Math 648: Exercise sheet 2

Due F Jan 21.
Read the “notes on linear algebra”.
IV, §6, 4,5,6
My questions. Always, $R$ is a PID.
(i) Let $M$ be a finitely generated $R$-module with invariant factor sequence $d_1, \ldots, d_s$. Prove that $M$ cannot be generated by less than $s$ elements.
(ii) Find the invariant factor sequence AND the primary decomposition for the following $R$-modules:
(a) an $n$-dimensional vector space $V$ over a field $k$, with $R = k$.
(b) the same vector space as in (a) but with $R = k[x]$, where $x$ acts on $V$ by $xv_i = v_{i+1}$, where $v_1, \ldots, v_n$ is a $k$-basis of $V$ and $v_{n+1} = 0$.
(c) $\mathbb{Z}_{2000}$ with $R = \mathbb{Z}$.
(iii) Let $M$ be a finitely generated $R$-module with $p^aM = 0$ for some prime $p \in R$ and some $a \geq 1$. Suppose that $x \in M$ has order exactly $p^a$. Show that $M = N \oplus Rx$ for some submodule $N$ of $M$.
(iv) Let $p$ be a prime and $A$ be a non-trivial cyclic group of $p$-power order. Prove that the equation $px = 0$ has $p$ solutions in $A$. More generally, prove that if $A$ is the direct sum of $s$ non-trivial cyclic groups of $p$-power order then the equation $px = 0$ has $p^s$ solutions.
(v) Let $k$ be a finite field of characteristic $p$. Let $k^* = k \setminus \{0\}$, viewed as a finite Abelian group. Using (iv), prove that each primary component of $k^*$ is cyclic. Deduce that $k^*$ is cyclic.
(vi) Let $p$ be a prime and $A$ be a finite Abelian group with invariant factor sequence $p^{a_1}|\ldots|p^{a_s}$. Let $B$ be a subgroup of $A$. Prove that $B$ is a direct sum of cyclic groups of orders $p^{b_1}, \ldots, p^{b_s}$ where $0 \leq b_1 \leq \cdots \leq b_s$ and $b_i \leq a_i$ for each $i$. $p$-group.