Example 7.4 A worker-operated machine produces a defective item with probability 0.01 if the worker follows the machine’s operating instructions exactly, and with probability 0.03 if he does not. If the worker follows the instructions 90% of time, what proportion of all items produced by the machine will be defective? Given that a defective item is produced, what is the conditional probability of the event that the worker exactly follows the machine operating instructions?

Solution: Define

$D$: Machine produces a defective item.

$F$: Worker follows instructions.

Then, we have following information:

$\mathbb{P}(D|F) = 0.01 \quad \mathbb{P}(F) = 0.9$

$\mathbb{P}(D|F^c) = 0.03 \quad \mathbb{P}(F^c) = 0.1$

According to the law of total probability, we have

$\mathbb{P}(D) = \mathbb{P}(D|F)\mathbb{P}(F) + \mathbb{P}(D|F^c)\mathbb{P}(F^c)$

$= 0.01(0.9) + 0.03(0.1) = 0.012.$

According to the Bayes’ Rule, we have

$\mathbb{P}(F|D) = \frac{\mathbb{P}(F)\mathbb{P}(D|F)}{\mathbb{P}(D)} = \frac{0.9(0.01)}{0.012} = 0.75$

$\mathbb{P}(F^c|D) = 0.25$

Example 8.1 Whether a grant proposal is funded quite often depends on the reviewers. Suppose a group of research proposals was evaluated by a group of experts as to whether the proposals were worthy of funding. When these same proposals were submitted to a second independent group of experts, the decision
to fund was reversed in 30% of the cases. If the probability that a proposal is judged worthy of funding by the first peer review group is 0.2, what are the probabilities of these events?
(a) A worthy proposal is approved by both groups.
(b) A worthy proposal is disapproved by both groups.
(c) A worthy proposal is approved by one group.

Solution: Define
\[ A_1 := \{ \text{The worthy proposal is approved by first group} \}; \]
\[ A_2 := \{ \text{The worthy proposal is approved by second group} \}; \]
\[ D_1 := \{ \text{The worthy proposal is disapproved by first group} \}; \]
\[ D_2 := \{ \text{The worthy proposal is disapproved by second group} \}. \]

Then, \( P(A_1) = 0.2 \), and \( P(D_1) = 0.8 \). Since the decision to fund was reversed in 30%, we have
\[ P(D_2|A_1) = 0.3 \quad P(A_2|D_1) = 0.3 \]

According to the complement relationship, we have
\[ P(A_2|A_1) = 0.7 \quad P(D_2|D_1) = 0.7 \]

Thus,
(a) \( P(A_1 \cap A_2) = P(A_1)P(A_2|A_1) = 0.2(0.7) = 0.14 \)