Immigration Superprocesses with Dependent Spatial Motion and Non-critical Branching

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Abstract

A class of immigration superprocess with dependent spatial motion is constructed by a passage to the limit from a sequence of superprocesses with positive jumps. A non-critical branching is then obtained by using a Girsanov transform of Dawson’s type, which also gives a state-dependent spatial drift.

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1 Introduction

Let \(B(\mathbb{R})\) be the totality of all bounded Borel functions on \(\mathbb{R}\) and let \(C(\mathbb{R})\) denote its subset comprising of continuous functions. Let \(M(\mathbb{R})\) denote the space of finite Borel measures on \(\mathbb{R}\) endowed with the topology of weak convergence. We write \(\langle f, \mu \rangle\) for \(\int fd\mu\) and for a function \(F\) on \(M(\mathbb{R})\) let

\[
\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{r \to 0^+} \frac{1}{r} [F(\mu + r\delta_x) - F(\mu)], \quad x \in \mathbb{R},
\]

if the limit exists. Let \(\frac{\delta^2 F(\mu)}{\delta \mu(x)\delta \mu(y)}\) be defined in the same way with \(F\) replaced by \(\frac{\delta F}{\delta \mu(y)}\) on the right hand side. Suppose that \(h\) is a continuously differentiable function on \(\mathbb{R}\) such that both \(h\) and \(h'\) are square-integrable. Then the function

\[
\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R},
\]

(1.1)

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is twice continuously differentiable with bounded derivatives $\rho'$ and $\rho''$. Suppose that $c \in C(\mathbb{R})$ is Lipschitz and $\sigma \in B(\mathbb{R})^+$. We may define an operator $\mathcal{L}$ by

$$\mathcal{L}F(\mu) = \frac{1}{2} \int_{\mathbb{R}} a(x) \frac{d^2 F(\mu)}{d\mu(x)^2} \mu(dx) + \frac{1}{2} \int_{\mathbb{R}^2} \rho(x-y) \frac{d^2 F(\mu)}{d\mu(x)d\mu(y)} \mu(dx) \mu(dy) + \frac{1}{2} \int_{\mathbb{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta\mu(x)^2} \mu(dx), \quad (1.2)$$

which acts on a class of functions on $M(\mathbb{R})$ to be specified. A Markov process with generator $\mathcal{L}$ was constructed in Dawson et al [2], generalizing the construction of Wang [9, 10]. The process generated by $\mathcal{L}$ is naturally called an \textit{superprocess with dependent spatial motion} (SDSM) with parameters $(a, \rho, \sigma)$, where $a(\cdot)$ represents the rate of the underlying motion, $\rho(\cdot)$ represents the interaction between the “particles” and $\sigma(\cdot)$ represents the branching density. We shall also call the process simply a $(a, \rho, \sigma)$-\textit{superprocess}. We refer the reader to [2, 9, 10] for detailed descriptions of the model. Given $\lambda \in M(\mathbb{R})$, we may define another operator $\mathcal{J}$ by

$$\mathcal{J}F(\mu) = \mathcal{L}F(\mu) + \int_{\mathbb{R}} \frac{\delta F(\mu)}{\delta\mu(x)} \lambda(dx). \quad (1.3)$$

A Markov process generated by $\mathcal{J}$ can be called an \textit{SDSM with immigration} with parameters $(a, \rho, \sigma, \lambda)$ or simply a $(a, \rho, \sigma, \lambda)$-\textit{superprocess}, where $\lambda$ represents the immigration rate.

In this work, we give a construction of the $(a, \rho, \sigma, \lambda)$-superprocess by a passage to the limit from a sequence of SDSM’s with positive jumps. From the $(a, \rho, \sigma, \lambda)$-superprocess we shall use Girsanov transform of Dawson’s type to derive an $M(\mathbb{R})$-valued diffusion process with generator

$$\mathcal{J}^bF(\mu) = \mathcal{J}F(\mu) - \int_{\mathbb{R}} b(x) \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) - \int_{\mathbb{R}^2} \rho(x-y)b'(y) \frac{d}{dx} \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) \mu(dy), \quad (1.4)$$

where $b \in C^1(\mathbb{R})$. Note that the generator $\mathcal{J}^b$ not only involves a non-critical branching given by the second term on the right hand side, it also involves a state-dependent drift in the spatial motion represented by the last term. This is different from the classical case where the Girsanov transform does not effect the spatial motion; see Dawson [1].

## 2 Function-valued dual processes

As in Dawson et al [2], we shall define a function-valued dual process and investigate its connection to the solution of the martingale problem for the immigration SDSM. For $\mu \in M(\mathbb{R})$ and a subset $\mathcal{D}(\mathcal{J})$ of the domain of $\mathcal{J}$, we say an $M(\mathbb{R})$-valued cádlág process $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-\textit{martingale problem} if

$$F(X_t) - F(X_0) - \int_0^t \mathcal{J}F(X_s)ds, \quad t \geq 0, \quad (2.1)$$
is a martingale for each \( F \in \mathcal{D}(\mathcal{F}) \). Let \( G^m \) denote the generator of the interacting particle system introduced in [2], and let \((P^m_t)_{t \geq 0}\) denote the transition semigroup generated by the operator \( G^m \). Observe that, if \( F_{m,f}(\mu) = \langle f, \mu^m \rangle \) for \( f \in C_0^2(\mathbb{R}^m) \), then

\[
\mathcal{J} F_{m,f}(\mu) = F_{m,G^m f}(\mu) + \frac{1}{2} \sum_{i,j=1, i \neq j}^m F_{m-1, \Phi_{ij} f}(\mu) + \sum_{i=1}^m F_{m-1, \Psi_i f}(\mu),
\]

with \( \Phi_{ij} f \in C_0^2(\mathbb{R}^{m-1}) \) defined by

\[
\Phi_{ij} f(x_1, \cdots, x_{m-1}) = \sigma(x_{m-1}) f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}),
\]

where \( x_{m-1} \) is in the places of the \( i \)th and the \( j \)th variables of \( f \) on the right hand side, and \( \Psi_i f \in C_0^2(\mathbb{R}^{m-1}) \) defined by

\[
\Psi_i f(x_1, \cdots, x_{m-1}) = \int_{\mathbb{R}} f(x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_{m-1}) \lambda(dx), \quad x_j \in \mathbb{R},
\]

where \( x \in \mathbb{R} \) is the \( i \)th variable of \( f \) on the right hand side.

Let \( \{ M_t : t \geq 0 \} \) be a nonnegative integer-valued cádlág Markov process with transition intensities \( \{q_{ij} \} \) such that \( q_{i,i} = -q_{i,i} = i(i+1)/2 \) and \( q_{i,j} = 0 \) for all other pairs \((i, j)\). Let \( \tau_0 = 0 \) and \( \tau_{M_0+1} = \infty \), and let \( \{ \tau_k : 1 \leq k \leq M_0 \} \) be the sequence of jump times of \( \{ M_t : t \geq 0 \} \). Let \( \{ \Gamma_k : 1 \leq k \leq M_0 \} \) be a sequence of random operators which are conditionally independent given \( \{ M_t : t \geq 0 \} \) and satisfy

\[
P\{ \Gamma_k = \Phi_{i,j} | M(\tau^-_k) = l \} = \frac{1}{l(l+1)}, \quad 1 \leq i \neq j \leq l,
\]

and

\[
P\{ \Gamma_k = \Psi_i | M(\tau^-_k) = l \} = \frac{2}{l(l+1)}, \quad 1 \leq i \leq l.
\]

Let \( B \) denote the topological union of \( \{ B(\mathbb{R}^m) : m = 1, 2, \cdots \} \) endowed with pointwise convergence on each \( B(\mathbb{R}^m) \). Then

\[
Y_t = P_{\tau_{k+1}}^{M_{k+1}} P_{\tau_k}^{M_k-1} \cdots P_{\tau_1}^{M_1} X_0, \quad \tau_k \leq t < \tau_{k+1}, 0 \leq k \leq M_0,
\]

defines a Markov process \( \{ Y_t : t \geq 0 \} \) taking values from \( B \). Clearly, \( \{(M_t, Y_t) : t \geq 0\} \) is also a Markov process. To simplify the presentation, we shall suppress the dependence of \( \{ Y_t : t \geq 0 \} \) on \( \sigma \) and let \( F^m_{m,f} \) denote the expectation given \( M_0 = m \) and \( Y_0 = f \in C(\mathbb{R}^m) \), just as we are working with a canonical realization of \( \{(M_t, Y_t) : t \geq 0\} \).

**Theorem 2.1** Let \( \mathcal{D}(\mathcal{F}) \) be the set of all functions of the form \( F_{m,f}(\mu) = \langle f, \mu^m \rangle \) with \( f \in C_0^2(\mathbb{R}^m) \). Suppose that \( \{ X_t : t \geq 0 \} \) is a continuous \( M(\mathbb{R}) \)-valued process and that \( E \{ \langle 1, X_t \rangle^m \} \) is locally bounded in \( t \geq 0 \) for each \( m \geq 1 \). If \( \{ X_t : t \geq 0 \} \) is a solution of the \((\mathcal{J}, \mathcal{D}(\mathcal{F}))\)-martingale problem with \( X_0 = \mu \), then

\[
E \langle f, X_t^m \rangle = E^\sigma_{m,f} \left[ \langle Y_t, \mu^{M_t} \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s (M_s + 1) ds \right\} \right]
\]

for any \( t \geq 0, f \in B(\mathbb{R}^m) \) and integer \( m \geq 1 \). Consequently, the \((\mathcal{J}, \mathcal{D}(\mathcal{F}))\)-martingale problem has at most one solution possessing locally bounded moments of all degrees.
Moreover, the transition semigroup of $\eta$ is Feller.

**Proof.** The general equality follows by bounded pointwise approximation once it is proved for $f \in C^2_\beta(\mathbb{R}^m)$. Set $F_\mu(m, f) = F_{m,f}(\mu) = \langle f, \mu^m \rangle$. From the construction (2.7), it is not hard to see that $\{ ( M_t, Y_t) : t \geq 0 \}$ has generator $\mathcal{L}^\ast$ given by

$$
\mathcal{L}^\ast F_\mu(m, f) = F_\mu(m, G^m f) + \frac{1}{2} \sum_{i,j=1, i \neq j}^m [F_\mu(m-1, \Phi_{ij} f) - F_\mu(m, f)] + \sum_{i=1}^m [F_\mu(m-1, \Psi_i f) - F_\mu(m, f)].
$$

In view of (2.2) we have

$$
\mathcal{L} F_{m,f}(\mu) = \mathcal{L}^\ast F_\mu(m, f) + \frac{1}{2} m(m+1) F_\mu(m, f). 
$$

(2.9)

Guided by (2.9) one can prove (2.8) using similar calculations as in [2]. To show the last assertion of the theorem, we may first consider the special case $\sigma(x) \equiv \sigma_0$ for a constant $\sigma_0$. In this case, (2.1) implies that $\{ (1, X_t) : t \geq 0 \}$ is a one-dimensional diffusion with generator $2^{-1}\sigma_0 x^2 dx^2 + (1, \lambda) dx / dx$. As in [5, pp.236-237] one sees that

$$
E \exp \{ z(1, X_t) \} = [1 - \sigma_0 z t/2]^{-2(1,\lambda)/\sigma_0} \exp \left\{ \frac{(1, \mu) z}{1 - \sigma_0 z t/2} \right\}, \quad t \geq 0, |z| < 2/\sigma_0 t.
$$

The remaining arguments are similar to those in the proof of Theorem 2.2 in [2].

\[\square\]

## 3 SDSM with discrete immigration

Suppose that $(P_t)_{t \geq 0}$ is a Feller transition semigroup on some metric space $E$ which has a Hunt process realization $\xi$. Suppose that $K(x, dy)$ is a bounded kernel on $E$. We assume that $K(x, \cdot)$ depends on $x \in E$ continuously. Let $\beta(x) = K(x, E)$. Let $K_0(x, dy) = \beta(x)^{-1} K(x, dy)$ if $\beta(x) > 0$ and $K_0(x, dy) = \delta_x(dy)$ if $\beta(x) = 0$. By the concatenation argument described in Sharpe [7, p.82] it is not hard to construct a Markov process $\eta$ with the following properties:

(3A) The process evolves in $E$ according to the law given by the transition probabilities of $\xi$ until the random time $\tau_1$ with $P\{ \tau_1 > t \} = \exp \{- \int_0^t \beta(y_0) ds \}.$

(3B) At time $\tau_1$ the particle jumps from $\eta_{\tau_1}^-$ to a new place in $E$ according to the probability distribution $K(\eta_{\tau_1}^-, dy)$, and then moves randomly according to the transition probabilities of $\xi$ again until the random time $\tau_1 + \tau_2$ with $P\{ \tau_2 > t \} = \exp \{- \int_{\tau_1}^{\tau_1+\tau_2} \beta(y_0) ds \};$ and so on.

**Lemma 3.1** Suppose that $\xi$ has generator $(A, \mathcal{D}(A))$, where $\mathcal{D}(A) \subset C(E)$. Then $\eta$ has generator $(B, \mathcal{D}(B))$, where $\mathcal{D}(B) = \mathcal{D}(A)$ and

$$
B f(x) = A f(x) + \int_E |f(y) - f(x)| K(x, dy), \quad x \in E, f \in \mathcal{D}(B). \tag{3.1}
$$

Moreover, the transition semigroup of $\eta$ is Feller.
Observe that measure $\lambda$.

Since $(3.2)$ we get from

$$\exp \{-\xi\} \quad \text{for} \quad \exp \{\lambda\} \quad \text{imply that}
$$

For a fixed non-trivial measure $\lambda \in M(\mathbb{R})$ we consider a random variable $\zeta$ in $\mathbb{R}$ with distribution $\lambda(1)^{-1}\lambda$. For $\mu \in M(\mathbb{R})$, let $K(\mu, d\nu)$ denote the distribution of the random measure $X := \mu + \theta^{-1}\delta\zeta$.

Observe that

$$\int_{M(\mathbb{R})} [F(\nu) - F(\mu)] K(\mu, d\nu) = \lambda(1)^{-1} \int_{\mathbb{R}} [F(\mu + \theta^{-1}\delta\zeta) - F(\mu)] \lambda(dy). \quad (3.3)$$

This equation follows as we think about the behavior of the particle. It either moves according to $\xi$ without jumping until time $t$, or it first jumps at some time $s \in (0, t]$. The first event happens with probability $\exp \{-\int_0^t \beta(\xi_u) du\}$ and the second happens with probability $\exp \{-\int_0^t \beta(\xi_u) du\} \beta(\xi_s) ds$, giving the two terms of on the right hand side. For $f \in \mathcal{D}(A)$, we get from (3.2) that

$$Bf(x) = \lim_{t \downarrow 0} t^{-1} \mathbf{P}_x \left[ f(\xi_t) - f(\xi_0) \right] + \lim_{t \downarrow 0} t^{-1} \int_0^t \mathbf{P}_x \left[ f(\xi_t) - f(\xi_0) \right] K(\xi_s, Q_{t-s}f) ds
$$

$$= \lim_{t \downarrow 0} t^{-1} \mathbf{P}_x \left[ f(\xi_t) - f(\xi_0) \right] + \lim_{t \downarrow 0} t^{-1} \int_0^t \mathbf{P}_x \left[ f(\xi_t) - f(\xi_0) \right] K(\xi_s, Q_{t-s}f) ds
$$

$$= Af(x) - \beta(x)f(x) + K(x, f)
$$

$$= Af(x) + \int_{\mathbb{R}} [f(y) - f(x)] K(x, dy).$$

Since $(A, \mathcal{D}(A))$ generates a Feller transition semigroup, so does $(B, \mathcal{D}(B))$; see e.g. [4, p.37]. □
For $\theta > 0$ we can define the generator $\mathcal{J}_{\theta}$ by

$$
\mathcal{J}_{\theta} F(\mu) = \mathcal{L} F(\mu) + \theta \int_{\mathbb{R}} [F(\mu + \theta^{-1}\delta_x) - F(\mu)] \lambda(dx).
$$

(3.4)

By the result in [2], $\mathcal{L}$ generates a Feller semigroup on $M(\mathbb{R})$, then so does $\mathcal{J}_{\theta}$ by Lemma 3.1. We shall call the process generated by $\mathcal{J}_{\theta}$ a SDSM with discrete immigration with parameters $(a, \rho, \sigma, \lambda)$ and unit mass $1/\theta$. Intuitively, the immigrants come to $\mathbb{R}$ by cliques with mass $1/\theta$ with time-space configuration given by a Poisson random measure with intensity $\theta ds \lambda(dx)$. A more general immigration model for superprocesses with independent spatial motions has been considered in Li [6].

4 SDSM with continuous immigration

In this section, we construct a solution of the $(\mathcal{L}, D(\mathcal{L}))$-martingale problem by using an approximation by the SDSM with discrete immigration. Observe that, if

$$
F_{f,\{\phi_i\}}(\mu) = f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_n, \mu \rangle), \quad \mu \in M(\mathbb{R}),
$$

(4.1)

for $f \in C^2_b(\mathbb{R}^n)$ and $\phi_i \in C^2_b(\mathbb{R})$, then

$$
\mathcal{J} F_{f,\{\phi_i\}}(\mu) = \frac{1}{2} \sum_{i=1}^n f_i' \langle \phi_1, \mu \rangle \langle a \phi_i', \mu \rangle
+ \frac{1}{2} \sum_{i,j=1}^n f_i'' \langle \phi_1, \mu \rangle \langle \phi_n, \mu \rangle \int_{\mathbb{R}^2} \rho(x-y) \phi_i'(x) \phi_j'(y) \mu(dx) \mu(dy)
+ \frac{1}{2} \sum_{i,j=1}^n f_i'' \langle \phi_1, \mu \rangle \langle \phi_n, \mu \rangle \langle \sigma \phi_i \phi_j, \mu \rangle
+ \sum_{i=1}^n f_i' \langle \phi_1, \mu \rangle \langle \phi_n, \mu \rangle \langle \phi_i, \lambda \rangle.
$$

(4.2)

Let $\{\theta_k\}$ be any sequence such that $\theta_k \to \infty$ as $k \to \infty$. For $k \geq 1$, let $\{X_{t}^{(k)} : t \geq 0\}$ be a càdlàg SDSM with discrete immigration with parameters $(a, \rho, \sigma, m)$, unit $1/\theta_k$ and initial state $X_{0}^{(k)} = \mu_k \in M_{\theta_k}(\mathbb{R})$.

**Lemma 4.1** If the sequence $\{\langle 1, \mu_k \rangle\}$ is bounded, then $\{X_{t}^{(k)} : t \geq 0\}$ form a tight sequence in $D([0, \infty), M(\mathbb{R}))$.

**Proof.** Let $H(\nu) = \langle 1, \nu \rangle$. By (3.4), it is not hard to see that $\mathcal{J}_{\theta_k} H(\nu) = \langle 1, \lambda \rangle$. It follows that

$$
E_{\mu_k} \{\langle 1, X_{t}^{(k)} \rangle\} = \langle 1, \mu_k \rangle + \langle 1, \lambda \rangle t,
$$

$t \geq 0$.

Then $\{\langle 1, X_{t}^{(k)} \rangle - \langle 1, \lambda \rangle t : t \geq 0\}$ is a martingale. By a martingale inequality, for $u > 0$ and $\eta > \langle 1, \lambda \rangle u$ we have

$$
\mathbb{P} \left\{ \sup_{0 \leq t \leq u} \langle 1, X_{t}^{(k)} \rangle > 2\eta \right\} \leq \mathbb{P} \left\{ \sup_{0 \leq t \leq u} |\langle 1, X_{t}^{(k)} \rangle - \langle 1, \lambda \rangle t| > \eta \right\}.
$$

6
Let \( J \) be the generator of \( \{X_t^{(k)}: t \geq 0\} \) and let \( F_{f,\phi} \) be given by (4.1) with \( f \in C_0^2(\mathbb{R}^n) \) and with each \( \phi \in C_0^2(\mathbb{R}) \) bounded away from zero. Then

\[
F_{f,\phi}(X_t^{(k)}) - F_{f,\phi}(X_0^{(k)}) - \int_0^t JF_{f,\phi}(X_s^{(k)})ds, \quad t \geq 0,
\]

is a martingale and the desired tightness follows from the result of [4, p.145].

Now suppose that all functions in \( C_0(\mathbb{R}) \) are extended to \( \hat{\mathbb{R}} \) by continuity. If \( \sigma \in C_0(\mathbb{R})^+ \), we may regard \( F \) given by (4.1) and the right hand side of (4.2) as functions on \( M(\hat{\mathbb{R}}) \). Let \( \hat{J}F(\mu) \) be defined by the right hand side of (4.2) as a function on \( M(\hat{\mathbb{R}}) \). Let \( D(\hat{J}) \) be the totality of all functions of the form (4.1) with \( f \in C_0^2(\mathbb{R}^m) \) and with each \( \phi \in C_0^2(\mathbb{R}) \) bounded away from zero. Suppose that \( \mu_k \to \mu \in M(\hat{\mathbb{R}}) \) as \( k \to \infty \) and let \( Q_\mu \) be a limit point of the distributions of \( \{X_t^{(k)}: t \geq 0\} \). As in the proof of Lemma 4.2 in [2], we may see that \( Q_\mu \) is supported by \( C([0, \infty), M(\hat{\mathbb{R}})) \) and

\[
F_{f,\phi}(w_t) - F_{f,\phi}(w_0) - \int_0^t \hat{J}F_{f,\phi}(w_s)ds, \quad t \geq 0,
\]

is a martingale for each \( F_{f,\phi} \in D(\hat{J}) \), where \( \{w_t: t \geq 0\} \) denotes the coordinate process of \( C([0, \infty), M(\hat{\mathbb{R}})) \).

Lemma 4.2 Let \( Q_\mu \) be given as the above. Then for \( n \geq 1, t \geq 0 \) and \( \mu \in M(\mathbb{R}) \) we have

\[
Q_\mu \{(1, w_t)^n\} \leq (1, \mu)^n + n[(n - 1)\|\sigma\|^2/2 + (1, \lambda)] \int_0^t Q_\mu \{(1, w_s)^{n-1}\}ds.
\]

Consequently, \( Q_\mu \{(1, w_t)^n\} \) is a locally bounded function of \( t \geq 0 \). Let \( D(\hat{J}) \) be the union of all functions of the form (4.1) with \( f \in C_0^2(\mathbb{R}^m) \) and \( \phi \in C_0^2(\mathbb{R}) \) and all functions of the form \( F_{m,f}(\mu) = (f, \mu^m) \) with \( f \in C_0^2(\mathbb{R}^m) \). Then (4.3) under \( Q_\mu \) is a martingale for each \( F \in D(\hat{J}) \).

Proof. For any \( k \geq 1 \), take \( f_k \in C_0^2(\mathbb{R})) \) such that \( f_k(z) = z^n \) for \( 0 \leq z \leq k \) and \( f_k(z) \leq n(n - 1)z^{n-2} \) for all \( z \geq 0 \). Let \( F_k(\mu) = f_k((1, \mu)) \). It s easy to see that

\[
\hat{J}F_k(\mu) \leq n[(n - 1)\|\sigma\|^2/2 + (1, \lambda)](1, \mu)^{n-1}.
\]

Since

\[
F_k(X_t) - F_k(X_0) - \int_0^t \hat{J}F_k((1, X_s))ds, \quad t \geq 0,
\]

is a martingale, we get

\[
Q_\mu f_k((1, X_t)^n) \leq f_k((1, \mu)) + n[(n - 1)\|\sigma\|^2/2 + (1, \lambda)] \int_0^t Q_\mu ((1, X_s)^{n-1})ds
\]

\[
\leq (1, \mu)^n + n[(n - 1)\|\sigma\|^2/2 + (1, \lambda)] \int_0^t Q_\mu ((1, X_s)^{n-1})ds.
\]
Then the desired estimate follows by Fatou’s Lemma. The last assertion is immediate. □

By the martingale problem (4.3) and the last lemma, it is easy to find that for each \( \phi \in C^2_b(\mathbb{R}) \),
\[
M_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \langle \phi, m \rangle t - \frac{1}{2} \int_0^t \langle a \phi'', w_s \rangle ds, \quad t \geq 0,
\]
is a \( Q_\mu \)-martingale with quadratic variation process
\[
\langle M(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', w_s \rangle^2 dz. \tag{4.5}
\]

For a continuous branching density function \( \sigma \in C_\partial(\mathbb{R})^+ \), the existence of a SDSM with immigration is given by the following

**Theorem 4.1** Let \( \mathcal{D}(\mathcal{J}) \) be the union of all functions of the form (4.1) with \( f \in C^2_0(\mathbb{R}^n) \) and \( \phi_i \in C^2(\mathbb{R}) \) and all functions of the form \( F_{m, f}(\mu) = \langle f, \mu \rangle \) with \( f \in C^2(\mathbb{R}^m) \). Let \( \{w_t : t \geq 0\} \) denote the coordinate process of \( C([0, \infty), M(\mathbb{R})) \). Then for each \( \mu \in M(\mathbb{R}) \) there is a unique probability measure \( Q_\mu \) on \( C([0, \infty), M(\mathbb{R})) \) such that \( Q_\mu \{w_0 = \mu\} = 1 \), the moments \( Q_\mu \{(1, w_t)^m \} \) are locally bounded and \( \{w_t : t \geq 0\} \) under \( Q_\mu \) is a solution of the \( (\mathcal{J}, \mathcal{D}(\mathcal{J})) \)-martingale problem.

**Proof.** Let \( Q_\mu \) be as in Lemma 4.2. By Theorem 2.1, the \( (\mathcal{J}, \mathcal{D}(\mathcal{J})) \)-martingale problem has at most one solution possessing locally bounded moments of all degrees. Then the desired result follows once it is proved that
\[
Q_\mu \{w_t(\{\partial\}) = 0 \text{ for all } t \in [0, u]\} = 1, \quad u > 0. \tag{4.6}
\]

Let \( M(ds, dx) \) denote the stochastic integral relative to the martingale measure defined by (4.4) and (4.5). As in [2], we have
\[
\langle \phi, w_t \rangle = \langle \hat{P}_t \phi, \mu \rangle + \int_0^t \langle \hat{P}_{t-s} \phi, \lambda \rangle ds + \int_0^t \int_{\mathbb{R}} \hat{P}_{t-s} \phi(x) M(ds, dx)
\]
for \( t \geq 0 \) and \( \phi \in C^2_b(\mathbb{R}) \). For any fixed \( u > 0 \), we have that
\[
M^u(\phi) := \langle \hat{P}_{u-t} \phi, w_t \rangle - \langle \hat{P}_u \phi, \mu \rangle - \int_0^t \lambda(\hat{P}_{u-s} \phi) ds = \int_0^t \int_{\mathbb{R}} \hat{P}_{u-s} \phi M(ds, dx), \quad t \in [0, u],
\]
is a continuous martingale with quadratic variation process
\[
\langle M^u(\phi) \rangle_t = \int_0^t \langle \sigma(\hat{P}_{u-s} \phi)^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \hat{P}_{u-s} \phi', w_s \rangle^2 dz
\]
\[
= \int_0^t \langle \sigma(\hat{P}_{u-s} \phi)^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)(\hat{P}_{u-s} \phi)', w_s \rangle^2 dz.
\]
By a martingale inequality we have
\[ Q_\mu \left\{ \sup_{0 \leq t \leq u} \left| \langle \hat{P}_u t \phi, w_t \rangle - \langle \hat{P}_u \phi, \mu \rangle - \int_0^t \lambda(\hat{P}_u s) ds \right|^2 \right\} \]
\[ \leq 4 \int_0^u Q_\mu \{ \langle \sigma(\hat{P}_u s) \phi, w_s \rangle \} ds + 4 \int_0^u ds \int_{\mathbb{R}} Q_\mu \{ \langle h(z - \cdot) \hat{P}_u s \phi', w_s \rangle^2 \} dz \]
\[ \leq 4 \int_0^u \langle \sigma(\hat{P}_u s) \phi, \mu \rangle ds + 4 \int_{\mathbb{R}} h(z)^2 dz \int_0^u Q_\mu \{ \langle 1, w_s \rangle \langle \hat{P}_u s \phi', w_s \rangle \} ds \]
\[ \leq 4 \int_0^u \langle \sigma(\hat{P}_u s) \phi, \mu \rangle ds + 4 \| \phi' \|^2 \int_{\mathbb{R}} h(z)^2 dz \int_0^u Q_\mu \{ \langle 1, w_s \rangle^2 \} ds. \]

Choose a sequence \[ \{ \phi_k \} \subset C^2(\mathbb{R}) \] such that \( \phi_k(\cdot) \to 1_{\{ \theta \}}(\cdot) \) boundedly and \( \| \phi_k' \| \to 0 \) as \( k \to \infty \). Replacing \( \phi \) by \( \phi_k \) in the above and letting \( k \to \infty \) we obtain (4.6).

For a general \( \sigma \in B(\mathbb{R})^+ \), we may choose a bounded sequence of functions \( \{ \sigma_k \} \subset C_0(\mathbb{R})^+ \) such that \( \sigma_k \to \sigma \) pointwise out of a Lebesgue null set. Suppose that \( \{ \mu_k \} \subset M(\mathbb{R}) \) and \( \mu_k \to \mu \in M(\mathbb{R}) \) as \( k \to \infty \). For each \( k \geq 1 \), let \( \{ X^{(k)}_t \} : t \geq 0 \} \) be an immigration SDSM with parameters \( (a, \rho, \sigma_k, m) \) and initial state \( \mu_k \in M(\mathbb{R}) \) and let \( Q_k \) denote the distribution of \( \{ X^{(k)}_t : t \geq 0 \} \) on \( C([0, \infty), M(\mathbb{R})) \). By the arguments in the proofs of Theorems 5.1 and 5.2 in [2] we get

**Theorem 4.2**

As \( k \to \infty \), the sequence \( Q_k \) converges to a probability \( Q_\mu \) on \( C([0, \infty), M(\mathbb{R})) \). Let \( D(J) \) be as in Theorem 4.1 for the more general \( \sigma \in B(\mathbb{R})^+ \). Then \( Q_\mu \) is the unique probability measure on \( C([0, \infty), M(\mathbb{R})) \) such that \( Q_\mu \{ w_0 = \mu \} = 1 \) and \( \{ w_t : t \geq 0 \} \) under \( Q_\mu \) solves the \( (J, D(J)) \)-martingale problem. Consequently, \( \{ w_t : t \geq 0 \} \) under \( Q_\mu \) is a diffusion process with transition semigroup \( (Q_t)_{t \geq 0} \) defined by

\[ \int_{M(\mathbb{R})} \langle f, \nu^m \rangle Q_t(\mu, d\nu) = E^\sigma_{m,f} \left[ \langle Y_t, \mu^M \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s(M_s + 1) ds \right\} \right]. \]

This gives the existence of the SDSM with continuous immigration for a bounded measurable branching density \( \sigma \in B(\mathbb{R})^+ \). Clearly, we have that for each \( \phi \in C^2(\mathbb{R}) \),

\[ M_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \langle \phi, m \rangle t - \frac{1}{2} \int_0^t \langle a \phi'' , w_s \rangle ds, \quad t \geq 0, \]

is a \( Q_\mu \)-martingale with quadratic variation process

\[ \langle M(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2 , w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} h(z - \cdot) \phi', w_s \rangle^2 dz. \]

Conversely, if \( Q_\mu \) is the unique probability measure on \( C([0, \infty), M(\mathbb{R})) \) such that (4.8) is a martingale with quadratic variation process (4.9), by Itô’s formula one can show that \( Q_\mu \) is a solution of the \( (J, D(J)) \)-martingale problem. Then (4.8) and (4.9) give an alternate definition of the immigration SDSM.
5 Non-critical branching mechanism

Let $Q_\mu$ denote the distribution on $C([0, \infty), M(\mathbb{R}))$ of an $(a, \rho, \sigma, \lambda)$-superprocess with initial state $\mu \in M(\mathbb{R})$. Let $M(ds, dx)$ denote the martingale measure defined by (4.8) and (4.9). Then for any $b \in C^1(\mathbb{R})$ the stochastic integral

$$M_t(b) := \int_0^t b(x)M(ds, dx), \quad t \geq 0,$$

is well-defined and

$$\langle M(b) \rangle_t = \int_0^t \langle \sigma b^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) b', w_s \rangle^2 dz. \quad (5.2)$$

We consider the exponential martingale

$$Z_t(b) := \exp \left\{ - M_t(b) - \frac{1}{2} \langle M(b) \rangle_t \right\}, \quad t \geq 0. \quad (5.3)$$

Fix a constant $T > 0$ and let $Q_\mu^b(dw) = Z_T(w, b)Q_\mu(dw)$. By Girsanov’s theorem,

$$N_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \int_0^t \langle \phi'', w_s \rangle ds - \int_0^t \langle \sigma b \phi, w_s \rangle ds$$

$$+ \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) b', w_s \rangle \langle h(z - \cdot) \phi', w_s \rangle dz, \quad 0 \leq t \leq T, \quad (5.4)$$

is a $Q_\mu^b$-martingale with quadratic variation process

$$\langle N(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', w_s \rangle^2 dz, \quad 0 \leq t \leq T. \quad (5.5)$$

As usual, the coordinate process $\{w_t : 0 \leq t \leq T\}$ under $Q_\mu^b$ is a diffusion process; see e.g. [5, pp.190-197]. We call the new process a $(a, \rho, \sigma, b, \lambda)$-superprocess. Intuitively, the term $\int_0^t \langle \sigma b \phi, w_s \rangle ds$ in (5.5) represents a linear growth with growth rate $\sigma(\cdot)b(\cdot)$. Girsanov transformations of this type were introduced by Dawson [1] to get non-critical superprocesses for critical ones. Note that we have on the right hand side of (5.5) an extra term

$$\int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) b', w_s \rangle \langle h(z - \cdot) \phi', w_s \rangle dz, \quad (5.6)$$

which may be interpreted as a spatial drift with state-dependent coefficient

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(z - y)b'(y)h(z - \cdot)w_s(dy)dz = \int_{\mathbb{R}} b'(y)\rho(y - \cdot)w_s(dy). \quad (5.7)$$

This is different from the classical case where the Girsanov transform does not effect the spatial motion; see [1]. Let $\mathcal{D}(J^b)$ be the union of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^m)$ and $\phi_i \in C^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C^2(\mathbb{R}^m)$.

**Theorem 5.1** The $(a, \rho, \sigma, b, \lambda)$-superprocess solves the $(\mathcal{J}^b, \mathcal{D}(\mathcal{J}^b))$-martingale problem.
Proof. If $F_{f_i}$ is given by (4.1), we have

$$\mathcal{J}^bF_{f_i}(\mu) = \frac{1}{2} \sum_{i=1}^{n} f_i'(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle) (a \phi_i'' - 2b \phi, \mu)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} f_{ij}''(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle) \int_{\mathbb{R}^2} \rho(x - y) \phi_i'(x) \phi_j'(y) \mu(dx) \mu(dy)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} f_{ij}''(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle) (\sigma \phi_i, \phi_j, \mu)$$

$$- \sum_{i=1}^{n} f_i'(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle) \int_{\mathbb{R}^2} \rho(x - y) b'(y) \phi_i'(x) \mu(dx) \mu(dy)$$

$$+ \sum_{i=1}^{n} f_i'(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle) (\phi_i, \lambda). \quad (5.8)$$

Based on (5.4) and (5.5), it is easy to check by Itô’s formula that

$$F_{f_i}(w_t) - F_{f_i}(w_0) - \int_{0}^{t} \mathcal{J}^bF_{f_i}(w_s) ds, \quad 0 \geq t \leq T, \quad (5.9)$$

is a martingale under $Q^b_{\mu}$. Then the theorem follows by an approximation of an arbitrary $F \in \mathcal{D}(\mathcal{L})$. □

If $F_{m.f}(\mu) = \langle f, \mu^m \rangle$ for $f \in C_0^2(\mathbb{R}^m)$, then

$$\mathcal{J} F_{m.f}(\mu) = F_{m,G^m_b f}(\mu) + \frac{1}{2} \sum_{i,j=1}^{m} F_{m-1, \phi_{ij} f}(\mu)$$

$$+ \sum_{i=1}^{m} F_{m-1, \phi_i f}(\mu) + \sum_{i=1}^{m} F_{m+1, \Gamma_i f}(\mu), \quad (5.10)$$

where

$$G^m_b f(x_1, \cdots, x_m) = G^m f(x_1, \cdots, x_m) - \sum_{i=1}^{m} b(x_i) f(x_1, \cdots, x_m), \quad (5.11)$$

and

$$\Gamma_i f(x_1, \cdots, x_m, x_{m+1}) = -\rho(x_{m+1} - x_i) b'(x_{m+1}) f_i'(x_1, \cdots, x_m). \quad (5.12)$$

In view of this expression of the generator, we may construct a dual process which gives expressions for the moments of the $(a, \rho, \sigma, b, \lambda)$-superprocess.

References


