Abstract

This document addresses the problem that our elementary school math education system is not as successful as many people would like it to be, and it is not as successful as it could be. This document was written specifically for use in a Math Methods course for preservice elementary school teachers. However, it is also designed for use by inservice elementary school teachers and for students enrolled in a Math for Elementary Teachers course.

A person’s math expertise is a combination of math content knowledge and skills, and math maturity. In the United States, math content to be taught and learned is specified in a wide variety of state and national standards, and it is strongly based on the standards developed by the National Council of Teachers of Mathematics (NCTM, n.d.).

Math maturity focuses on areas such as understanding, problem solving, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning and making connections (being able to use one’s mathematical knowledge over a wide range of disciplines and in novel settings), and interest (including intrinsic motivation) in math.

Recent progress in Brain Science and Computer Science, when combined with thousands of years of progress in the teaching and learning of math, provides a foundation for significant improvements in our math education system. The goal of this document is to improve your preparation to be a good math teacher—and thus, to improve the math education of your future students. The main focus is on helping you better understand and apply some ideas about math maturity and some of the modern related topics on brain and mind science.
Increasing the Math Expertise of Elementary School Students and Their Teachers

Introduction

This document was specifically written for students enrolled in a Math Methods course for preservice elementary school teachers. Its goal is to improve your preparation to be a good math teacher—and thus, to improve the math education of your future students.

At the current time, reading and math are “the” two most emphasized components of the elementary school curriculum. Throughout our country there is a top down movement to establish high standards for student achievement in these two areas, and to improve our educational system so that these high standards are met.

Both areas evoke considerable controversy. In essence, the issues are what the standards should be (what students should learn), and how students should be taught. In reading, there is considerable agreement about goal of having students achieve an adequate level of reading fluency (speed, accuracy, comprehension) by the end of the third grade so that they can begin to make effective use of reading as an aid to learning throughout the curriculum. The controversy tends to lie in teaching methods, such as phonics versus whole language.

In math, both the content and the pedagogy issues remain unresolved. However, there is considerable agreement that the results being produced by our math education system, whether the approach is “back to basis” or “new-new math,” is not nearly as successful as many people would like.

For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them. Though the same topics are taught and retaught year after year, the students do not learn them. Numerous scientific studies have shown that traditional methods of teaching mathematics not only are ineffective but also seriously stunt the growth of students' mathematical reasoning and problem-solving skills. Traditional methods ignore recommendations by professional organizations in mathematics education, and they ignore modern scientific research on how children learn mathematics (Battista, 1999).

Reading, writing, and mathematics are closely connected. The Sumerians (who lived in the area that is now Iraq) developed writing about 5,200 years ago (Acosta, n.d.). This soon led to the development of schools and formal schooling to teaching reading, writing, and arithmetic. It also led to a steady accumulation of human knowledge, including math.

Mathematics is one of humanity's great achievements. By enhancing the capabilities of the human mind, mathematics has facilitated the development of science, technology, engineering, business, and government. (Kilpatrick, Swafford, and Findell, 2002)

Math is so important in our society that children begin to learn math well before they enter kindergarten, and math is a required part of the school curriculum well into high school. Most colleges require students to take some math, and most likely you have taken a sequence of courses titled Math for Elementary Teachers. Your informal and formal studies and use of math have led to your current level of math expertise.

In this document, I divide math expertise into two major components: math content and math maturity. Much of the math coursework you have taken focused on math content—for example, learning many different arithmetic, algebraic, and geometric procedures and how to use these procedures to solve a wide range of math problems.
Math maturity focuses on areas such as understanding, solving problems you have not previously encountered, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning (being able to use one’s math knowledge and make math connections over a wide range of disciplines and in novel settings), and interest (including intrinsic motivation) in math. I will discuss math maturity more in a later section.

To be an effective teacher of math, you need both math content knowledge and math maturity. In addition, you need to know how to teach math—that is, you need math pedagogical knowledge designed to help your students learn math content and gain in their math maturity. Research by Liping Ma (2002) and others suggests that the majority of elementary school teachers in the United States are relatively ill prepared in math pedagogy.

Elementary school teachers tend to teach math in the way that they were taught. That is, much of what you know about being a teacher of elementary school mathematics you learned while you were in elementary school. This creates a cycle in which the next generation of students is taught in much the same manner as the previous generation. This cycle can and must be broken if the quality of math education that our students receive is to be significantly improved. You, personally, can make a significant difference for your students. The ideas presented in this document will help you.

Interspersed throughout this document are a number of activities. These are designed for use in personal reflection, in small group discussions, in whole class discussions, and in written assignments.

Activity 1a. Think about your own elementary school math education experiences and what you have observed in visits to elementary school since then. What seems to be working well and what does not seem to be working well? Be as specific as possible.

Activity 1b. You are currently taking a math pedagogy course. Prior to this you have had years and years of instruction in mathematics. When I think about this, I conclude that most of what you will know about math pedagogy will have come from what you happened to pick up through your years and years of math coursework. Share your thinking about this situation. What might you be learning in your teacher education program of study that will help to break the model of teachers teaching math in the way that they were taught?

Academic Disciplines

Elementary school teachers are responsible for teaching a wide range of disciplines such as art, language arts, math, music, science, and social science. Although the focus of this document is on math, let’s begin by taking a more general approach. What is a discipline, and how does one distinguish between disciplines?

Each discipline can be defined by its unique combination of:

- The types of problems, tasks, and activities it addresses.
- Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
- Its history, culture, language (including notation and special vocabulary), and methods of teaching, learning, and assessment.
- Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.
When you read this list, did you just “bleep” over the details, or did you pause at each bulleted item and reflect on its meaning to you and to our educational system? Did you select a discipline that you know well and check on your insights into how each of the listed items fits or fails to fit the discipline? Did you think about what items you think should be added to the list, and what might be deleted?

Just for the fun of it, think about your knowledge of the accumulated accomplishments in medicine. You know quite a bit about a wide range of diseases, germs, bacteria, viruses, a wide range of drugs and vaccines, various types of surgery, and so on. You know some things about DNA, cloning, and genetic engineering. Your accumulated knowledge in medicine is well beyond that of the best physicians and medical researchers of a few hundred years ago.

Now, contrast that with your knowledge of the accumulated accomplishments in math. Can you name some of the accumulated accomplishments of math, and how does your list compare to your knowledge of medicine? (Remember, Isaac Newton and others developed calculus about 350 years ago, and its mathematical foundations go back a long time before then.)

What might you conclude from this activity? Medicine is an important and ongoing part of your life. You have learned a lot about the discipline of medicine through your informal efforts and the efforts of our schools. This is because medicine is relevant to your everyday life. Think about what aspects of math are relevant to your everyday life. Think about what aspects of math are relevant to the everyday lives of elementary school students. What might you and other elementary school teachers do to make math more relevant across the entire curriculum and in the lives of your students?

This document contains a relatively high density of Big Ideas. If you read this document in the same manner and at the same rate as you read a short story or a novel, you will gain very little from it. To gain appreciable benefit from reading this document you will need to read in a reflective manner, pausing frequently to think about what you already know and how it fits in with what you are reading. You will need to construct meaning from the written materials that integrates into and adds to your current knowledge and understanding. That is, you will need to practice constructivist learning (Ryder, n.d.).

In essence, that is what the learning theory called constructivism is all about. Constructivism is a learning theory applicable to learning in each discipline. It is particularly important in the teaching and learning of math. Thus, you might want to spend a little time thinking about your preparation to help your future students learn math in a constructivist manner (Math Forum, n.d.).

The activities in this document are intended to encourage you to think, and to think about your thinking. Thinking about your thinking is called metacognition. It is an important component of formal and informal education at all grade levels.

More specifically, the ideas in this document are intended to encourage you to think about what you, personally, can do to improve our educational system. If this document does not lead to you, personally, making changes designed to improve upon the “traditional” curriculum and methods of instruction, then this document will have failed as an aid to improving your education.

Teaching is a very challenging and demanding profession. Good teachers are always learning and growing professionally. You may find it useful to make a copy of the bulleted list so you can
refer back to it as you develop lesson plans and as you engage in your everyday activities as a (constructivist) teacher.

Moreover, you should structure your professional career as a teacher to allow significant time for learning. There is a huge amount of research and practitioner knowledge on the craft and science of teaching and learning. Bransford (2000) provides an excellent overview of this field, and the book can be read free from the Website listed in the reference. On September 30, 2004 the National Science Foundation announced it had committed $36.5 million to fund three major research centers in the area of Learning About Learning (NSF, 2004). Quoting from this announcement:

How do we learn? This most fundamental ability comes about through the complex interplay of genes, brain-based neural mechanisms, developmental trajectories, and social and physical environments. These processes of learning are just beginning to be understood. A deeper understanding of learning will allow scientists and educators to devise methods for improving how humans learn and develop machines that can perform tasks intelligently and independently.

NSF has launched the new Science of Learning Centers to meet the challenge of learning about learning. Their goal is to make new discoveries about the foundations of learning across a wide range of learning situations—from processes at the cellular level to complex processes engaging different brain areas, to behaviors of individuals, to interactions in the classroom, to learning in informal settings, to learning performed by computer algorithms.

Activity 2a. Spend some time thinking about the bulleted “discipline” list from the point of view of your preparation to teach the various subjects you will teach in elementary school. Select the discipline that you feel you know best, and summarize your discipline-specific knowledge and skills from the point of view of the four bulleted items. Then do a compare and contrast with a second discipline. Share some of your insights and feelings from doing this activity.

Activity 2b. Think about the elementary school math curriculum that was in place when you were in school and/or that you have observed in more recent visits to schools. Discuss some of the aspects of the discipline of mathematics that are in the curriculum, and some aspects that you feel should be added to the curriculum.

Activity 2c. Suppose that you stopped reading this document right after reading Activity 2c. Name some things that you have learned that can make a significant contribution to improving the quality of education that your future students will receive.

The Discipline of Mathematics

It is not easy to give a useful and simple answer to the question: What is mathematics? Many mathematicians and math educators have attempted to answer this question. Here are a few examples.

I remember discussing with some colleagues, early in our careers, what it was like to be a mathematician. Despite obvious individual differences, we had all developed what might be called the mathematician's point of view—a certain way of thinking about mathematics, of its value, of how it is done, etc. What we had picked up was much more than a set of skills; it was a way of viewing the world, and our work. We came to realize that we had undergone a process of acculturation, in which we had become members of, and had accepted the values of, a particular community. As the result of a protracted apprenticeship into mathematics, we had become mathematicians in a deep sense (by dint of world view) as well as by definition (what we were trained in, and did for a living). (Schoenfeld, 1992.)

Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in
systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, 1992).

Mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, logic, application, and principles. Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, mathematical education should be centered on encouraging students to think for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof—the demand for succinct argument that from a logical foundation for the veracity of a claim (Padraig & McLoughlin, 2002).

The terms fluency and proficiency are often used in talking about goals and expertise in mathematics. The following definition is quoted from Adding It Up (2001), a report written for the National Academy of Sciences.

Mathematical proficiency, as we see it, has five components, or strands:

• conceptual understanding—comprehension of mathematical concepts, operations, and relations
• procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
• strategic competence—ability to formulate, represent, and solve mathematical problems
• adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
• productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

Warning!!! The materials that you have just read in this section reflect many hundreds of hours of thinking by some of the world’s leading math educators. Did you read it in a reflective manner? Did you work to construct your own meaning? What aspects of the presented ideas will you remember five minutes from now, a day from now, a year from now?

Many people argue that our math education system is not as good as it could be. For example:

Developing proficiency in mathematics is important for all students. However, when considered in light of current standards, or compared with performance in other countries, evidence on student achievement in mathematics makes clear the need for substantial improvement. U.S. students do not, as a group, achieve high levels of mathematical proficiency. The nation must seek to narrow the achievement gaps between white students and students of color, between middle-class students and students living in poverty; gaps that have persisted over the past decade. To address these problems, the federal government and the nation’s school systems have made and are continuing to make significant investments in the improvement of mathematics education. However, the
knowledge base on which these efforts are founded has often been weak and speculative (Ball, 2002).

If you look at state and national assessments of math and science competence among our country's elementary and secondary schools today, you'll discover small pockets of excellence amid a broad swath of mediocrity. In fact, only a minority of U.S. students are meeting math and science proficiency benchmarks.

International assessments from the Trends in International Mathematics and Science Study (TIMSS) show U.S. students are at or below the international average and significantly behind their peers in Japan and Canada. TIMSS compared our most advanced students with those from 15 other nations, and the brightest U.S. students scored dead last against international competitors in advanced math and physics assessments (Ruetters, 2002).

There is a significant battle going on between the “back to basics” math education reformers, and the “new math (new-new math) reformers. The Mathematically Correct (n.d.) Website presents arguments against the ideas of the new-new math education reformers. Quoting from their Website:

Mathematics achievement in America is far below what we would like it to be. Recent "reform" efforts only aggravate the problem. As a result, our children have less and less exposure to rigorous, content-rich mathematics .

The advocates of the new, fuzzy math have practiced their rhetoric well. They speak of higher-order thinking, conceptual understanding and solving problems, but they neglect the systematic mastery of the fundamental building blocks necessary for success in any of these areas. Their focus is on things like calculators, blocks, guesswork, and group activities and they shun things like algorithms and repeated practice. The new programs are shy on fundamentals and they also lack the mathematical depth and rigor that promotes greater achievement.

The Standards produced by the National Council of Teachers of Mathematics (NCTM, n.d.) represent the sense of direction of new-new math reform. The NCTM Standards are divided into five content standards and five process standards. None of the ten standards say anything about computation in their titles. The ten NCTM Standards contain a total of 33 goals. Exactly one of the 33 goals talks about computation—the traditional focus of much of the elementary school math curriculum! This particular goal statement is the Numbers and Operations standard, and it says “compute fluently and make reasonable estimates.”

More general information about proposed math reforms is available in Mathematically Sane (n.d.).

Activity 3a. As an elementary school teacher, you will likely encounter both the back to basics approach to math education and the new-new math approach to math education. Think about how easy it is to fall back into the mode of teaching the way you were taught (thus, revert to a focus on computation and the other basics), versus learning and teaching a new-new math curriculum. Spend some time making a list of topics and ideas that you feel are new-new math, and topics and ideas that you feel are stressed by the back to basics movement.

Activity 3b. Review the “what is math” quotations given in this section. Think about which (if any) of the ideas in these quotations can be integrated into elementary school math. Think about this from the point of view of “The way the twig is bent is the way the tree will grow.” Argue for and against the idea that elementary school math should place much less emphasis on paper and pencil computation and much more emphasis on topics that lay a different type of foundation for students as they continue to study math in middle school, high school, and beyond.
Activity 3c. Name one big and important idea from this section that you are apt to remember and make use of during your student teaching and your first year on the job. What is it about this idea that resonates with you and is likely to stay with you?

**Mathematical Expertise**

One of your goals as a teacher is to help your students increase their levels of expertise within the various disciplines you teach. An expertise scale like the one pictured in Figure 1 is applicable to you in each academic discipline you are preparing to teach. Roughly speaking, such a scale covers all the way from a beginner (novice) to a person who is considered to be a world-class expert. For most preservice and inservice teachers, their current level of expertise in any particular discipline is well above the novice level and well below the world-class level.

![Single Topic Expertise Scale for a Teacher](image)

Figure 1: Expertise scale.

One of the Big Ideas in education is self-assessment. You can practice self-assessment on various aspects of your current preparation to be an elementary teacher. This will help you to decide where to place the most emphasis as you continue your studies, do your student teaching, and then proceed in your chosen profession. But, you should also think about the training, education, and experience that you have received in the past in doing self-evaluation. If it has not been up to the standards that you feel are appropriate, then you will certainly want to remedy this situation for the students you teach in the future. Helping students to increase their skills in self-evaluation is an important goal in education.

The previous section contains various definitions of math. As noted earlier, math expertise can be broken into two major components, as illustrated in Figure 2.
There is considerable agreement about the scope and sequence of PK-12 math education in
the US, for example, and the undergraduate college math curriculum is relatively standardized
throughout this country. Clearly, one measure of a person’s progress toward increasing Math
Expertise is the level of coursework that has been completed, the grades received in these
courses, and the quality and rigor of the coursework. However, math can be learned through
other ways than just taking courses. Moreover, there is a large amount of math that is not
included in the commonly available coursework.

And (long pause, drum roll), many students forget most of the math they have “covered” in
their math courses. Teachers of math tend to be driven by the need to “cover” the curriculum, to
“get through” the book and the planned lessons. They do this even though they know that
students will forget most of what is covered. As I reflect about this situation, I tend to feel
embarrassed about the teaching that I have done over the years. Perhaps you have been in
courses that seem to speed up near the end of the term in order to “cover” the “required”
curriculum.

**Math Maturity**

There is much less agreement on the meaning of Math Maturity, even though the term is
widely used by mathematicians and math educators. Math Maturity is **not** primarily knowledge of
specific math content areas or skill in memorizing and accurately using arithmetic and other math
procedures.

Here is a list of some possible components of Math Maturity. Note that one can argue that
each is “merely” a component of Math Content Knowledge. However, when people use the term
Math Maturity they tend to be interested in those aspects of the topics listed below that are not
dependent on specific Math Content Knowledge.

1. An understanding of the math that one has had adequate opportunities to learn. A good
   way to think about this is in terms of lower-order versus higher-order knowledge and
   skills. Bloom’s Taxonomy, developed about 50 years ago, is still a useful aid in
   understanding lower-order and higher-order (Bloom’s Taxonomy, n.d.).

2. Considering mathematics as a language suggests three related components of
   Mathematical Maturity:
A. Mathematical speaking and listening fluency.

B. Mathematical reading and writing fluency.

C. Thinking and reasoning in the language of mathematics. Gary Marcus (2004, p. 124) discusses roles of language and thought. Thought and language are loosely connected. Mathematicians and other people clearly develop and make use of mental representations that are not words. For example, Albert Einstein, when describing his discovery of special relativity said: “Words and sentences, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thoughts are certain signs or images, more or less clear, that I can reproduce and recombine at will.” (Marcus, 2004, p219.)

3. Ability to pose and represent math problems, and to ask insightful mathematical questions. This includes the ability to recognize math aspects of a problem situation in a wide range of disciplines and represent them mathematically.

4. Ability to effectively use one’s Math Content Knowledge to solve or help solve the types of math problems that arise in (3) above. Making connections within mathematics, and transfer of one’s math learning to other disciplines.

5. Ability to learn mathematics, and to build upon one’s current mathematical knowledge. In the field of reading, people talk about learning to read and then reading to learn. In our current education system approximately 70% of students learn to read well enough by the end of the third grade so that they can use their reading knowledge as a significant aid to learning in other disciplines. We can think about “learning to math and them mathing to learn.”

6. Other factors affecting Math Maturity include attitude, interest, motivation, focused attention, perseverance, and acceptance of and fitting into the “culture” of mathematics.

   Lets look at a few examples to help illustrate some of the components and concepts of Math Maturity. Consider your content knowledge of multi-digit multiplication and long division. You have memorized and extensively practiced paper and pencil algorithms for these two mathematical operations. How do these two algorithms relate to your understanding of arithmetic, math in general, connections within math, transfer outside of math, and so on? How might one go about teaching multiplication and division so that students “really understand” these two topics? Or, perhaps you believe that learning math is mainly an activity of memorizing without understanding, and that the math education you received while in elementary school was just fine? How might handheld calculators fit into this discussion?

   As a second example, consider the idea of dividing by fractions. Probably you have memorized an algorithm that is summarized by “invert and multiply.” Think about your understanding of this procedure. Can you explain it, use it, and justify or prove it? Is it applicable in algebra, or does it just work in arithmetic? Can you give practical examples that are meaningful to yourself and to the students you will teach of when and why one might want to divide by a fraction?

   Next, consider what you know about plane geometry. The chances are that you had a year length course on this topic while in high school, and that you had additional instruction on this topic while taking a Math for Elementary Teachers course. Think about what you remember, and
attempt to divide it into the “content” and “maturity.” This is not an easy task. But, for example, perhaps you remember the general concept that there are theorems and that you studied proofs of many different theorems. Perhaps you remember that there are paper and pencil straight edge and compass constructions. There is a good chance that you remember some of the vocabulary, but that you may not remember how to prove very many theorems or do very many of the constructions. What are your current thoughts on why one might want to learn to state and prove some theorems in plane geometry and to be able to do straight edge and compass constructions?

**Activity 4a.** Think about the mathematics instruction you received while in school and college. Focus specifically on those aspects of your math education that seemed to be designed to increase your math maturity. Name some of these activities and analyze their effectiveness. For example, have you received specific instruction on how to read math, how to learn math, and how to retrieve math information from reference books and the Web? Can you explain what a variable is, what a function is, and the importance of these two topics in math?

**Activity 4b.** Likely you remember that the square roots of 16 are +4 and –4. Think about your content knowledge of square root and your math maturity in this topic area. In your thinking, pay special attention to the fact that paper and pencil algorithms for calculating square roots were once a standard part of the second year high school math curriculum, and that such students also learned to make use of math tables and “extrapolation” to determine square roots. These two computational techniques have been dropped from the curriculum because of inexpensive and readily available handheld calculators.

**Problem Solving**

Here is a definition of the word *problem* that I have found useful in my teaching of preservice and inservice teachers at all grade levels and in a variety of subject areas:

1. You (personally) have a problem if the following four conditions are satisfied:
   1. You have a clearly defined given initial situation.
   2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single goal problems.
   3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. There may be specified limitations on resources, such as rules, regulations, and guidelines for what you are allowed to do in attempting to solve a particular problem.
   4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, and energy, to achieve the desired final goal.

The fourth component of this definition is particularly important. Unless a student has ownership—an appropriate combination of intrinsic and extrinsic motivation—the student does not have a problem. Motivation, especially intrinsic motivation, is a huge topic in its own right, and I will not attempt to explore it in detail in this paper. Perhaps it suffices to say that many teachers are not very successful in helping their students to develop intrinsic motivation in their math studies. As students progress through elementary school and into secondary school, the math problem solving that they study seem to have less and less meaning and intrinsic motivation for them.

As noted earlier in this document, problem solving lies at the core of each discipline. Perhaps you have heard people ask questions such as “Why do I need to study math?” or “Why do I need to study xxxx?” While there are many possible answers to such questions, a unifying answer is that by doing so you will be able to solve a variety of problems that you cannot currently solve.
You will learn about some of the important accomplishments within the discipline, some of its history, and some of its language. As you learn the language and notation, you will get better in making use of and building on the accumulated knowledge of the discipline. You will learn to precisely represent problems to be solved and tasks to be accomplished so that you can communicate your needs and interests to other people and to Information and Communications Technology (ICT) systems. ICT provides powerful information retrieval systems (an aid to building on the previous work of others) as well as tools that can solve or greatly aid in solving a wide range of problems.

George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. *The Goals of Mathematical Education* (Polya, 1969) is a talk that he gave to a group of elementary school teachers.

To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.

... However, we have a higher aim. We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures.

Polya’s comments focus on problem solving and abstraction. Later in this document I will present some ideas from Piaget and others on cognitive developmental theory. Problem solving and abstraction lie at the Formal Operations end of the Piagetian scale for cognitive development. As we teach math, we are attempting to help students move up this cognitive development scale.

One of the most important ideas in problem solving is to build on the previous work of yourself and others. That is, one way to solve a problem is to retrieve from your own memory either a solution to the problem or a method for solving the problem. Another way is to retrieve this information from another person, from a book, from a machine such as a cash register, or from a calculator or a computer. If you are repeatedly faced by a particular problem or type of problem, it is very useful to memorize one or more solutions to the problem, or a general method for solving the problem in a timely fashion.

However, even though the human brain has a huge capacity for memorization, it takes most people a long time to memorize “stuff” and most people tend to forget the “stuff” that they do not frequently use. Rote memorization of data, information, and knowledge that you need to use frequently is a very important aspect of informal and formal education. However, it is a totally inadequate substitute for learning how to make use of your memorized knowledge to solve
problems. Also, it is a totally inadequate substitute for learning to access and effectively use data, information, and knowledge stored in other people’s brains and in machines such as calculators and computers.

The research literature on problem solving is quite large, and math education includes a number of strategies for attaching math problems. This is a large topic and an important component of any math or math education course. While this topic is beyond the scope of this document, all readers should be interested in Polya’s (1957) general strategy for attempting to solve any math problem. I have reworded his strategy so that it is applicable to a wide range of problems in a wide range of disciplines—not just in math. This six-step strategy can be called the Polya Strategy or the Six Step strategy. Note that there is no guarantee that use of the Six Step strategy will lead to success in solving a particular problem. You may lack the knowledge, skills, time, and other resources needed to solve a particular problem, or the problem might not be solvable.

1. Understand the problem. Among other things, this includes working toward having a well-defined (clearly defined) problem. You need an initial understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary. You need to make a personal commitment (Ownership) to solving the problem.

2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task?

3. Think carefully about possible consequences of carrying out your plan of action. Focus major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 as a consequence of this thinking.

4. Carry out your plan of action. Do so in a thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that this reflective thinking leads to increased expertise.

5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:
   A. If the problem has been solved, go to step 6.
   B. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.
   C. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.

6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think
about how your increased knowledge and skills can be used in other problem-solving situations. (Work to increase your reflective intelligence!)

Many of the steps in this six-step strategy require careful thinking. However, there are a steadily growing number of situations in which much of the work of step 4 can be carried out by a computer. The person who is skilled at using a computer for this purpose may gain a significant advantage in problem solving, as compared to a person who lacks computer knowledge and skill.

I find the diagram given in Figure 3 to be particularly useful when I talk about computers and math problem solving at the precollege level. With some effort, this diagram can be modified to fit problem solving in other disciplines.

![Figure 3. Math problem solving.](image)

The six steps illustrated are 1) Problem posing and problem recognition; 2) Mathematical modeling; 3) Using a computational or algorithmic procedure to solve a computational or algorithmic math problem; 4) Mathematical "unmodeling"; 5) Thinking about the results to see if the Clearly-defined Problem has been solved; and 6) Thinking about whether the original Problem Situation has been resolved. Steps 5 and 6 also involve thinking about related problems and problem situations that one might want to address or that are created by the process or attempting to solve the original Clearly-defined Problem or resolve the original Problem Situation.

In some sense, all of the steps except (3) involve higher-order knowledge and skills. They require a significant level of math maturity. Step (3) lends itself to a rote memory approach. It is highly desirable that students develop speed and accuracy in certain types of mathematical operations. However, the human mind is not good at memorizing math procedures and then carrying them out rapidly and accurately with the assistance of pencil and paper. On the other hand, calculators and computers are really good at carrying out math procedures.

PK-12 teachers who teach math tend to estimate that about 75% of the math education curriculum focuses on (3). [Note: This is an estimate I have made based upon working with a very large number of teachers. I don’t know of any published research that backs up my assertion.] This leaves about 25% of the learning time and effort focusing on the remaining five
steps. Appropriate use of calculators and computers as tools, and Computer-Assisted Learning, could easily decrease the time spent on (3) to 50% or less of the total math education time. This would allow a doubling of the time devoted to instruction and practice on the higher-order knowledge and skill areas.

**Activity 5a.** You know that there are 50 states in the United States, that each has a geographical location, Governor, state capital, two Senators, a number of Representatives, and so on. Think about what data about each state is worthwhile for most students to memorize. As you do this, think about the concepts such as geographical location, state capital, government and governmental officials, and so on. If a person learns the concepts, then information about specific details can be retrieved relatively quickly from the Web or other resources. What are your current thoughts on what to memorize and what to “understand” and be able to look up? What would it take to change your current position?

**Activity 5b.** Think about some “real world” math problems that you have encountered recently. How did you go about solving these problems? For example, which did you solve by quick recall of memorized information, on which did you seek help on, on which did you make use of calculators or computers, and what other approaches did you use?

**Mind Science; Intelligence**

Historically, the study of the human brain (one of a person’s organs) and the study of the human mind (think of the mind as a product of the brain) have been distinct disciplines. Computer-oriented people tend to think of the brain as hardware (wetware) and the mind as software.

This document contains one section on the mind and a separate section on the brain. In essence, the study of the mind is part of the field of psychology, while the study of the brain is part of the discipline of neuroscience. In recent years, the mind and brain disciplines have begun to merge, so these two sections contain some overlap.

Intelligence is the ability to learn and to take actions that make use of one’s learning. Clearly, intelligence is not limited just to humans. However, the ability to learn a natural language such as English demonstrates a very high level of intelligence on the intelligence scale of all life on earth. That is, your students vary in intelligence but are all highly intelligent on the scale of all intelligent creatures on earth.

For many years, psychologists studying the human brain/mind have tried to measure its capabilities. Quite a bit of this work has focused on defining intelligence and measuring a person’s intelligence.

The concept that intelligence could be or should be tested began with a nineteenth-century British scientist, Sir Francis Galton. Galton was known as a dabbler in many different fields, including biology and early forms of psychology. After the shake-up from the 1859 publishing of Charles Darwin's "The Origin of Species," Galton spent the majority of his time trying to discover the relationship between heredity and human ability (History of I.Q., n.d.).

Howard Gardner (1993), David Perkins (1995), and Robert Sternberg (1988) are researchers who have written widely sold books about intelligence. Of these three, Howard Gardner is probably the best known by PK-12 educators, because his theory of Multiple Intelligences has proven quite popular with such educators (Mckenzie). However, there are many researchers who have contributed to the extensive and continually growing collection of research papers on the intelligence (Yekovich 1994). The following definition of human intelligence is a composite from various authors, especially Gardner, Perkins, and Sternberg. Intelligence is a combination of the abilities to:
1. Learn. This includes all kinds of informal and formal learning via any combination of experience, education, and training.

2. Pose problems. This includes recognizing problem situations and transforming them into more clearly defined problems.

3. Solve problems. This includes solving problems, accomplishing tasks, and fashioning products.

Ways to measure intelligence were first developed more than 120 years ago, and this continues to be an active field of research and development. A very simplified summary of the current situation consists of:

1. There are a variety of IQ tests that produce one number or a small collection of numbers as measures of a person’s intelligence. Most of these tests place a high emphasis on the linguistic and mathematical/logical aspects of intelligence. (Increases in Math Content Knowledge and in Math Maturity tend to contribute to scoring higher on IQ tests.)

2. The “one number” approach (the general intelligence, or “g” factor) was developed by Charles Spearman in 1904, and it still has considerable prominence.

3. Many people have proposed and discussed the idea of multiple intelligences. In the past two decades, the work of Howard Gardner has helped to publicize this idea. Logical/mathematical and spatial are two of the eight Multiple Intelligences identified by Gardner (n.d.), and that relate to learning and using mathematics. As noted earlier in this document, linguistic intelligence is also related to mathematics.

4. Intelligence comes from a combination of nature and nurture. There have been a number of studies of possible genetic differences that might affect IQ between “White” Americans and “African” Americans. In an analysis of this research literature, Niabett (1998) reports, “The studies most directly relevant to the question of whether the Black/White IQ gap is genetic in origin provide no evidence for a correlation between IQ and African (rather than European) ancestry.” That is, the differences are due to “nurture,” not genetics.

5. Significant decreases in the intelligence of children result from starvation, lack of needed vitamins and minerals, and exposure to various poisons such as lead and mercury (Nutrition, n.d.).

While Howard Gardner and Robert Sternberg have garnered a lot of publicity during the past couple of decades for their work on intelligence, many really important ideas have been developed by other people. One of these is the idea that “g” can be divided into two major components: fluid intelligence (gF) and crystallized intelligence (gC).

Cognitive psychologists have re-framed the "fluid" and "crystallized" aspects of cognition into a model of a human cognitive system made-up of a long term memory which constitutes a knowledge base ("crystallized intelligence") for the person, a working memory which engages various processes ("fluid intelligence") that are going on at a given time using information picked-up from both the long term memory’s knowledge base, and a sensory system that picks-up information from the external world that the person is in. Today, over thirty years of research has
validated the usefulness of this simple three-part model for thinking about human cognition (Healy & McNamara, 1996).

In casual conversations about intelligence and IQ, people tend to forget about the meaning of the “Q” in IQ. The human brain grows considerably during a person’s childhood, with full maturity being reached in the early 20s. Both gF and gC increase during this time. Recent research suggests that gF then begins a slow decline. However, with appropriate education and cognitive experiences, gC continues to grow well into a person’s 60s (McArdle, et al. 2002). Quoting from the McArdle article:

The theory of fluid and crystallized intelligence … proposes that primary abilities are structured into two principal dimensions, namely, fluid (Gf) and crystallized (Gc) intelligence. The first common factor, Gf, represents a measurable outcome of the influence of biological factors on intellectual development (i.e., heredity, injury to the central nervous system), whereas the second common factor, Gc, is considered the main manifestation of influence from education, experience, and acculturation. Gf-Gc theory disputes the notion of a unitary structure, or general intelligence, as well as, especially in the origins of the theory, the idea of a structure comprising many restricted, slightly different abilities

Robert Sternberg is well known for his triarchic model of intelligence. Very roughly speaking, he divides intelligence into the three parts: creativity, street smarts, and school smarts. Here is a somewhat different way of explaining his theory. Think of creativity as being gF, while street smarts and school smarts are two broad categories in which one develops gC. If a person is raised in a preliterate hunter-gather community living in a jungle, the person will develop a high level of “hunter-gather living in a jungle” street smarts. Since the person will not be exposed to reading, writing, and books, the person will not develop an appreciable level of school smarts.

Gene Maier (n.d.) was one of the founders of the Math Learning Center that has offices in Salem and Portland, Oregon, and he served as its President for many years. One of his areas of interest is “folk math” versus school math. He notes that many people (including cabinet makers, carpenters, millwrights, and lots of people with little or no formal education) make routine use of math to help solve the types of problems they encounter on the job and in their day-to-day lives. By and large they make use of folk math (their math-oriented street smarts) rather than school math.

The street smarts versus school smarts analysis helps to explain why children raised in poverty (low socioeconomic environments) tend to be a year behind average in school smarts by the time they begin school. Their early childhood learning focuses on gaining street smarts knowledge and skills that help them survive and prosper in a poverty environment. Sanchez (2004) provides an interesting analysis of such data from the state of Colorado.

It is interesting to carry this line of thought a little further. Some children grow up in an environment that is school smarts mathematically “rich.” I am an example of such a person, since both my father and mother were on the faculty in the Department of Mathematics at the University of Oregon. I grew up in a culture that placed high value on knowing and using math. This environment helped to “grow” my math oriented gF and gC.

My conclusion is that one of the reasons for the relatively poor success of our formal, school smarts math education system is that the math environment many of our children grow up in before they start school and the math environment they encounter both at home and in school during the early years of their formal education is not particularly “rich” in its support of school mathematical development. This idea illustrated in the following quote from an American
Teacher attitudes and knowledge may also account for much of the inequitable treatment of preschool mathematics, science, and technology. The field of early childhood education has struggled for much of the second half of this century to establish a reputation of professionalism. However, the knowledge base deemed essential for teachers’ scientific and professional status derives almost exclusively from the child study movement and the field of developmental psychology. Few states require early childhood educators to have formal professional knowledge in the content areas as a condition of certification. Consequently, the experiences in science, mathematics, and technology that many early childhood educators bring with them to the classroom are limited by their personal histories as learners in those domains. Thus, children become accustomed to a female teacher’s comments that “boys...know more about how that thing works than I do.” Such teachers are more likely to use computers as crutches rather than as a valued educational tool. Some female teachers’ willingness to display their own lack of knowledge in computer technology also reinforces the gender stereotype that computers are not essential for girls’ development.

I also conclude that many people grow up rather weak in their folk math development, because they are not raised and taught in environments that are explicitly designed to foster cognitive growth of street smarts mathematics (folk math). They find that much of the school math they learn is not particularly to their outside of school needs.

Activity 6a. What are your personal thoughts on nature versus nurture as determiners of intelligence? What personal knowledge and experience do you have that supports your position? How does your position fit into the way you plan to work with young students?

Activity 6b. Think about the math that you routinely use in your day-to-day life. Give examples of the folk math aspects that you see in this use of math. Give some ideas about what schools might do to increase the folk math knowledge and skills of students.

**Developmental Theory**

You are probably familiar with the four-stage Piagetian Developmental Scale shown in Figure 4 (Huitt and Hummel, 1998).

<table>
<thead>
<tr>
<th>Approximate Age</th>
<th>Stage</th>
<th>Major Developments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1. Birth to 2 years</td>
<td>Sensorimotor</td>
<td>Infants use sensory and motor capabilities to explore and gain understanding of their environments.</td>
</tr>
<tr>
<td>Level 2. 2 to 7 years</td>
<td>Preoperational</td>
<td>Children begin to use symbols. They respond to objects and events according to how they appear to be.</td>
</tr>
<tr>
<td>Level 3. 7 to 11 years</td>
<td>Concrete operations</td>
<td>Children begin to think logically. This stage is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume. Increasing intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking—mental actions that are reversible—develops.</td>
</tr>
<tr>
<td>Level 4. 11 years and beyond</td>
<td>Formal operations</td>
<td>Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, thinking logically about abstract propositions, testing hypotheses, and gaining and using higher-order knowledge and skills.</td>
</tr>
</tbody>
</table>

Figure 4. Piaget's Stages of Cognitive Development
Piaget’s stages of cognitive development are not specific to any particular discipline. However, a math-oriented reader of Figure 4 might decide that Concrete Operations and Formal Operations seem to be somewhat math oriented. Piaget was particularly interested in math aspects of cognitive development. Later in this section I explore a still more math-oriented cognitive development scale.

Cognitive development is dependent on both nature and nurture. Roughly speaking, a child’s progress though the first two Piagetian Developmental stages is more strongly dependent on nature, while progress in the latter two stages is more strongly dependent on nurture. However, nature versus nurture is not that simple. Marcus (2004) argues that the two are so thoroughly intertwined that it is hopeless to attempt to separate them. Moreover, his arguments provide strong support for the value of high quality informal and formal education.

Although the Piagetian scale has only four labeled levels, it is a continuous scale. It is a common mistake to think of a person either being at Formal Operations or not being at Formal Operations. It is much more accurate to think of a person making progress in moving through a stage and gradually moving into the early part of the next stage. Moreover, the rate of movement strongly depends on formal and informal education and the environment in which one operates. Thus, a person may be well into Formal Operations in a one discipline such as history, and not yet have reached the beginnings of Formal Operations in another discipline such as math.

There are a variety of instruments used to measure cognitive development, and with such an instrument one can define a specific score as being the minimum score to be labeled “Formal Operations.” When that is done, researchers find that only about 35% of children in industrialized societies have achieved Formal Operations by the time they finish high school (MacDonald, n.d.).

However, data from similar cross-sectional studies of adolescents do not support the assertion that all individuals will automatically move to the next cognitive stage as they biologically mature. Data from adult populations provides essentially the same result: Between 30 to 35% of adults attain the cognitive development stage of formal operations (Kuhn, Langer, Kohlberg & Haan, 1977). For formal operations, it appears that maturation establishes the basis, but a special environment is required for most adolescents and adults to attain this stage. Huit, W., & Hummel, J. (2003). Piaget's theory of cognitive development. Educational Psychology Interactive. Valdosta, GA: Valdosta State University. Retrieved 9/16/04 from http://chiron.valdosta.edu/whuitt/col/cogsys/piaget.html.

These findings suggest that we need to take a careful look at the cognitive expectations in courses in all disciplines and at all grade levels. For example, the study of causality and the generating and testing of hypotheses are key ideas in the discipline of history and in the sciences. A ninth grade history or science course is apt to have a significant emphasis on these ideas. But, these ideas are part of Formal Operations. Unless they are presented and explored in a careful and appropriate Concrete Operations manner, they will be well over the heads of most of the ninth graders. Needless to say, this difficulty grows as one attempts to teach such ideas to still less cognitively developmentally mature students.

The same sort of analysis is applicable to our math curriculum. About 50 years ago, the Dutch educators Dina and Pierre van Hiele focused some of their research efforts on defining a Piagetian-type developmental scale for Geometry (van Hiele, n.d.). Their five-level scale is shown in Figure 5. Notice that the van Hieles, being mathematicians, labeled their first stage Level 0.
<table>
<thead>
<tr>
<th>Stage</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0 (Visualization)</td>
<td>Students recognize figures as total entities (triangles, squares), but do not recognize properties of these figures (right angles in a square).</td>
</tr>
<tr>
<td>Level 1 (Analysis)</td>
<td>Students analyze component parts of the figures (opposite angles of parallelograms are congruent), but interrelationships between figures and properties cannot be explained.</td>
</tr>
<tr>
<td>Level 2 (Informal Deduction)</td>
<td>Students can establish interrelationships of properties within figures (in a quadrilateral, opposite sides being parallel necessitates opposite angles being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle). Informal proofs can be followed but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.</td>
</tr>
<tr>
<td>Level 3 (Deduction)</td>
<td>At this level the significance of deduction as a way of establishing geometric theory within an axiom system is understood. The interrelationship and role of undefined terms, axioms, definitions, theorems and formal proof is seen. The possibility of developing a proof in more than one way is seen.</td>
</tr>
<tr>
<td>Level (Rigor)</td>
<td>Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples.</td>
</tr>
</tbody>
</table>

Figure 5. Van Hiele five-level developmental scale for geometry.

The van Hieles’ scale is mainly a school math (as distinguished from folk math) scale. The van Hieles’ work suggested that the typical high school geometry course was being taught at a developmental level considerably above that of the typical students taking such courses. Think carefully about your math experiences as you took algebra and geometry courses in high school. Did some of this coursework seem over your head (“I just don’t get it.”), forcing you into memorize, regurgitate, and forget mode? The same general question applies to students studying math at all grade levels. When students “just don’t seem to get it,” the chances are good that the content and the way it is being presented are at an inappropriate cognitive developmental level for the student.

It is evident that moving up the van Hiele geometry cognitive developmental scale requires learning quite a bit of school-math geometry. For most students, this means that progress in moving up this scale is highly dependent on their teachers and the math curriculum.

Figure 6 represents my current thinking on a six-level Piagetian-type scale for school mathematics (as distinguished from folk math). It is an amalgamation and extension of ideas of Piaget and the van Hieles.

<table>
<thead>
<tr>
<th>Stage Name</th>
<th>Math Developments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1. Piagetian and Math sensorimotor.</td>
<td>Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments. Research on very young infants suggests some innate ability to deal with small quantities such as 1, 2, and 3. As infants gain crawling or walking mobility, they can display innate spatial sense. For example, they can move to a target along a path requiring moving around obstacles, and can find their way back to a parent after having taken a turn into a room where they can no longer see the parent.</td>
</tr>
<tr>
<td>Level 2. Piagetian and Math preoperational.</td>
<td>During the preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. The children are making rapid progress in receptive and generative oral language. They accommodate to the language environments (including math as a language) they spend a lot of time in, so can easily become bilingual or...</td>
</tr>
</tbody>
</table>
During the preoperational stage, children learn some folk math and begin to develop an understanding of number line. They learn number words and the idea that to name the number of objects in a collection, one can “count” them, with the answer being the last number used in this counting process.

A majority of children discover or learn “counting on” and counting on from the larger quantity as a way to speed up counting of two or more sets of objects. Children gain increasing proficiency (speed, correctness, and understanding) in such counting activities.

In terms of nature and nurture in mathematical development, both are of considerable importance during the preoperational stage.

### Level 3
Piagetian and Math concrete operations.

During the concrete operations stage, children begin to think logically. In this stage—characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume—intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops (mental actions that are reversible).

While concrete objects are an important aspect of learning during this stage, children also begin to learn from words, language, and pictures/video, learning about objects that are not concretely available to them.

For the average child, the time span of concrete operations is approximately the time span of elementary school (grades 1-5). During this time, learning math is somewhat linked to having previously developed some knowledge of math words (such as counting numbers) and concepts. However, the level of abstraction in the written and oral math language quickly surpasses a student’s previous math experience. That is, math learning tends to proceed in an environment in which the new content materials and ideas are not strongly rooted in verbal, concrete, mental images and understanding of somewhat similar ideas that have already been acquired.

There is a substantial difference between developing general ideas and understanding of conservation of number, length, liquid, mass, weight, area, volume, and learning the mathematics that corresponds to this. These tend to be relatively deep and abstract topics, although they can be taught in very concrete manners.

### Level 4
Piagetian and Math formal operations.

Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, and gaining and using higher-order knowledge and skills.

Math maturity supports the understanding of and proficiency in math at the level of a high school math curriculum. Beginnings of understanding of math-type arguments and proof.

Piagetian and Math formal operations includes being able to recognize math aspects of problem situations in both math and non-math disciplines, convert these aspects into math problems (math modeling), and solve the resulting math problems if they are within the range of the math that one has studied. Such transfer of learning is a core aspect of Level 4.

### Level 5
Abstract mathematical operations.

Mathematical content proficiency and maturity at the level of contemporary math texts used at the senior undergraduate level in strong programs, or first year graduate level in less strong programs. Good ability to learn math through some combination of reading required texts and other math literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level math problems posed by others (such as in the text books and course assignments). Pose and solve problems at the level of one’s math reading skills and knowledge. Follow the logic and arguments in mathematical proofs. Fill in details of proofs when steps are left out in textbooks and other representations of such proofs.

### Level 6
Mathematician.

A very high level of mathematical proficiency and maturity. This includes speed, accuracy, and understanding in reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level mathematics. Pose and solve original math problems at the level of contemporary research frontiers.
You, and each of the students you teach, are at some place on this six-level continuous scale. As you teach math, think carefully about what you are doing that will help your students move up this scale. As you study math, think carefully about how this helps you move up the scale.

One think to consider is the use of math manipulatives in math teaching and learning. Math manipulatives are fit in well with students who are at the Preoperational and Concrete Operations level. Thus, they are now extensively used in schools.

The ready availability of computers in schools has facilitated the development of computer-based manipulatives (virtual manipulatives), and these are now commonly used in school. Douglas Clement (1999) has written an excellent analysis of physical manipulatives versus virtual manipulatives. Many useful virtual manipulative materials are available free at the Website Virtual Manipulatives (n.d.).

It is worth noting that when an architect develops a computerized model of a building to be constructed, the architect is constructing a virtual representation (a virtual manipulative) as an aid to visualizing and designing the building, and communicating its features to a client. Such computer aids to visualization and problem solving are now common tools in many different disciplines, including all of the sciences and in math.

Activity 7a. The chances are good that you are at the Formal Operations level on the four-level Piagetian Cognitive Developmental Scale. Think about where you fall on the six-stage mathematical cognitive developmental scale. Share your insights into your mathematical self that result from this activity.

Activity 7b. Can a teacher be an effective teacher of elementary school mathematics if the teacher is not at Level 4 (Piagetian and Math formal operations) on the six-stage mathematical cognitive developmental scale? Present arguments on each side of this issue, as well as suggestions for an elementary school math teacher who is not at this math cognitive developmental level.

Activity 7c. Explore and share your insights into how math manipulative fit into helping students learn math while at various states in their mathematical development. What do you know about uses of and the effects of using physical manipulatives versus virtual manipulatives (that is, computerized manipulatives)?

Brain Science

Research using brain imaging is beginning to make significant contributions to our understanding of learning and using math. For example, by five years ago brain imaging showed different parts of the brain being used in exact calculations than being used in estimations or approximate calculations (Dehaene et al. 1999). Brain imaging has identified regions of the brain associated with different types of dyscalculia (Stanescu-Cosson et al., 2000; Pearson, 2003).

A useful analogy is that dyscalculia is to learning math as dyslexia is to learning to read. Perhaps five-percent of students have a difference in their brain structures that makes it very difficult to learn math. Research on dyslexia suggests that a somewhat larger percentage of students have a difference in brain structure that makes it very difficult to learn to become fluent readers.

Mind research on gF suggests that this component of g increases into early adulthood. A recently published longitudinal brain imaging study reports results that seem to be consistent with this gF result (Gogtay et al., 2004).
We now have theory and instrumentation that helps us gain increased understanding of the human brain. We have steadily increasing knowledge of the human genome, noninvasive tools for brain imaging, and tools and skills for manipulation of individual genes. This progress has raised the nature versus nurture discussion to an entirely new level. We are gaining increased understanding of nature, and we now have the ability to change nature. Here are two quotes from Marcus (2004):

…in our world, nature’s contribution to development comes not by providing a finely detailed sketch of a finished product, but by providing a complex system of self-regulating recipes. These recipes provide for many different things—from the construction of enzymes and structural proteins to the construction of motors, transporters, receptors, and regulatory proteins—and thus there is no single, easily characterizable genetic contribution to the mind. In the ongoing everyday functioning of the brain, genes supervise the construction of neurotransmitters, the metabolism of glucose, and the maintenance of synapses. In early development, they help to lay down a rough draft, guiding the specialization and migration of cells as well as the initial pattern of wiring. In synaptic strengthening, genes are a vital participant in a mechanism by which experience can alter the wiring of the brain (thereby influencing the way that an organism interprets and responds to the environment). (Marcus, 2004, p168.)

In the coming decades, we will all collectively as a society need to decide what we think about biotechnology and what applications we are and are not willing to allow. The debates we have now, about cloning and stem cell research, pale in comparison to debates we are likely to encounter as the technology for manipulating genes advances. We are already at the point where it is possible to screen embryos for the predisposition to certain life-threatening illnesses; as we unravel more and more of the genome, we will be able to detect more and more disorders (or predispositions to disorders) well in advance of birth. Ultimately, if we so choose, we may be able to directly manipulate embryonic genomes—add a gene here, delete a gene there. The genes of a child might eventually be more a matter of choice than of chance (Marcus, 2004, p174).

The next four subsections of this document are Big Ideas that are quite important in math education.

**Brain Versus Computer**

In the early days of computers, people often referred to such machines as *electronic brains*. Even now, more than 50 years later, many people still use this term. Certainly a human brain and a computer have some characteristics in common. However:

- Computers are very good at carrying out tasks in a mechanical, “non-thinking” manner. They are millions of times as fast as humans in tasks such as doing arithmetic calculations or searching through millions of pages of text to find occurrences of a certain set of words. Moreover, they can do such tasks without making any errors.
- Human brains are very good at doing the thinking and orchestrating the processes required in many different very complex tasks such as carrying on a conversation with a person, reading for understanding, posing problems, and solving complex problems. Humans have minds and consciousness. A human’s brain/mind capability for “meaningful understanding” is far beyond the capabilities of the most advanced computers we currently have.

**Big Idea # 1:** There are many things that computers can do much better than human brains, and there are many things that human brains can do much better than computers. Our educational system can be significantly improved by building on the relative strengths of brains and
computers, and decreasing the emphasis on attempting to “train” students to compete with
computers. We need to increase the focus on students learning to solve problems using the
strengths of their brains and the strengths of ICT.

**Chunks and Chunking**

Here are three different types of human memory:

- Sensory memory stores data from one’s senses, and for only a short time. For
example, visual sensory memory stores an image for less than a second, and auditory
sensory memory stores aural information for less than four seconds.
- Working memory (short term memory) can store and actively process a small number
of chunks. It retains these chunks for less than 20 seconds.
- Long-term memory has large capacity and stores information for a long time.

Research on working memory indicates that for most people the size of this memory is about
7 ± 2 chunks (Miller, 1956). This means, for example, that a typical person can read or hear a
seven-digit telephone number and remember it long enough to key into a telephone keypad. The
word *chunk* is very important. When I was a child, my home phone number was the first two
letters of the word diamond, followed by five digits. Thus, to remember the number (which I still
do, to this day) I needed to remember only six chunks. But, I had to be able to decipher the first
chunk, the word “diamond.”

Long-term memory has a very large capacity, but this does not work like computer memory.
Input to computer memory can be very rapid (for example, the equivalent of an entire book in a
second), and a computer can store such data letter perfect for a long period of time. The human
brain can memorize large amount of music, poetry, or other text. But, this is a long and slow
process for most people. By dint of hard and sustained effort, an ordinary person can memorize
nearly letter perfect the equivalent of a few books. However, the typical person is not very good
at this. At the current time, the Web contains the equivalent of tens of millions of books.

On the other hand, the human brain is very good at learning meaningful chunks of
information. Think about some of your personal chunks such as constructivism, multiplication,
democracy, transfer of learning, and Mozart. Undoubtedly these chunks have different meanings
to me than they do for you. As an example, for me, the chunk “multiplication” covers
multiplication of positive and negative integers, fractions, decimal fractions, irrational numbers,
complex numbers, functions (such as trigonometric and polynomial), matrices, and so on. My
breadth and depth of meaning and understanding was developed through years of undergraduate
and graduate work in mathematics.

It is useful to think of a chunk as a label or representation (perhaps a word, phrase, visual
image, sound, smell, taste, or touch) and a collection of pointers. A chunk has two important
characteristics:

1. It can be used by short-term memory in a conscious, thinking, problem-
solving process.
2. It can be used to retrieve more detailed information from long-term memory.

**Big Idea # 2:** Our education system can be substantially improved by taking advantage of our
steadily increasing understanding of how the mind/brain learns and then uses its learning in
problem solving. Chunking information to be learned and used is a powerful aid to learning and
problem solving. However, even if two people receive the same education about a topic, and use the same label for a chunk that they form on that topic, their chunks will be quite different. (This is a key idea in constructivism.)

**Rate of Learning**

Howard Gardner has identified eight domains or types of intelligence, including math/logic and spatial. One aspect of having varying levels of intelligence in these various domains is that a person is likely to have some differences in rates of learning and in learning potentials in these various domains.

Differences in rates of learning are very evident for students with special needs. Our school system does not deal very well with varying rates of learning, even though it puts a lot of money into special education. A solid example of this is provided by the math learning of students who are classified as learning disabled.

The background literature of special education has long shown that students with mild disabilities (a) demonstrate levels of achievement approximating 1 year of academic growth for every 2 or 3 years they are in school (Cawley & Miller, 1989); (b) exit school achieving approximately 5th- to 6th-grade levels (Warner, Alley, Schumaker, Deshler, & Clark, 1980); and (c) demonstrate that on tests of minimum competency at the secondary level, their performance is lower for mathematics than it is for other areas (Grise, 1980). Crawley et al., 2001).

On the other end of the cognitive ability scale, there are quite a few students in school who can learn math much faster than the average student. A typically elementary school class will likely have several students who learn math at least 50% faster than average. That is, these students are capable of making one and a half (or more) years of school math progress per school year.

Rate of learning is also highly dependent on the nature and quality of instruction. There has been quite a bit of research on the value of providing students with individual tutors. Research by Benjamin Bloom (1984), the same person who was responsible for Bloom’s Taxonomy, indicates that with individual tutoring a “C” student can become an “A” student. Similar research forms the basis of very small class and individual tutoring approaches to helping students who are making slow progress in learning math and reading.

**Big Idea # 3.** Students vary tremendously in their rates of learning and their abilities to learn various disciplines. Individualization of instruction and individualization of programs of instruction (such as in an IEP) make a significant contribution to improving the rate of learning of a student. This type of research helps to explain the success of computer assisted instruction that can provide a type of highly interactive, somewhat individualized type of instruction.

**Augmentation to Brain/Mind**

Reading and writing provide an augmentation to short term and long-term memory for personal use and that can be shared with others. Data and information can be stored and retrieved with great fidelity. As Confucius noted about 2,500 years ago, “The strongest memory is not as strong as the weakest ink.”

Writing onto paper provides a passive storage of data and information. The “using” of such data and information is done by a human’s brain/mind.
Computers add a new dimension to the storage and retrieval of data and information. Computers can process (carry out operations on) data and information. Thus, one can think of a computer as a more powerful augmentation to brain/mind than is provided by static storage on paper or other hardcopy medium.

**Big Idea # 4**: ICT provides a type of augmentation to one’s brain/mind. The power, capability, and value of this type of augmentation continues to grow rapidly. Certainly this is one of the most important ideas in education at the current time. At the current time our formal educational system has yet to understand the idea of ICT as an augmentation to the mind/brain.

Here is some of my current thinking about the ideas listed above. From my point of view, this is a wide-open area, ripe for research progress.

In thinking about chunks and learning, I see two approaches. In the first approach, a clear framework is provided. Think of the framework as scaffolding for a chunk along with a label for the chunk. One learns the framework and then fits new knowledge and experiences into the framework. In the second approach, one creates their own framework. This is less efficient initially, but perhaps more productive over the long run in the task of helping students learn to learn and to take increasing responsibility for their own learning.

To illustrate, suppose I want to know a modest amount about something that others have carefully studied. Since part of a discipline is how to teach and learn it, I decide to take advantage of this accumulated knowledge. I have the discipline taught to me by an expert teacher.

But now, suppose that I want to extend my knowledge to “my” world and to situations not covered in the standard curriculum. Now, I hope that I have learned to learn on my own. I hope that I have the creativity and skill to discover, invent, find, and so on, and fit my new learning into the old framework. I hope that I can restructure the old framework so that it better fits the new and my needs.

There is one more important piece to this. Suppose that the area that I want to study is one in which computers provides powerful aids to solving its problems. Then I want my chunk to include a link to the capabilities and limitations of computers as an aid to solving the problems. I want to have the knowledge and skills to make use of this computer augmentation to my brain.

**Activity 8a.** Select a math topic that you feel it is important for an elementary school student to learn. Think about this from the point of view of being a “chunk” that the student will construct in his or her brain/mind. What is a good name for this chunk? What is a good mental image or picture for this chunk? How do you expect this chunk to grow in breadth and depth over time? What are some aspects of this chunk that you expect will serve the student over a lifetime?

**Activity 8b.** Think carefully about your rate and ease of learning math versus your rate and ease of learning some other discipline that is in the elementary school curriculum. How have you and the school curriculum accommodated these rates of learning during your many years of being a student?

**Information and Communication Technology (ICT)**

Information and Communications Technology (ICT) includes calculators, computers, telecommunications, the Web, digital still and video cameras, digital phones, audio and video recorders and players, and so on. This section provides additional ideas on ICT and math education.

ICT and math share much in common. For example, both provide powerful aids to problem solving across a wide range of academic disciplines. Both have tremendous breadth and depth. Much of the theory of the discipline of Computer and Information Science is rooted in mathematics.
ICT is a large, vibrant, and rapidly growing field. The International Society for Technology in Education (ISTE) has developed national standards for students, teachers, and school administrators. These standards have been widely adopted and serve to provide a good sense of direction for the ICT preparation of teachers and their students (ISTE NETS, n.d.).

Listed below are some math-related ICT topics. The intent of this list is to provide you with a hint of the breadth and depth of this discipline. More detail on a number of these topics is available in my math Website http://darkwing.uoregon.edu/~moursund/Math/.

1. The discipline of mathematics is now commonly divided into three major components: pure math, applied math, and computational math. “Computational” is also a new aspect of many other disciplines. For example, one of the winners of the 1998 Nobel Prize in Chemistry received the award for his past 15 years of work in Computational Chemistry. Computer-based modeling and simulation, based on computational mathematics, is now a common component of each discipline that makes use of mathematics. Such modeling and simulation is very slowly working its way into the math curriculum.

2. Computer algebra systems (CAS) provide very powerful tools to carry out a wide range of mathematical procedures. The idea that a handheld calculator can add, subtract, multiply, divide, and calculate square roots is easy to understand. Nowadays, the chunks named add, subtract, etc. that most adults in this country carry in their mind/brain include a calculator component.

It is now common for students taking high school math courses to learn to use graphing, equation-solving calculators that have some built-in rudiments of CAS. Ideas such as function, equation, and graphing are very important ideas in math. As you work with elementary school students you are laying the foundations for their future learning of these topics. Among other things this means you are helping students to develop chunks in their mind/brain that can grow to include these topics. For more information see http://darkwing.uoregon.edu/~moursund/Math/computational_math.htm.

3. Computer-assisted learning (CAL) is gradually improving. We now have Highly Interactive Intelligent Computer-Assisted Learning (HIICAL) systems that are quite good. The meaning of “quite good” can be debated. Research in this area tends to compare test scores of students taught by conventional instructional methods versus test scores of students taught by HIICAL. There is now a significant amount of such software that, on average, leads to better test scores than does conventional instruction. See http://www.uoregon.edu/~moursund/dave/second_order.htm.

HIICAL software can be developed that integrates the power of computer-assisted instruction with the power of CAS systems. That is, we are gradually seeing a merger of powerful computer tools and powerful aids to learning and using the tools. Such software has the potential to lead to major changes in math education. The goal might become to educate students so that they function well mathematically in a world in which such systems are readily available.
4. It is helpful to think about math training versus math education. Most of what an animal trainer does falls into the category of training, as contrasted with education. Education has a focus on understanding; training has a focus on rote performance.

Our educational system consists of a mixture of training and education, and it is not easy to draw a clear distinction between the two. Research in computer-assisted learning suggests that this approach to teaching and learning is much more effective in training than it is in education. Suppose, for example, that we want students to memorize the single digit multiplication facts and to be able to retrieve these facts with great speed and accuracy. This can be considered as a training task, and CAL is quite effective in this teaching/learning situation. Even the simplest of HIICAL designed for such training is able to individualize instruction, detect student weaknesses and address these weaknesses, and assess student speed and accuracy. From those points of view, such a CAL system is definitely more effective than a teacher working with a whole class. As we look to the future of math education, we will see HIICAL becoming a common component.

This document has previously mentioned math manipulatives. Such manipulatives can be used in both training and education modes. However, the current focus on using math manipulatives is on education for understanding and problem solving, rather than on training. For a list or resources on virtual manipulatives for use in math education see.

5. Artificial intelligence (AI) is a branch of the discipline of Computer and Information Science. It focuses on developing hardware and software systems that solve problems and accomplish tasks that—if accomplished by humans—would be considered to be a display of intelligence. As I look toward the future, I see a steady increase in situations where people and AI systems work together to solve problems and accomplish tasks.

   What is artificial intelligence? It is often difficult to construct a definition of a discipline that is satisfying to all of its practitioners. AI research encompasses a spectrum of related topics. Broadly, AI is the computer-based exploration of methods for solving challenging tasks that have traditionally depended on people for solution. Such tasks include complex logical inference, diagnosis, visual recognition, comprehension of natural language, game playing, explanation, and planning (Horvitz, 1990).

   AI is of steadily growing importance in education (Moursund, 2004e). Elementary school students already have a mind/brain chunk in this area, based on the robots and computers they see on television, their electronic toys, and so on. One of your jobs as a teacher is to shape this chunk so that it is more accurate and so that it can better accommodate future learning. For example, a handheld calculator has some intelligence. Think about how this intelligence is similar to and different from human intelligence in math.

6. Distance education is a rapidly growing field. If we use a rather broad definition of distance education, then it is already in common use in elementary schools. When a student uses the Web to retrieve information, this is a form of distance education. When a student uses a help feature in a software package, this is a form of distance education. Much of the CAL that students is accessed through a computer that is remotely located; thus, much of current CAL is a type of distance education.
Imagine the situation in which HIICAL that covers all of the math curriculum is routinely available to students at home, at school, and wherever else they have access to the Internet. Such a system would also provide access to CAS, large numbers of math resource books, and other aids to learning and using math. While the progress in this direction seems relatively slow, I believe that this situation will be a standard part of many educational systems within the next two decades.

7. In light of goals for students learning math content and gaining in math maturity, how authentic is math assessment? Outside of school testing situations, people who need to make appreciable use of math tend to make use of calculators, computers, and many specialized devices (such as a global positioning system, computerized laser measuring and surveying systems) as aids to math problem solving. This suggests that authentic assessment in math should be moving in the direction of open book, open notes, open calculator, open computer, and similar forms of assessment. Some progress in this direction has occurred in the use of calculators, but little progress is occurring other aspects of authentic math assessment. See [http://darkwing.uoregon.edu/~moursund/PBL/part_7.htm](http://darkwing.uoregon.edu/~moursund/PBL/part_7.htm) for more information about authentic assessment.

Activity 9a. Many leaders in the field of ICT in education argue that the development of writing, the development of the movable type printing press, and the development of computers are the three most important developments in the history of education. Compare and contrast current and potential roles of ICT in education relative to the contributions made by writing and the printing press.

Activity 9b. Make a list of things that you can do much better than ICT systems, things that ICT systems can do much better than you, and things that you and ICT systems working together can do much better than either can do alone. Analyze your list from the point of view of our current elementary school and teacher education systems.

Conclusions Thoughts

Math education is a large, complex, and challenging discipline. The formal teaching of math began at the time of the first formal teaching of reading and writing, a little more than 5,000 years ago. During the past 5,000 years the collected mathematical knowledge of the human race has grown immensely. A number of ideas that challenged the mathematical geniuses of their time have trickled down into the precollege school math curriculum—indeed, even into elementary school.

As the agriculture age has given way to the industrial age and now the information age, the math-related demands placed on people have grown. In information age societies such as the United States, there are now much higher math education expectations than there were in the industrial age or the agricultural age. As our society continues to raise its math education expectations, it is not achieving the math learning gains that it would like.

Because math knowledge and skills are so important in our information age society, you can expect to see continued efforts to “reform” our math education system. This document supports the idea that with appropriate informal and formal teaching and support, students (on average) can gain greater Math Content Knowledge and greater Math Maturity than they are currently obtaining. However, such math education goals leave us with many challenging issues. Here are a few examples:
1. It is likely that well over half of parents and elementary school teachers have not achieved Math Formal Operations. Their levels of School Math Maturity and School Math Content Knowledge are low. Thus, on average, children growing up in our society tend gain their first 10 to 12 years (birth through grade school) of informal and formal math education in what I would call relatively poor math education environments. If we want to significantly improve our math education system, we will have to make significant progress toward addressing this problem.

2. Our current math education curriculum is often described as being “a mile high and an inch deep’ (Ruetters, 20023). I have some trouble understanding what this means, as I don’t use linear measure when I am taking about the breadth and depth of a curriculum. However, what I think it means is that many people are concerned about how our curriculum has expanded in breadth, and that it lacks the depth needed for students to gain understanding and a number of other aspects of increasing math maturity. Our curriculum is not well designed in terms of helping students learn to make connections and to transfer their math knowledge and skills to areas outside of the formal math curriculum.

3. ICT brings new dimensions to both School Math and Folk Math. We have yet to appropriately understand and implement a math education system that adequately takes into consideration the capabilities of ICT as aids to teaching, learning, and using math.

   For example, consider computer tools that are routinely used by graphic artists. They are based on a very large amount of mathematics. However, very few graphic artists feel the need to have studied this underlying mathematics, and few people who teach graphic arts use of computers have appreciable insights into the underlying mathematics. The issue here is somewhat similar to the issue of children using calculators rather than paper and pencil algorithms, or researchers using statistical packages of computer programs without having mastered the underlying mathematics.

   But, the issue is also quite different. The goal of a graphic artist is to solve a graphic artist problem (complete a graphic artist task). The graphic artist has non-mathematical knowledge and skills that can provide feedback on progress toward solving the problem or accomplishing the task.

   It turns out that this example identifies a major hole in the overall math curriculum. We are not very successful in helping students understand math at a level where they can detect their own errors. People who routinely use math are able to detect their errors because they have knowledge (intuition, deep insights) into the problems that they are addressing. Even though our math curriculum makes considerable use of “word problems” that provide some context for the problem to be solved, it is rare that a student has a sufficient grasp of the problem setting and meaning so that the student can detect errors in math thinking and in carrying out needed math procedures.

4. There are a variety of math topics that require a student to be at or near math Formal Operations in order to gain a significant understanding of the topic. Examples include probability, ratio and proportion, and algebra. Roughly
speaking, if many of the students you are teaching “just don’t seem to get it” for certain topics, then there is a good chance that they are not developmentally ready for the topic.

5. One of the most important ideas in math education is learning to build upon and make effective use of the accumulated knowledge in the discipline of math. ICT is a powerful aid to learning, a powerful aid to information retrieval, and a powerful aid to carrying out many of the types of procedures that are important in solving math problems. Our current math education system is not doing well in making effective use of this range of ICT uses.

Activity 10a. List and reflect on two or three Big Ideas from this document that may have an impact on you as you become a teacher. Explain why these ideas seem particularly relevant and important to you.

Activity 10b. Suggest one or more topics that you feel should be added to this document, and explain why you feel they should be added.

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