<u>Example 5.1</u> A student prepares for a quiz by studying a list of ten problems. She only can solve six of them. For the quiz, the instructor selects five questions at random from the list of ten. What is the probability of event A that the student can solve all five problems on the exam?

<u>Solution:</u> The number of simple events in the sample space is

$$C_5^{10} = \frac{10!}{5!(10-5)!}.$$

The event A that the student can solve all five problems can be described in the way that the five problems are selected from the six problems she learned. Thus, the number of simple events in the event A is

$$C_5^{6} = \frac{6!}{5!(6-5)!}$$

and

$$\mathbb{P}(A) = \frac{C_5^6}{C_5^{10}}$$

<u>Example 5.2</u> We want to choose 5 people from 20 people to organize a traveling group. Are there how many different ways to choose a group?
<u>Solution:</u> In this example order is not important, therefore, there are C_5^{20}.

<u>Example 5.3</u> Suppose that 10 defective computers are included in a shipment of 1000 computers.If you test 20

computers in this 1000 computers, what is the probability of event A that you found two defective computers?
Solution:

$$\mathbb{P}(A) = \frac{C_2^{10} C_{18}^{990}}{C_{20}^{1000}}.$$

Definition 5.2 The intersection of events A and B, denoted by $A \cap B$, is all the simple events belonging to both A and B. The union of events A and B, denoted by $A \cup B$, is all the simple events belonging to either A or B. The complement of event A, denoted by A^c, is all the simple events belonging to sample space Ω but A.

Example 5.4 Consider the random experiment of rolling a fair die. Define $A = \{E_1, E_2, E_3, E_4\}$, $B = \{E_1, E_3, E_5\}$ What are $A \cap B$, $A \cup B$, and A^c?
A more general definition of probability is defined as follows:
Definition 5.3 Given a sample space Ω, if we define a function $\mathbb{P}$ on Ω by:
(1)The empty set is denoted by ϕ. $\mathbb{P}(\phi) = 0$.
(2) For a sequence of mutually exclusive events A_i, let $A = \cup_{i=1}^{\infty} A_i$, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

(3) For each event B, $0 \leq \mathbb{P}(B) \leq 1$ and $\mathbb{P}(\Omega) = 1$.
Then, $\mathbb{P}$ is called a probability on Ω.

<u>Example 5.5</u> Roll a fair die. Define $E_i = \{$ Observe a i $\}$, $A = \{$ Observe an odd number$\}$, and $B = \{$ Observe a number less or equal to 4$\}$. Find the probabilities $\mathbb{P}(A \cup B)$, $\mathbb{P}(A \cap B)$, and $\mathbb{P}(A^c)$.

$$\mathbb{P}(A \cup B) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) + \mathbb{P}(E_4) + \mathbb{P}(E_5) = \frac{5}{6}$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(E_1) + \mathbb{P}(E_3) = \frac{2}{6}$$
$$\mathbb{P}(A^c) = \mathbb{P}(E_2) + \mathbb{P}(E_4) + \mathbb{P}(E_6) = \frac{3}{6}$$

<u>Example 5.6</u> Consider an experiment of flipping a coin as many times as necessary until a head turns up. Define a probability on the sample space. Define $A = \{$ The time to first time observe a head is bigger than one$\}$. Find $\mathbb{P}(A)$.
<u>Solution:</u> Let E_i be the event of first time observing a head at i^{th} flipping. Then, $\Omega = \{E_1, E_2, \cdots\}$. Define

$$\mathbb{P}(E_i) = \frac{1}{2^i}$$

**Then, $\mathbb{P}(\Omega) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. If we define that for any given event A, $\mathbb{P}(A)$ is equal to the sum of the probabilities of the simple events in A. Then, $\mathbb{P}$ is a probability on Ω. $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}(E_1) = 1 - \frac{1}{2} = \frac{1}{2}$.
Probability of an event A is a number to indicate how big the occurring possibility of event A. Sometimes**

given an event, say B, occurs, it can affect the occurring probability of another event A.

Example 5.7 **Consider an experiment of rolling a fair die. Define the typical classical probability on the sample space. Define $A = \{$ Observe a number less or equal to 3$\}$. If we already knew that A occurred, then what is $\mathbb{P}(E_1)$.**
Solution: **If we denote that the "probability" of E_1, given that A has occurred by $\mathbb{P}(E_1|A)$. Then**

$$\mathbb{P}(E_1|A) = \frac{1}{3}$$

Definition 5.4 **The conditional probability of an event A, given that event B has occurred, is**

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B))}{\mathbb{P}(B)} \quad \text{if} \quad \mathbb{P}(B) \neq 0$$

The conditional probability of an event B, given that event A has occurred, is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B))}{\mathbb{P}(A)} \quad \text{if} \quad \mathbb{P}(A) \neq 0$$

Definition 5.5 **Two events A and B are said to be independent if and only if**

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

or

$$\mathbb{P}(B|A) = \mathbb{P}(B).$$

Otherwise, the two events are said to be dependent.

Example 5.8 Consider an experiment of rolling two fair dice . Define the typical classical probability on the sample space.

$$\{(E_1, E_1), (E_1, E_2), \cdots, (E_6, E_6)\}$$

Define $A = \{$ **Observe 1 on the first die**$\}$**. Define** $B = \{$ **Observe 2 on the second die**$\}$**. Events** A **and** B **are independent?**
Solution: First,

$$\mathbb{P}(A) = \mathbb{P}((E_1, E_1)) + \mathbb{P}((E_1, E_2)) + \mathbb{P}((E_1, E_3)) + \mathbb{P}((E_1, E_4)) + \mathbb{P}((E_1, E_5)) + \mathbb{P}((E_1, E_6)) = \tfrac{6}{36}.$$

and

$$\mathbb{P}(B) = \mathbb{P}((E_1, E_2)) + \mathbb{P}((E_2, E_2)) + \mathbb{P}((E_3, E_2)) + \mathbb{P}((E_4, E_2)) + \mathbb{P}((E_5, E_2)) + \mathbb{P}((E_6, E_2)) = \tfrac{6}{36}.$$

Second,

$$\mathbb{P}(A \cap B) = \mathbb{P}((E_1, E_2)) = \frac{1}{36}.$$

Therefore,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/36}{6/36} = \frac{1}{6} = \mathbb{P}(A)$$