Quasi-split iquantum group definitions

1. Let Q be a loop-free quiver, vertex set I, $\#(i \to j)$ arrows from vertex i to vertex j.

2. The corresponding Cartan matrix $A = (a_{i,j})_{i,j\in I}$ is

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j \\ -\#(i \to j) - \#(j \to i) & \text{if } i \neq j. \end{cases}$$

3. Fix a realization: free Abelian groups X and Y, a perfect pairing $Y \times X \to \mathbb{Z}$, linearly independent simple coroots $h_i \in Y$ and simple roots $\alpha_j \in X$ such that $h_i(\alpha_j) = a_{i,j}$.

4. Let $\tau : I \to I$ be an involution such that $\#(i \to j) = \#(\tau j \to \tau i)$. Let $\tau : X \to X$ be an involution such that $\tau(\alpha_i) = \alpha_{\tau i}$ and $\tau^*(h_i) = h_{\tau i}$. Let

$$\varsigma_i := \begin{cases} -1 & \text{if } i = \tau i \\ \#(i \to \tau i) & \text{if } i \neq \tau i. \end{cases}$$

5. Let $X^\imath := X/\operatorname{im}(\mathrm{id} + \tau)$ be the iweight lattice. For $\lambda \in X^\imath$ with pre-image $\hat{\lambda} \in X$ let

$$\lambda_i := (h_i - h_{\tau i})(\hat{\lambda}) \in \mathbb{Z}.$$

6. The (modified) iquantum group $\dot{\mathrm{U}}^\imath$ is the locally unital $\mathbb{Q}(q)$-algebra, mutually orthogonal distinguished idempotents 1_λ ($\lambda \in X^\imath$), generators $b_i 1_\lambda = 1_{\lambda-\alpha_i} b$ ($i \in I, \lambda \in X^\imath$), relations

$$\sum_{n=0}^{1-a_{i,j}} (-1)^n b_i^{(n)} b_j b_i^{(1-a_{i,j}-n)} 1_\lambda = \delta_{i,\tau j} \prod_{r=1}^{-a_{i,j}} (q^r - q^{-r}) \cdot \frac{(-1)^{a_{i,j}} q^{\lambda_i - \varsigma_i - \binom{a_{i,j}}{2}} - q^{\binom{a_{i,j}}{2} + \varsigma_i - \lambda_i}}{q - q^{-1}} b_i^{(-a_{i,j})} 1_\lambda$$

for $i \neq j$ in I and $\lambda \in X^\imath$. Here, $b_i^{(n)} 1_\lambda$ is the divided power defined by the recurrence relation

$$b_i b_i^{(n)} 1_\lambda = \begin{cases} [n+1] b_i^{(n+1)} 1_\lambda + [n] b_i^{(n-1)} 1_\lambda & \text{if } i = \tau i \text{ and } n \equiv h_i(\hat{\lambda}) \pmod 2 \\ [n+1] b_i^{(n+1)} 1_\lambda & \text{otherwise.} \end{cases}$$

Divided powers generate a $\mathbb{Z}[q, q^{-1}]$-form $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ for $\dot{\mathrm{U}}^\imath$.

7. Additional 2-iquantum group parameters: A ground field $\Bbbk$ of characteristic $\neq 2$.
A normalization homomorphism $c_i : X \to \Bbbk^\times$ such that

- $c_i(\alpha_j) = (-1)^{\#(j \to i)}$ for all $i, j \in I$;
- $c_{\tau i}(\tau(\lambda)) = (-1)^{h_i(\lambda)} c_i(\lambda)$ for all $\lambda \in X$ and $i \in I$ with $i \neq \tau i$.

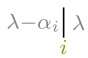
Let $\zeta_i := \pm 2^{\varsigma_i}$ choosing the signs of ζ_i and $\zeta_{\tau i}$ so that exactly one is minus. Let

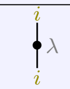
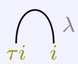
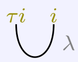
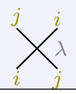
$$\gamma_i(\lambda) := \begin{cases} c_i\big(\hat{\lambda} - \tau(\hat{\lambda})\big) & \text{if } i \neq \tau i \\ (-1)^{h_i(\hat{\lambda})} & \text{if } i = \tau i, \end{cases}$$

Most important, let

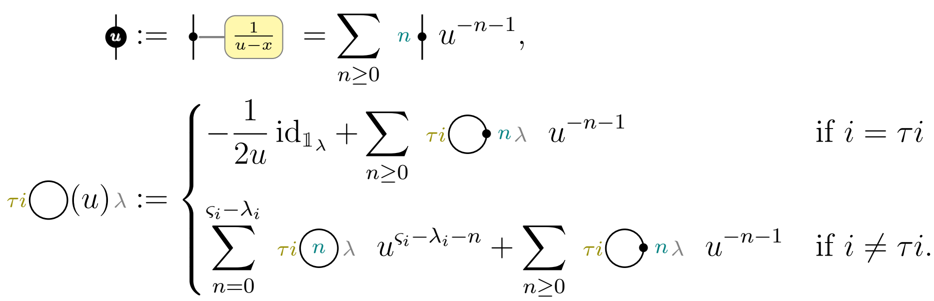
$$Q_{i,j}(x,y) := \begin{cases} (x-y)^{\#(i\to j)} (y-x)^{\#(j \to i)} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases} \quad R_{i,j}(x,y) := \begin{cases} Q_{i,j}(x,y) & \text{if } i \neq j \\ 1/(x-y)^2 & \text{if } i = j, \end{cases}$$

$$Q^\iota_{i,j}(x,y) := (-1)^{\delta_{i,\tau j}} Q_{i,j}(x,y),$$

8. The 2-iquantum group $\mathfrak{U}^\imath$ is the graded $\Bbbk$-linear 2-category with object set $X^\imath$, generating 1-morphisms $B_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}B_i : \lambda \to \lambda - \alpha_i$ for $\lambda \in X^\imath$ and $i \in I$, identity 2-endomorphisms denoted by strings , and generating 2-morphisms

Generator	Degree
 $: B_i\mathbb{1}_\lambda \Rightarrow B_i\mathbb{1}_\lambda$	2
 $: B_{\tau i}B_i\mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$1+\varsigma_i-\lambda_i$
 $: \mathbb{1}_\lambda \Rightarrow B_{\tau i}B_i\mathbb{1}_\lambda$	$1+\varsigma_i-\lambda_i$
 $: \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$ for $0 \le n \le \varsigma_i-\lambda_i$	$2n$
 $: B_iB_j\mathbb{1}_\lambda \Rightarrow B_jB_i\mathbb{1}_\lambda$	$-a_{i,j}$

Relations are expressed using dot and bubble generating functions (formal series in u^{-1}):



Also x, y, z denote dots on strings in order from left to right. Defining relations: