearrowright \underset{\text{affine}}{X}$, $x_0 \in \overline{\mathbb{G}_m x}$

$\Rightarrow$ either $\lim\limits_{\lambda\to 0} \lambda\cdot x = x_0$ or $\lim\limits_{\lambda\to\infty} \lambda\cdot x = x_0$

(both limits exist only if $\mathbb{G}_m x = x$ otherwise $\mathbb{P}^1 \to X$)

Suppose X (sep.) variety, $\mathbb{G}_m \curvearrowright X$, $X = \bigcup$ $\mathbb{G}_m$-inv. open affines

$x_0 \in \overline{\mathbb{G}_m x}$, $U \ni x_0$ $\mathbb{G}_m$-inv. open affine $\Rightarrow$ $x \in U$

$\Rightarrow$ either $\lim\limits_{\lambda\to 0} \lambda\cdot x = x_0$ or $\lim\limits_{\lambda\to\infty} \lambda\cdot x = x_0$

Hilbert-Mumford criterion

$\underset{\text{conn. red.}}{G} \curvearrowright \underset{\text{aff. var.}}{X}$ $/\mathbb{C}$, suppose $Gy \subset \overline{Gx}$

y: (unique) closed orbit

Then $\exists\, \nu : \mathbb{G}_m \to G$ 1PS s.t. $\lim_{t \to 0} \nu(t) x \in Gy$.

Cor. $\underset{\text{conn. red.}}{G} \curvearrowright V$ lin., $\overline{Gv} \ni 0 \Rightarrow \exists\, \nu : \lim_{t\to 0} \nu(t) v = 0$.

Cor. Gx closed $\Leftrightarrow$ $\forall\, \nu : \mathbb{G}_m \to G$ 1PS if $\lim_{t \to 0} \nu(t) \cdot x$ exists then it is in Gx

In part., if $\lim_{t\to 0} \nu(t) x$ exists only for $\nu = 1$ then Gx is closed.

Cartan decomp-n. Recall: $\exists\, K \subset G$ max. comp. subgr. s.t. $\forall$ compact $K_1 \subset G$, $\exists\, g K_1 g^{-1} \subset K$.

Ex. $U(n) \subset GL_n(\mathbb{C})$ satisfies this property:

suppose $K_1 \subset GL(V)$ compact

fix a pos. Herm. form H on V.

average$_{K_1}$ $\rightsquigarrow$ K_1-invar. H-form

so $K_1 \subset g\, U(n)\, g^{-1}$

Let $T \subset G$ be max. torus

$(\mathbb{C}^*)^n \cong T$, $(S^1)^n \cong K_1 \subset T$ max. comp. in K, can conjugate T so $K_1 \subset K$

Thm. Then $G = KTK$.

Ex. $G = GL_n(\mathbb{C})$, $K = U(n)$, T = diag. matr.

$\forall g = k_1 t k_2$, where k_1, k_2 unitary, t diag. with >0 entries

$\uparrow$ sing. value decomposition

Pf of HM-criterion.

Step 1. Enough to prove: $\exists\, x' \in Gx : \overline{Tx'} \cap Gy \neq \emptyset$

Indeed, by Cor. about closures of toric orbits then $\exists\, \nu: \mathbb{G}_m \to T$ 1PS s.t. $\lim\limits_{t\to 0} \nu(t)x' \in Gy$. If $x' = gx$, then $\lim\limits_{t\to 0}(g^{-1}\nu(t)g)x =$

$g^{-1}\lim\limits_{t\to 0}\nu(t)x' \in Gy$

Step 2. Enough to prove the same w.r.to classical top. and assuming $y \in \overline{Gx}$ w.r.to classical topology

Indeed, $\overline{S}^{Zar}$ contains $\overline{S}^{class.}$ (for $S \subset X$ a subset) & $\overline{S}^{Zar} = \overline{S}^{class}$ if S is Zar. open in $\overline{S} = \overline{S}^{Zar}$

Indeed, can assume S irred., so $Y = \overline{S}$ is irred.

<u>Thm</u> $U \overset{\neq\emptyset}{} $ Zar. open in irred. var-ty $Y/\mathbb{C}$ $\Rightarrow$ U dense in Y in class. topology.

Mumford's "red book"

Easy case: $Y = \mathbb{C}^n$. Need: if f is $\neq 0$ polyn. then hypersurf. $(f=0)$ has no interior. But otherwise, Taylor series of f at some point is $\equiv 0$, impossible.

Hence, the assertion is true if Y is Zar. open in $\mathbb{C}^n$.

<u>Gen. case</u> (Mumford gives a more elementary proof)

Let $\pi: \tilde{Y} \to Y$ be res-n of sing., replace $U \subset Y$ by $\pi^{-1}(U) \subset \pi^{-1}(\tilde{Y})$. Can assume Y smooth.

$\exists\, Y = \bigcup Y_i$ open covering & $\tilde{Y}_i \xrightarrow{f_i} Y_i$ fin. étale s.t. $\tilde{Y}_i \underset{\text{open}}{\subset} \mathbb{C}^n$

$f_i^{-1}(U)$ dense in $\tilde{Y}_i \Rightarrow U \cap Y_i$ dense in Y_i (in class. top.)

Step 3. Use Cartan dec. $G = KTK$, so $y \in \overline{KTKx}$ (in class. top.)

<u>Claim</u>: $\overline{KTKx} = K \cdot \overline{TKx}$. Indeed, the action map

$$\begin{array}{ccc} K \times X & \xrightarrow{a_K} & X \\ \downarrow\wr & & \| \\ K \times X & \xrightarrow{p_2} & X \end{array} \qquad \begin{array}{c}(k,x)\\ \downarrow \\ (k,kx)\end{array}$$

is proper in class. topology (since p_2 is)

$\left.\begin{array}{l} X \text{ loc. compact} \\ a_K \text{ proper} \end{array}\right\} \Rightarrow a_K(\text{closed})$ is close

$\Rightarrow K \cdot \overline{TKx}$ is closed

So can assume $y \in \overline{TKx}$ (since the statement depends only on Gy)

Step 4. Consider $\pi_T : X \to X/\!\!/T$ cont. in class. top.

$\Rightarrow \pi_T(y) \in \pi_T(TKx) = \pi_T(Kx) \Rightarrow \exists k \in K :$

$$\pi_T(y) = \pi_T(kx)$$

$\Rightarrow \overline{Ty} \cap \overline{Tkx} \neq \emptyset$ (both contain unique closed T-orbit in $\pi_T^{-1}\pi_T(y)$)

$\Rightarrow \underset{\substack{\shortparallel \\ Gy}}{\overline{Gy}} \cap \overline{Tkx} \neq \emptyset$

Ex. $G = SL_2(\mathbb{C}) \curvearrowright S^n(\mathbb{C}^2) \leftarrow f(x,y)$ homog. of deg n.

Claim: $\overline{Gf} \ni 0, \iff$ f has lin. factor of mult. $> \frac{n}{2}$.
$\quad f \neq 0$

Pf: $\forall\ \nu : \mathbb{G}_m \hookrightarrow G$ is conjugate in G to $\nu_0 : \lambda \mapsto \mathrm{diag}(\lambda, \lambda^{-1})$
1PS

$$\nu_0(\lambda)(x^{n-i} y^i) = \lambda^{n-2i} \cdot x^{n-i} y^i$$

$\lim_{\lambda \to 0} \nu_0(\lambda)(f) = 0 \iff$ only $x^{n-i}y^i$ with $i < \frac{n}{2}$ appear

$\iff$ mult. $(x) > \frac{n}{2}$

$\lim g\nu_0(\lambda)g^{-1}(f) = 0 \iff$ mult. (x) in $g^{-1}(f) > \frac{n}{2}$

$\iff$ mult. (gx) in $f > \frac{n}{2}$

Claim 2. Gf closed, $\iff$ either each lin. factor has mult. $< \frac{n}{2}$
$\quad f \neq 0$ $\quad$ or $n = 2k$, $f = \ell_1^k \ell_2^k$

Pf: $\lim_{\lambda \to 0} \nu_0(\lambda) f$ exists $\iff$ only $x^{n-i}y^i$ with $i \leq n/2$ appear

$\iff$ mult. $(x) \geq \frac{n}{2}$

so mult. $\forall$ lin. factor $< \frac{n}{2} \Rightarrow Gf$ closed.

Assume $n = 2k$.

$f = x^k \cdot f_1$ & Gf closed $\Rightarrow \lim \nu_0(\lambda) f = c \cdot x^k y^k \Rightarrow$ can assume $f \sim \underset{\neq 0}{c} \cdot x^k \cdot y^k$

$$\lim_{\lambda \to 0} \mathrm{diag}(\lambda, \lambda^{-1}) \cdot (ax + by)^k (cx + dy)^k = \lim_{\lambda \to 0} (a\lambda x + b\lambda^{-1} y)^k (c\lambda x + d\lambda^{-1} y)^k$$

exists $\iff b = 0$ or $d = 0$

Say $b = 0 \Rightarrow \lim = (ad\, xy)^k = (xy)^k$, so Gf closed.

$\underline{\text{Ex}}$. $G = GL_2(\mathbb{C}) \curvearrowright \mathfrak{gl}_2 = \mathrm{Mat}_2(\mathbb{C})$ adjoint action.

Suppose $G\cdot x$ not closed $\Rightarrow$ changing g to conj.

get $y = \lim \mathrm{diag}(t^m, t^n) x$ exists & $y \not\sim x$, $m > n$

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} t^m & \\ & t^n \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t^{-m} & \\ & t^{-n} \end{pmatrix} = \begin{pmatrix} a & t^{m-n} b \\ t^{-m+n} c & d \end{pmatrix}$$

so $c = 0$, $y = \begin{pmatrix} a & \\ & d \end{pmatrix}$; $y \not\sim x \Rightarrow a = d, b \neq 0$

so Gx closed $\Leftrightarrow$ x semisimple.

$\underline{\text{Ex}}$. $G = GL_n(\mathbb{C}) \curvearrowright \mathrm{Mat}_n(\mathbb{C})$.

$v(t) = \mathrm{diag}(t^{n_1} \ldots, t^{n_2} \ldots, \ldots)$ $\quad n_1 > n_2 > \ldots$

$\lim$ exists $\rightsquigarrow x$ block Δ-r, $y \not\sim x \rightsquigarrow$ non semisimple.

$\underline{\text{Kempf-Ness thm.}}$ G red. group, $G \curvearrowright V$ linearly

$K \subset G$ max. compact as before,

$\langle \cdot , \cdot \rangle$ K-inv-t hermitian sc. prod. on V.

$|v| = \sqrt{\langle v, v\rangle}$

$\mu_K = \mu : V \to \mathrm{Lie}(K)^* : \langle \mu(v), x\rangle := i\langle xv, v\rangle \in \mathbb{R}$

dual $/\mathbb{R}$

Note : x skew-hermitian so $\langle xv, v\rangle = -\langle xv, v\rangle$

$\underline{\text{moment map}}$. $\underline{\mu \text{ is } K\text{-equivariant.}}$

$\underline{\text{Ex}}$. $G = T$ - cpx torus $\cong (\mathbb{C}^*)^n \curvearrowright V \rightsquigarrow \chi_1, \ldots, \chi_N \in X^\cdot(T)$

$\mathrm{Lie}(S_1) \hookrightarrow \mathrm{Lie}(\mathbb{C}^*) \Rightarrow \forall \chi \in X^\cdot(T)$, $i d\chi\big|_{\mathrm{Lie}(K) \cong (S^1)^n}$ takes values in $\mathbb{R}$

$\mathbb{R} \xrightarrow{i} i\mathbb{R} \subset \mathbb{C}$

can view $i\cdot d\chi \in \mathrm{Lie}(K)^*$

Also $|\chi(k)| = 1$ for $k \in K$.

so $\forall$ $\langle v, v\rangle = \sum |z_i|^2$ is K-invariant, $v = \sum z_i v_i$

$x \cdot v = \sum \langle d\chi_i, x\rangle \cdot z_i v$, $v = \sum z_i v_i$

$$\Rightarrow \mu(v) = \sum_j |z_j|^2 \cdot \underbrace{i d\chi_j}_{\in \mathrm{Lie}(K)^*}$$

Thm (Kempf-Ness) Gv is closed $\iff$ $g \mapsto |gv|$ attains min.

$\iff$ $Gv \cap \mu^{-1}(0) \neq \emptyset$

In this case $Gv \cap \mu^{-1}(0)$ is a single K-orbit

Proof: Can assume $v \neq 0$.

1) $\Rightarrow$ 2) Assume Gv is closed.

$0 \notin Gv \Rightarrow \exists \varepsilon > 0 : |\ | \geq \varepsilon$ on $Gv \Rightarrow$

$|\ |^2$ attains minimum on Gv (since $Gv \cap \{|v| \leq C\}$ is compact)

2) $\Rightarrow$ 3) Let $v_0 \in Gv$ be s.t. $|v_0|$ is minimal.

$\forall x \in Lie(K) \subset Lie(G)$

$$\frac{d}{dt} |\exp(itx) v_0|^2 = 0 \text{ at } t = 0$$

$$\iff \langle ixv_0, v_0 \rangle + \underbrace{\langle v_0, ixv_0 \rangle}_{= \overline{\langle ixv_0, v_0\rangle}} = 0$$

so $\langle xv_0, v_0 \rangle = 0$ i.e. $\mu(v_0) = 0$

3) $\Rightarrow$ 1)

Assume $\mu(v) = 0$, but Gv is not closed

HM-criterion: $\exists \nu : \mathbb{G}_m \to G$, compatible with K

i.e. $\nu(S^1) \subset K$, s.t. $\lim_{t \to 0} \nu(t) v$ exists $\notin Gv$

Then

$v = \sum_{i \geq 0} v_i$ comp-s w.r.to ν, $\lim_{t\to 0} \nu(t) v = v_0$

$x = d\nu(\frac{d}{dt})$, $\langle \mu(v), x \rangle = 0 \Rightarrow \langle xv, v \rangle = 0$

$$= \sum_{m \geq 0} m |v_m|^2$$

$$\Rightarrow v = v_0$$

Now Suppose $v_1, v_2 \in Gv \cap \mu^{-1}(0)$, $G = KTK \Rightarrow$

can assume $v_2 = t v_1$, $t \in T$; $\mu_K^{-1}(0) \subset \mu_{T \cap K}^{-1}(0)$

Reduce to the case $G = T$. $T/_{T \cap K} \simeq \mathbb{R}^n_{>0} \xleftarrow[\exp]{\simeq} \mathbb{R}^n$

Consider the function $f : \mathbb{R}^n \simeq T/_{T \cap K} \to \|tv\|^2$.

$V = \bigoplus^{N} \chi_i$ as T-repr., (v_i) orthonormal, $v = \sum z_i v_i$, $z_i \neq 0$

$f(\underset{\in \mathbb{R}^n}{x}) = \sum |z_i|^2 e^{2\ell_i(x)}$, $\ell_i = d\chi_i|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}$

$St_v = \{t \mid tv = v\} = \{t \mid \chi_i(t) = 1\} \longrightarrow T/T \cap K$, image $= \cap \ker(\ell_i) = M$

$f(x) \in \mu^{-1}(0) \Rightarrow x$ crit. point of f; f descends to $\mathbb{R}^n/M$.

$\forall$ aff. line $L \subset \mathbb{R}^n/M$, $f|_L$ has positive 2nd deriv.

$\Rightarrow f$ on $\mathbb{R}^n/M$ is strictly convex with unique crit. point = minimum

$\Rightarrow v_1 = v_2$

Cor. $V/\!/G \simeq \mu^{-1}(0)/K$ as top. spaces

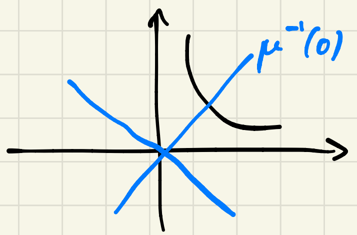
(this space is Hausdorff)

Ex. $\mathbb{C}^* \curvearrowright \mathbb{C}^n$, $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$, $\pm\mu = \sum |x_i|^2$

so $\mu^{-1}(0) = \{0\}$.

Ex. $\underset{\cup\, S^1}{\mathbb{C}^*} \curvearrowright \mathbb{C}^2$, $t \cdot (x, y) = (tx, t^{-1}y)$

$\pm\mu(x, y) = |x|^2 - |y|^2$



If consider same action on $X = \mathbb{C}^2 - 0$, all orbits will be closed, $X = \underset{x \neq 0}{U_1} \cup \underset{y \neq 0}{U_2}$

$U_1/\mathbb{C} \xrightarrow[xy]{\sim} \mathbb{A}^1$, $U_2/\mathbb{C} \xrightarrow[xy]{\sim} \mathbb{A}^1$

so $X/G \simeq \mathbb{A}^1 \cup_{\mathbb{A}^1 - 0} \mathbb{A}^1$ non-Hausdorff

GIT-quot-s for $G \curvearrowright$ aff. var-s.

Recall: $A = \bigoplus_{n \geq 0} A_n$, fin.-gen. $\rightsquigarrow$ $\mathrm{Proj}(A)$
$$\downarrow$$
$$\mathrm{Spec}(A_0)$$

$\mathrm{Proj}(A) = \bigcup_{\deg f > 0} \mathrm{Spec}\, A_{(f)}$, equipped with sheaves $\mathcal{O}(n)$

Ex. weighted proj. space $\mathbb{P}(d_1, \ldots, d_n) = \mathrm{Proj}\, \mathbb{C}[x_1, \ldots, x_n]$, $\deg x_i = d_i$

$$= \{ (x_1 : \ldots : x_n) \}_{0}^{*} \Big/ (x_1 : \ldots : x_n) = (\lambda^{d_1} x_1 : \ldots : \lambda^{d_n} x_n)$$

$(x_i \neq 0) = U_i \subset \mathbb{P}(d_1, \ldots, d_n)$ $\quad \gcd(d_1, \ldots, d_n) = 1$

$\simeq \mathbb{A}^{n-1} / \mu_{d_i}$ $\leftarrow$ wts $(d_1, \ldots, \hat{d_i}, \ldots, d_n)$

$m = \mathrm{lcm}(d_1, \ldots, d_n)$

$\Rightarrow \mathcal{O}(m)$ invert. line bdle

$\mathbb{C}[x_1, \ldots, x_n] \hookrightarrow \mathbb{C}[u_1, \ldots, u_n] : x_i \mapsto u_i^{d_i}$, $\deg u_i = 1$

finite $\mathbb{C}[x_1, \ldots, x_n]$-module

$\rightsquigarrow \mathbb{P}^{n-1} = \mathrm{Proj}\, \mathbb{C}[u_1, \ldots, u_n] \xrightarrow{\pi} \mathbb{P}(d_1, \ldots, d_n)$ finite surj. morphism

$\pi^* \mathcal{O}(m) \simeq \mathcal{O}(m)$ ample

$\Rightarrow \mathbb{P}(d_1, \ldots, d_n)$ projective.

$\Rightarrow \mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$ projective

$$\mathrm{Proj}(A) \hookrightarrow \mathrm{Spec}(A_0) \times \mathbb{P}(d_1, \ldots, d_n) \to \mathrm{Spec}(A_0)$$

if $A = A_0[x_1, \ldots, x_n]/J$, $\deg x_i = d_i$.

Rem. In fact, π is the quot. map w.r.to $\mu_{d_1} \times \cdots \times \mu_{d_n}$ action on $\mathbb{P}^{n-1}$!

$G \curvearrowright X$, $\theta : G \to \mathbb{G}_m$ char-r

red.gr. aff. var. (aff. sch. fin. type)

Assume θ of infinite order

$$\mathbb{C}[X]^{G, n\theta} := \{ f \mid g \cdot f = \theta^n(g) \cdot f \}$$

$\rightsquigarrow R_{X,G,\theta} := \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, n\theta}$ graded algebra.

deg=0 comp-t $= \mathbb{C}[X]^G$

<u>Lem.</u> $R_{X,G,\theta}$ is fin. gen. as $\mathbb{C}$-alg.

<u>Pf:</u> Consider the action of G on $X \times \mathbb{C}$

$g \cdot (x, z) = (gx, \theta(g) z)$.

Then $\mathbb{C}[X, \mathbb{C}]^G = \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, n\theta} \cdot z^n$

fin. gen. as proved before.

<u>Def</u>. $X /\!\!/_\theta G := \operatorname{Proj} R_{X,G,\theta}$ proj- over $\operatorname{Spec} \mathbb{C}[X]^G = X /\!\!/ G$.

<u>Def</u>. $x \in X$ is θ-semistable if $\exists n > 0$ & $f \in \mathbb{C}[X]^{G, n\theta}$: $f(x) \neq 0$

$X^{ss} = X^{\theta\text{-}ss} \subset X$ G-inv. open subset

<u>Thm.</u> 1) $\exists \pi^\theta : X^{\theta\text{-}ss} \to X /\!\!/_\theta G$ G-inv.

X irred. & reduced affine

$$\begin{array}{ccc} X^{\theta\text{-}ss} & \longrightarrow & X /\!\!/_\theta G \\ \downarrow & & \downarrow \\ X & \longrightarrow & X /\!\!/ G \end{array}$$

commutes.

X normal $\Rightarrow X /\!\!/_\theta G$ normal

2) π^θ is a <u>good quotient</u> (see below) $\Rightarrow$ categ. quotient

Recall criterion for being cat. quot.

<u>Def.</u> $G \curvearrowright X$, $\pi : X \to Y$ <u>good quotient</u>

1) G-inv., 2) surject., 3) $(\pi_* \mathcal{O}_X)^G \simeq \mathcal{O}_Y$,

4) π(G-inv. closed) is closed, $\pi(\cap Z_i) = \cap \pi(Z_i)$ ← enough for fin. #.

5) π affine

<u>Have seen</u>: X aff., G red. $X \to X/\!\!/G$ is good quot.

good quot. $\Rightarrow$ categ. quot.

<u>Def.</u> <u>Geometric quotient</u>: good quot. s.t. $\forall$ fiber = single orbit

Same proof as before:

$X \to Y$ good quot. $\Rightarrow$ $\forall$ fiber $\pi^{-1}(y) = \sqcup O$ s.t. $\overline{O} \supset O_0$ ← unique closed orbit in $\pi^{-1}(y)$

<u>Exercise</u>: 1) $X \xrightarrow{\pi} Y$ good quot. for $G \curvearrowright Y$

$\Rightarrow$ $\forall$ open $U \subset Y$, $\pi^{-1}(U) \to U$ is a good quot.

2) $Y = \cup U_i$, $\pi^{-1}(U_i) \to U_i$ good quot.

$\Rightarrow$ so is $\pi : X \to Y$.

Same for geom. quot.

<u>Pf. of Theorem.</u>

$$R = R_{X,G,\theta}, \quad R_n = \mathbb{C}[X]^{G, n\theta}$$

$f \in R_n$, $n > 0$ $\rightsquigarrow$ affine open $X_f \subset X$ where $f \neq 0$

$\rightsquigarrow$ affine open $\mathrm{Spec}(R_{(f)}) \subset \mathrm{Proj}(R)$

$R_{(f)}$ = deg-0 part in $R[f^{-1}]$

$R_{(f)} \xrightarrow{\sim} \mathbb{C}[X_f]^G$ clear if $\mathbb{C}[X]$ is a domain

so $\operatorname{Spec}(R_{(f)}) \simeq X_f /\!/ G$

So we have $\pi_f : X_f \to X_f /\!/ G$ good quot-s

Easy to check: glue into global $\pi^\theta : X \to \operatorname{Proj}(R)$.

Namely, $X_{f_1 f_2} \subset X_{f_1}$,

$$\begin{tikzcd} R_{(f_1 f_2)} \arrow[r] & \mathbb{C}[X_{f_1 f_2}] \\ R_{(f_1)} \arrow[u] \arrow[r] & \mathbb{C}[X_{f_1}] \arrow[u] \end{tikzcd}$$

commutes

Let $z \in X_f /\!/ G \subset \operatorname{Proj}(R)$. Claim: $(\pi^\theta)^{-1}(z) \subset X_f$

enough to check $(\pi^\theta)^{-1}(z) \cap X_{f_1} \subset X_{f f_1}$

x

$$\begin{array}{ccc} x \in X_{f_1} & \xrightarrow{\pi_{f_1}} & X_{f_1} /\!/ G \\ & & \cup \\ & & X_{f f_1} /\!/ G \ni z \end{array}$$

$X_{f f_1} /\!/ G$ = principal affine in $R_{(f_1)}$ corr. to $\dfrac{f^{\deg f_1}}{f_1^{\deg f}}$

$\pi_{f_1}^{-1}(X_{f f_1} /\!/ G)$ = corr. principal aff. in X_{f_1}, i.e. $X_{f f_1}$

Thus, $X_f = (\pi^\theta)^{-1}(X_f /\!/ G)$

By locality, $\pi^\theta : X \to \operatorname{Proj}(R)$ is a good quotient.

Ex. $\mathbb{G}_m \curvearrowright \mathbb{A}^n \quad \lambda\cdot(x_1,\dots,x_n) = (\lambda x_1,\dots,\lambda x_n)$

$\theta = 1$ or -1 $\quad R = \begin{cases} \mathbb{C}[x_1,\dots,x_n] & \text{with } \deg(x_i)=1 \\ \mathbb{C} \end{cases}$

$\theta = -1 \rightsquigarrow \mathbb{A}^n /\!\!/_\theta \mathbb{G}_m = \mathbb{P}^{n-1}$, semistable locus $= \mathbb{A}^n - 0$

Similarly with $d_1,\dots,d_n > 0 \rightsquigarrow \mathbb{P}(d_1,\dots,d_n)$.

Ex. Consider $GL(W) \curvearrowright \mathrm{Hom}(W,V) = X$

$$g\cdot A = A\circ g^{-1}$$

$$f \in \mathbb{C}[\mathrm{Hom}(W,V)]$$

$$g\cdot f(A) = f(g^{-1}A) = f(A\circ g)$$

$k = \dim W \le \dim V = n$, $\quad \ell \in (\Lambda^k V)^*$

$$\mathrm{Hom}(W,V) \xrightarrow{\Lambda^k} \mathrm{Hom}(\Lambda^k W, \Lambda^k V) \xrightarrow{\ell^*} (\Lambda^k W)^*$$

choose coord-s $\rightsquigarrow f_I : A \longmapsto (k\times k)$-minor corr. to $I \subset [1,n]$, $|I| = k$

$$f_I(A\circ g) = \det g \cdot f_I(A)$$

Take $\theta = \det : GL(W) \longrightarrow \mathbb{G}_m$

<u>Claim:</u> θ-semistable locus $=$ A s.t. $A : W \hookrightarrow V$ injective

A injective $\iff \forall I : \; f_I(A) \neq 0$
$\Downarrow$
θ-semist.

Assume $\operatorname{rk} A < k$ $\Rightarrow$ can choose bases so $A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \\ & & & \end{bmatrix}$

consider $\nu : \mathbb{G}_m \to SL(W)$
$$\nu(\lambda) = \operatorname{diag}(\lambda, \dots, \lambda, \lambda^{-k+1})$$

Then $\lim\limits_{\lambda \to 0} \nu(\lambda) A = 0$ so $\forall f : \; f(A \circ g) = (\det g)^n f(A)$, $n > 0$

$\forall \lambda : \; f(\nu(\lambda)A) = f(A) \Rightarrow f(0) = 0$

$\Rightarrow A$ is unstable.

$R \leftarrow S = S^{\cdot}(R_1)$
$\parallel$
$R_{X, GL(W), \det}$

$X^{ss} \to \operatorname{Proj} R$ surjective $\Rightarrow \operatorname{Proj} R \to \operatorname{Proj} S$ regular morphism

The action of $GL(W)$ on X^{ss} is free

$\Rightarrow$ all orbits are closed

$\Rightarrow$ fibers of $X^{ss} \to \operatorname{Proj} R$ $=$ orbits

but fibers of $X^{ss} \to \operatorname{Proj} S = \mathbb{P}^N$ are also orbits

since this can be identified with
$$X^{ss} \to G(k, n) \hookrightarrow \mathbb{P}^N$$
Plücker embedding

$\Rightarrow \operatorname{Proj} R \to G(k, n)$ is bijective,

$G(k, n)$ - smooth $\Rightarrow$ isom-sm, so $X /\!\!/_{\theta} GL(W) \simeq G(k, n)$

Toric GIT.

Consider $T \cong \mathbb{G}_m^n \curvearrowright V = \mathbb{A}^N$

$\leftrightarrow \chi_1, \dots, \chi_N \in X^*(T) \ni \theta \neq 0$

$v \in V \rightsquigarrow \mathrm{supp}(v) \subset X^*(T)$.

Thm. $v \in V^{\theta-ss} \iff \theta \in \mathbb{Q}_{\geq 0}\langle \mathrm{supp}(v) \rangle$

$\iff \mathbb{R}_{>0}\theta \cap \mathrm{conv.hull}(\mathrm{supp}\, v) \neq \emptyset$

Pf: suppose $n\theta = \sum_{\chi \in \mathrm{supp}(v)} n_\chi \cdot \chi$, $n_\chi \geq 0$.

For each $\chi \in \mathrm{supp}\, v$ $\exists\, i(\chi) : \chi = \chi_{i(\chi)}$ & $x_{i(\chi)} \neq 0$ on v

Take $f = \prod_{\chi \in \mathrm{supp}\, v} x_{i(\chi)}^{n_\chi}$, then $f \in \mathbb{C}[V]^{T, n\theta}$ & $f(v) \neq 0$

Conversely, if $\exists f \in \mathbb{C}[V]^{T, n\theta}$ s.t. $f(v) \neq 0$

then can assume f monomial, $f = \prod x_i^{n_i}$

$\Rightarrow \chi_i \in \mathrm{supp}(v) \Rightarrow n\theta \in \mathbb{Z}_{\geq 0}\langle \mathrm{supp}(v) \rangle$

Cor. $V^{\theta-ss} \neq \emptyset \iff \mathbb{R}_{>0}\theta \cap \mathrm{conv.hull}(\chi_1, \dots, \chi_N) \neq \emptyset$.

Recall: $V /\!\!/_\theta T$ is projective over $V /\!\!/ T$

$V /\!\!/ T = \{0\} \iff \mathbb{C}[V]^T = 0 \iff 0 \notin \mathrm{convex\ hull}(\chi_1, \dots, \chi_N)$

Note: There is a bigger torus action $\mathbb{G}_m^N \curvearrowright \mathbb{A}^N$, our T-action is given by hom-sm $T \to \mathbb{G}_m^N \iff$

$\mathbb{Z}^N \xrightarrow{\varphi} X^*(T)$. $\quad \varphi(\mathbb{Z}^N) \subset X^*(T)$, so $T \to T' \hookrightarrow \mathbb{G}_m^N$

$e_i \longmapsto \chi_i \qquad \varphi(\mathbb{Z}^N) = X^*(T')$

$\Rightarrow$ reduce to the case φ is surjective $\iff T \hookrightarrow \mathbb{G}_m^N$

$\mathbb{G}_m^N$ acts on $\bigoplus_{n\geq 0}\mathbb{C}[V]^{T,n\theta}$ $\Rightarrow$ on $V/\!\!/_{\theta}T = X$

T acts by rescaling $\Rightarrow$ $T_1 := \mathbb{G}_m^N/T \curvearrowright X$

with dense orbit.

Also, X is normal

$\rightsquigarrow$ X is a toric variety.

Note $0 \to X_\bullet(T) \xrightarrow{\varphi^\vee} \mathbb{Z}^N \to X_\bullet(T_1) \to 0$

$e_i \mapsto v_i$ $\qquad X_\bullet(T_1) = N$ dual to $M = X^\bullet(T_1)$

$\underset{X^\bullet(T)}{\theta} = \varphi\big(\sum \underset{\mathbb{Z}}{a_i} e_i^*\big) \rightsquigarrow P_{\vec{a}} \subset M_{\mathbb{R}} \subset \mathbb{R}^N$

$:= \{m \mid \underbrace{\langle m, v_i\rangle}_{x_i(m)} \geq -a_i\}$

polyhedron

$0 \to M_{\mathbb{R}} \to \mathbb{R}^N \xrightarrow{\varphi} X^\bullet(T)_{\mathbb{R}} \to 0$

$P_\theta := \{\vec{x} \in \mathbb{R}^N_{\geq 0} : \varphi(\vec{x}) = \theta\} \subset \mathbb{R}^N$

$P_{\vec{a}} + \vec{a}$

variety w. T_1-action from polyhedron:

$\boxed{\text{can also describe } S_P \text{ in terms of } P_\theta \cap \mathbb{Z}^N,\ 2P_\theta \cap \mathbb{Z}^N. \text{ etc.}}$

$P \subset M_{\mathbb{R}}$ rat. polyhedr. $\rightsquigarrow$ $C(P) \subset M_{\mathbb{R}} \times \mathbb{R}$ cone over $P \times \{1\}$

$\rightsquigarrow S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$, so $S_{P,n} = \mathbb{C}\text{-sp}(nP \cap \mathbb{Z})$

semigroup ring

$\rightsquigarrow \mathrm{Proj}(S_P)$, M-grading on S_P $\rightsquigarrow$ T_1-action on $\mathrm{Proj}(S_P)$

In our case, $nP_{\vec{a}} \cap \mathbb{Z} \longleftrightarrow$ monomials in $\mathbb{C}[V]^{T,n\theta}$

$\overset{\in}{m} \longmapsto m + n\vec{a}$

$S_{P_{\vec{a}}}$

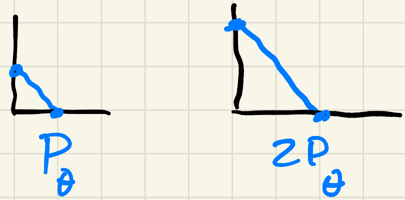
$\Rightarrow S_{P_{\vec{a}}} \simeq \bigoplus \mathbb{C}[V]^{T,n\theta} \Rightarrow \mathrm{Proj}(S_P) \simeq V/\!\!/_{\theta}T$

Ex. $\mathbb{G}_m \curvearrowright \mathbb{A}^N$ w. wts $1, \dots, 1$ $\qquad \mathbb{Z} \xrightarrow{\varphi^\vee} \mathbb{Z}^N$, $\; 1 \mapsto \sum e_i$,

$$N = \mathbb{Z}^N / \textstyle\sum e_i, \qquad \underset{\sum x_i = 0}{M \subset \mathbb{Z}^N}, \qquad \varphi : \mathbb{Z}^N \xrightarrow[\Sigma]{} \mathbb{Z}$$

$$\theta = 1 \rightsquigarrow P_\theta = \{ x_i \geq 0, \; \textstyle\sum x_i = 1 \}$$

E.g., $N = 2$,



etc.

$\rightsquigarrow \operatorname{Proj} k[y_0, y_1]$

$$N = 3 \rightsquigarrow P_\theta = \Delta \subset \{ \textstyle\sum_1^3 x_i = 0 \}$$

$$T = \mathbb{G}_m \times \mathbb{G}_m \curvearrowright \mathbb{A}^4 \qquad (\lambda, \mu) \cdot (x_1, x_2, y_1, y_2) = (\lambda x_1, \lambda x_2, \mu y_1, \mu y_2)$$

$\theta = (1, 1)$ $\qquad$ wts: $\mathrm{wt}(x_i) = (1, 0)$, $\mathrm{wt}(x_2) = (0, 1)$

v. semistable $\iff \operatorname{supp}(v) = \{ (1,0), (0,1) \}$

$\iff$ one of $x_i \neq 0$ & one of $y_i \neq 0$.

$$\mathbb{C}[V]^{T, n\theta} = \text{homog. of deg } n \text{ in } x \text{ \& in } y$$

$$\cong \mathbb{C}[x_1, x_2]_n \otimes \mathbb{C}[y_1, y_2]_n \rightsquigarrow \text{get Segre algebra}$$

$\implies$ GIT-quotient $= \mathbb{P}^1 \times \mathbb{P}^1$.

$$\underset{x_1 + x_2 = 0, \; y_1 + y_2 = 0}{M \subset \mathbb{Z}^4} \qquad P_\theta = \text{product of 2 intervals} = \text{square}$$

$\underline{\text{Ex}}$. $\mathbb{G}_m \curvearrowright \mathbb{A}^4$ x_{ij} $\lambda \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \lambda x_{11} & \lambda^{-1} x_{12} \\ \lambda^{-1} x_{21} & \lambda x_{22} \end{pmatrix}$

$wt(x_{11}) = wt(x_{22}) = 1, \quad wt(x_{12}) = wt(x_{21}) = -1$

$\theta_1 = 1 \rightsquigarrow V^{\theta_1\text{-ss}} = \{ v \mid (x_{11}, x_{22}) \neq (0,0) \}$

$\theta_2 = -1 \rightsquigarrow V^{\theta_2\text{-ss}} = \{ v \mid (x_{12}, x_{21}) \neq (0,0) \}$

$\mathbb{G}_m$-action on $V^{\theta_1\text{-ss}}$ is free,

$V^{\theta_1\text{-ss}} / \mathbb{G}_m \longrightarrow \mathbb{P}^1$

$v \longmapsto (x_{11} : x_{22})$

$\mathbb{C}[V]^{\mathbb{G}_m}$ spanned by
$u_1 = x_{11} x_{12}$, $x_{11} x_{21} = v_1$,
$u_2 = x_{21} x_{22}$, $x_{12} x_{22} = v_2$

$\underline{\text{Claim}}$: $V^{\theta_1\text{-ss}} / \mathbb{G}_m \simeq \underset{\mathbb{P}^1}{\mathrm{tot}}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$

$\pi \downarrow$ small resolution

$V /\!\!/ \mathbb{G}_m \leftarrow$ "conifold" $\{ u_1 u_2 = v_1 v_2 \} \subset \mathbb{A}^4$

π isom. away from 0, $\pi^{-1}(0) = \mathbb{P}^1 \left(x_{12} = x_{21} \big|_{\pi^{-1}(0)} \right)$

$\underline{\text{Pf}}$: over $x_{11} \neq 0$ coord-s on the quotient:
$u_1 = x_{11} x_{12}$, $v_1 = x_{11} x_{21}$ & $t = \dfrac{x_{22}}{x_{11}}$

over $x_{22} \neq 0$: u_2, v_2 & $s = \dfrac{x_{11}}{x_{22}} = t^{-1}$

transition between charts: $u_2 = v_1 \cdot t$, $v_2 = u_1 \cdot t$

Same descr. for $V^{\theta_2\text{-ss}}$ but with different $\mathbb{P}^1$

$V^{\theta_1\text{-ss}} / \mathbb{G}_m \dashrightarrow V^{\theta_2\text{-ss}} / \mathbb{G}_m$ isom. away from codim 2

This is called $\underline{\text{Atiyah flop}}$

Blow up of sing. point in $(u_1 u_2 = v_1 v_2)$

$X = \mathrm{tot}_Q\, \mathcal{O}(-1,-1)$, $Q = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$\dim V_1 = \dim V_2 = 2$

$$X = \{ V_1 \twoheadrightarrow L_1,\ L_2 \hookrightarrow V_2,\ f : L_1 \to L_2 \}$$
$$\downarrow$$
$$X_0 := \{ A : V_1 \to V_2 \mid \mathrm{rk}\, A \le 1 \}$$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr, bend left] & \\
X_1 \arrow[rr, dashed, "\text{Atiyah flop}"] \arrow[dr] & & X_2 \arrow[dl] \\
 & X_0 &
\end{tikzcd}$$

$X_1 = \{ V_1 \twoheadrightarrow L_1,\ A_1 : L_1 \to V_2 \}$, $X_2 = \{ L_2 \hookrightarrow V_2,\ A_2 : V_1 \to L_2 \}$

$$X_0 = \{ A : V_1 \to V_2 \mid \mathrm{rk}\, A \le 1 \}$$

Important property :

$$\omega_{X_1} \simeq \mathcal{O}_{X_1},\quad \omega_{X_2} \simeq \mathcal{O}_{X_2}$$

Indeed, $\omega_{X_0} \simeq \mathcal{O}_{X_0}$ since X_0 is a hypersurface in $\mathbb{A}^4$

$$\Rightarrow \mathrm{Pic}(X_1) \xrightarrow{\sim} \mathrm{Pic}(X_1 - \mathbb{P}^1) = \mathrm{Pic}(X_0 - \{0\})$$

$$\omega_{X_1} \longleftrightarrow \underbrace{\omega_{X_0 - \{0\}}}_{\text{trivial}}$$

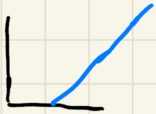
Ex. $\mathbb{G}_m \curvearrowright \mathbb{A}^2$, wts $= (1,-1)$, $\quad \mathbb{Z} \xrightarrow{\varphi^\vee} \mathbb{Z}^2$

$1 \longmapsto e_1 - e_2$

$\varphi : \mathbb{Z}^2 \to \mathbb{Z}$

$x_1 - x_2$

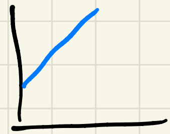
$\theta = 1$: $x_1 - x_2 = 1$, $x_2 \geq 0$



semist: $x \neq 0$

$\mathrm{Proj}\, \mathbb{C}[\underset{\deg = 0}{xy}, \underset{\deg = 1}{x}] \simeq \mathbb{A}^1$

$\theta = -1$: $x_1 - x_2 = -1$



semist: $y \neq 0$

$\mathrm{Proj}\, \mathbb{C}[\underset{\deg = 0}{xy}, \underset{\deg = 1}{y}] \simeq \mathbb{A}^1$

Ex. $T = T_1 \times \mathbb{G}_m \curvearrowright V$, $\quad T_1 \curvearrowright V$ by $\chi_1, \dots, \chi_N \in X^\cdot(T_1)$

$\mathbb{G}_m$ by rescaling

$\Rightarrow X^\cdot(T) = X^\cdot(T_1) \times \mathbb{Z}$, wts : $(\chi_i, 1)$

For $\theta = (0, 1)$, v is θ-semist. $\Longleftrightarrow$

$0 \in P_v := \text{conv. hull}\,(T_1\text{-wts}(v))$.

indeed, $n\theta = \sum n_i (\chi_i, 1) \Longleftrightarrow n = \sum n_i$

$0 = \sum n_i \chi_i$

Recall: T_1-orbit of v is closed $\Longleftrightarrow$

$0 \in$ rel. int. of P_v