GIT Exercises I.

1. For every positive integer n, give an example of a group Γ of order n and its representation V such that $\mathbb{C}[V]^G$ is *not* generated by invariants of degree less than n.

2. Let $\Gamma \subset GL(V)$ be a finite subgroup. Prove that
a) $\dim_{\mathbb{C}(V)^\Gamma} \mathbb{C}(V) = \sharp\Gamma$.
b) $\mathbb{C}(V) = \mathbb{C}(V)^\Gamma \otimes_{\mathbb{C}[V]^\Gamma} \mathbb{C}[V]$.
c) $\mathbb{C}(V)^\Gamma = \mathrm{Frac}(\mathbb{C}[V]^\Gamma)$.
d) $\dim_{\mathrm{Frac}(\mathbb{C}[V]^\Gamma)} \mathrm{Frac}(\mathbb{C}[V]^\Gamma) \otimes_{\mathbb{C}[V]^\Gamma} \mathbb{C}[V] = \sharp\Gamma$.

3. a) Prove that the *wreath product* $\mathfrak{S}_n \wr \mathbb{Z}/2\mathbb{Z} := \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ (the Weyl group of type B_n) in its natural n-dimensional representation V is a reflection group.
b) Find the generators of invariants in $\mathbb{C}[V]$ (in particular, their degrees).

4. The Weyl group of type B_n has an evident "sum" homomorphism onto $\mathbb{Z}/2\mathbb{Z}$. Its kernel K is the Weyl group of type D_n.
a) Prove that K in its n-dimensional representation V is a reflection group.
b) Find the generators of $\mathbb{C}[V]^K$ (in particular, their degrees).

5. More generally, $G(a,1,n) := \mathfrak{S}_n \wr \sqrt[a]{1}$ has an evident "product" homomorphism φ to the cyclic group $\sqrt[a]{1}$ of roots of unity. For a divisor p of a we define $G(a,p,n) \subset G(a,1,n)$ as the subgroup consisting of all g such that $\varphi(g)^{a/p} = 1$ (so that $\sharp G(a,p,n) = a^n n!/p$). The group $G(a,p,n)$ has a natural n-dimensional representation V.
a) Prove that $G(a,p,n)$ is a complex reflection group in V.
b) Find the generators of $\mathbb{C}[V]^{G(a,p,n)}$ (in particular, their degrees).

According to the Shephard-Todd classification, apart from the infinite series $G(a,p,n)$, there are 34 exceptional irreducible reflection groups (e.g. the group of symmetries of icosahedron).