The mn Rule

Consider an experiment that is performed in two stages. If the first stage can be accomplished in m different ways and for each of these ways, the second stage can be accomplished in n different ways, then there are total mn different ways to accomplish the experiment.

Example 4.4 **A driver wants to go from city A to B, then to c. If from A to B there 5 different ways and B to C there 6 different ways. Totally are there how many different ways from A to C?**

Solution: **By mn rule, there are 30 different ways.**

The Extended mn Rule

Consider an experiment that is performed in k stages. If the first stage can be accomplished in n_1 different ways and for each of these ways, the second stage can be accomplished in n_2 different ways,$\cdots$, and the k stage can be accomplished in n_k different ways, then there are total $n_1 n_2 \cdots n_k$ different ways to accomplish the experiment.

Example 4.5

There are 10 people stand in a line to take a photo. Two photos are said different if at least two people's positions are changed. How many different photos can we take?

Solution:

There are 10 different ways to choose a person to stand on the first position; After taking out the first person, we only have 9 remaining people. Then, there are 9 different ways to choose a person to stand on the second position; and so on. By the extended mn rule totally we can take $10 \times 9 \cdots \times 1 = 3628800$ different photos.

Example 4.6

Suppose that a family will certainly have 5 children, but each child being a boy or girl is totally uncertain with equal probability. How many simple events in the sample space? What are the probabilities of the following events:

a. $A = \{$Last baby is a boy$\}$.

b. $B = \{$The first and the second are girls$\}$.

Solution:

In this question, order is important. Consider that there are five positions in a line. Each position has two choices:boy or girl. By extended mn rule, the total number of simple events in the sample space is equal to $2^5 = 32$.

a. The last baby is a boy, then the last position is occupied and the first four positions have options. A has 2^4 different simple events. Therefore, the $\mathbb{P}(A) = 2^4/2^5 = 1/2$.

b. The first and second are girls, then the first and second positions are occupied and the remaining three

positions have options. B has 2^3 different simple events. Therefore, the $\mathbb{P}(B) = 2^3/2^5 = 1/4$.

Example 4.7

There are 10 people and 50 big rooms. Suppose that each person has equal probability 1/50 to go to any one of the 50 rooms. Let A={for 10 specified rooms, each of these 10 rooms has exactly one person }. Find out the probability of event A.

Solution:

By extended mn rule, the number of simple events in the sample space is 50^{10}. If we denote $n! = n \times (n-1) \times \cdots 1$, by the extended mn rule, the number of simple events in A is equal to 10!. Therefore,

$$\mathbb{P}(A) = 10!/(50^{10})$$

Definition 4.3 A permutation of n different objects is an ordering arrangement of this n objects.

Counting Rule for Permutations I The number of ways we can arrange n distinct objects is

$$P_n^n = n!.$$

Proof:

This is equivalent to the photo problem that there are n positions in a line and n different people. Two photos

are counted as different if at least two people's positions are different in the photos. How many different photos can we take? The first position can be occupied by one of n people. The second position can be occupied by remaining $n-1$ people, $\cdots$, the last position only can be occupied by the last person. Therefore, totally it is $n \times (n-1) \times \cdots, 1 = n!$.

Counting Rule for Permutations II **The number of ways we can arrange n distinct objects, taking them r at a time, is**

$$P_r^n = n!/(n-r)!.$$

Proof:

This is equivalent to the photo problem that there are r positions in a line and n different people. Two photos are counted as different if at least two people's positions or two people are different in the photos. How many different photos can we take? The first position can be occupied by one of n people. The second position can be occupied by remaining $n-1$ people, $\cdots$, the last position, r^{th} position, can be occupied by one of remaining $n-(r-1)$ people. Therefore, by extended mn rule, totally it is

$$n \times (n-1) \times \cdots, n-(r-1) = n \times (n-1) \times \cdots \times (n-r+1)$$
$$= \frac{n \times (n-1) \times \cdots \times (n-r) \times (n-r-1) \times \cdots \times 1}{(n-r) \times (n-r-1) \times \cdots \times 1}$$
$$= n!/(n-r)!$$

<u>Example 4.8</u>
There are three letters A, B, C. How many different ordered, two letter alphabets can we get?
<u>Solution:</u>

$$AB \quad AC \quad BC$$
$$BA \quad CA \quad CB$$

According to the permutation counting formula

$$P_2^3 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 6$$

<u>Example 4.9</u>
There are 26 letters $A, B, C, \cdots, X, Y, Z$. How many different ordered, five letter alphabets can we get?
<u>Solution:</u>
According to the permutation counting formula

$$P_5^{26} = \frac{26!}{(26-5)!} = 7893600$$

A more general definition of probability is as follows.

<u>Definition 4.4</u> Given a sample space $\Omega = \{E_1, E_2, \cdots, E_n\}$ with n simple events, where n is finite or infinite. if we define a function $\mathbb{P}$ on Ω by:
(1)The empty set is denoted by ϕ. $\mathbb{P}(\phi) = 0$.
(2) For each simple event E_i, its probability $\mathbb{P}(E_i)$ is defined and $0 \leq \mathbb{P}(E_i) \leq 1$. Thus, "equally likely" is a

special case of this definition.
(3) For each event A, $\mathbb{P}(A)$ is equal to the sum of the probabilities of simple events contained in A.
Then, $\mathbb{P}$ is called a probability on Ω.

Example 4.10
In a pocket there are 3 black and 2 white balls. Balls are identical except their colors. Randomly drawing a ball and observing its color. How many simple events in the sample space? Can you define a reasonable probability on the sample space? Is this equally likely?
Solution:

$$\Omega = \{W, B\}$$
$$\mathbb{P}(W) = \frac{2}{5}, \quad \mathbb{P}(B) = \frac{3}{5}.$$

Definition 5.1 A combination is an order ignored selection of objects from a larger group of objects.

We know that the number of different permutations of r different objects is

$$P_r^r = r!.$$

The P_r^n can be thought as the multiplication of two numbers:
(1) number of ways to select r different objects from n different objects. We denote this number by C_r^n
(2) number of ways to get different arrangements for each selected r objects. This is just P_r^r.

Therefore,

$$P_r^n = C_r^n \times P_r^r$$

or

$$C_r^n = P_r^n / P_r^r = \frac{n!}{r!(n-r)!}$$

<u>Counting Rule for Combination</u> The number of different combinations of n different objects that can be formed, taking them r at a time, is

$$C_r^n = \frac{n!}{r!(n-r)!}$$