Math 648: Exercise sheet 6

Reading: Look through the second part of the notes on linear algebra. The material on bilinear forms (from both parts of these notes) will not be needed in 600 algebra (though its quite essential material for all subjects). But the other material you should be familiar with.

Questions: All rings are assumed unital with $1 \neq 0$, as usual!

1. Let $R$ be a commutative ring. For a group $G$, recall that $RG$ is the group algebra over $R$, that is, the free $R$-module on generating set $G$, with multiplication induced by the multiplication in $G$ so that it is $R$-bilinear.
   (i) Explain how the map $G \mapsto RG$ on objects can be made into a functor from the category of groups to the category of rings.
   (ii) Find a right adjoint to this functor.

2. Let $G = \mathbb{Z}_n$, the cyclic group of order $n$. Let $R = \mathbb{C}G$ and $\omega \in \mathbb{C}$ be a primitive $n$th root of unity. Find a basis for the group algebra $\mathbb{C}G$ as a $\mathbb{C}$-vector space consisting of idempotents $e_1, \ldots, e_n$ which are orthogonal in the sense that $e_i e_j = 0$ for $i \neq j$.

3. Let $R$ be a ring and $M$ be a left $R$-module. Define the socle of $M$ to be the sum of all the simple $R$-submodules of $M$.
   (i) Prove that $soc_R(M)$ is a semisimple left $R$-module.
   (ii) Suppose moreover that $M$ is an Artinian left $R$-module, i.e. every descending chain of submodules of $M$ stabilizes. Show that $soc_R(M) \neq 0$.

4. In class, we showed that an $R$-module $M$ is semisimple if and only if it is the sum of its simple submodules. Use this to prove that every module over a division ring is free.

5. Let $A$ be a two dimensional $\mathbb{R}$-algebra. Prove that $A$ has a basis $\{1, u\}$ where $u^2$ equals 0, 1 or $-1$. Verify that no two of these are isomorphic.

6. Let $R = M_n(D)$, $D$ a division ring. Show that each row is a minimal right ideal of $R$, and each column is a minimal left ideal.

7. Show that if $R$ is left Artinian, then so is $M_n(R)$ for each $n \geq 1$ (hint: consider the functors we used when proving $R$ is Morita equivalent to $M_n(R)$).

8. Show that every finitely generated left module over a left Artinian ring has a composition series (a finite chain of submodules $0 < M_1 < \cdots < M_n = M$ with each factor $M_i/M_{i-1}$ being a simple module).
   By the way, it is true (by the same proof as we gave for groups) that for any $R$-module with a composition series, the composition factors in any two composition series are isomorphic after reordering ("Jordan-Holder theorem").

9. Here is an example of a ring which is left but not right Artinian: the set of all $2 \times 2$ matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$. Verify!