Math 648: Exercise sheet 1

Due F Jan 14.

I. Reread sections 1, 2, 3 of chapter II to compare what we’re doing now for PIDs with what was done there for \( \mathbb{Z} \). Then read section 7 of chapter IV where Hungerford gives proofs of the main results we’re working on. The proofs there are slightly shorter but more technical than ours...

II. IV, §7:1,2,3.

III. Do the following questions:

(i) Prove that every submodule of a finitely generated module over a PID is finitely generated.

(ii) Let \( A = \begin{bmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{bmatrix} \). Using the algorithm from class, find invertible matrices \( X \) and \( Y \) over \( \mathbb{Z} \) such that \( XAY \) has the form \( \text{diag}(d_1, d_2, d_3) \) with \( d_1 | d_2 | d_3 \).

(iii) Give an example of two matrices over \( \mathbb{Z} \) which are not equivalent over \( \mathbb{Z} \), but which are equivalent as matrices over \( \mathbb{Q} \).

(iv) Let \( R \) be a PID and \( A \) be an \( n \times n \) matrix over \( R \). Prove that \( A \) is invertible if and only if \( A \) is equivalent to the identity matrix.

(v) Find the elementary divisors of the matrix \( \begin{bmatrix} 2 & 1+i & 1-i \\ 8+6i & -4 & 0 \end{bmatrix} \) working over the ring \( \mathbb{Z}[i] \) of Gaussian integers.

(vi) Show that the matrix \( A = \begin{bmatrix} 2x & 0 \\ x & 2 \end{bmatrix} \) is not equivalent over \( \mathbb{Z}[x] \) to a diagonal matrix.

(Hint: consider the ideals \( J_i(A) \)).

(vii) Let \( R \) be a commutative, unital ring. Is it true \( R \) is indecomposable viewed as an \( R \)-module? What if \( R \) is a PID?

(viii) Let \( M \) be the set of all infinite sequences \((z_1, z_2, \ldots)\) of \( z_i \in \mathbb{Z} \). Regard \( M \) as a \( \mathbb{Z} \)-module with coordinate-wise addition and scalar multiplication defined by \( r(z_1, z_2, \ldots) = (rz_1, rz_2, \ldots) \) for \( r \in \mathbb{Z} \). Prove that \( M \cong \mathbb{Z} \oplus M \) as a \( \mathbb{Z} \)-module. Why could this never happen for finitely generated \( \mathbb{Z} \)-modules?