Math 647 exercise sheet 7

This is the last homework of term. I will collect it on the Friday of dead week.

I. Read: sections 1,2,3 from chapter III.

II. Section 2: 23,24.
Section 3: 1,5,7,10.
Section 5: 7,9.
Section 6: 1.

III. Try the following questions:
(i) Prove that $\mathbb{Z}[i]$ (Gaussian integers) is a Euclidean Domain. (Hint: Set $\phi(a + ib) = a^2 + b^2$. To verify the conditions, let $a, b \in \mathbb{Z}[i]$ with $b \neq 0$. To find $q \in \mathbb{Z}[i]$, show that you can choose an “integer vertex” $q$ of the square in the complex plane containing the complex number $a/b$ with $|a/b - q| < 1$.
(ii) Since $\mathbb{Z}[i]$ is a PID, it is a UFD so greatest common divisors exist. Use the division algorithm (see II.3 13) to find a GCD of $11 + 7i$ and $3 + 7i$ in the ring $\mathbb{Z}[i]$.
(iii) Show that $\mathbb{Z}[-5]$ is not a UFD (hint: consider factoring 6).
(iv) Let $R$ be a PID and $p \in R^*$. Prove that the following are equivalent:
(a) $p$ is prime;
(b) $p$ is irreducible;
(c) $(p)$ is a maximal ideal;
(d) $R/(p)$ is a field;
(e) $R/(p)$ is an integral domain.
(v) Let $R$ be a PID and $S$ an integral domain, and $f : R \to S$ be an epimorphism. Show that either $f$ is an isomorphism or that $S$ is a field.
(vi) Let $R$ be a commutative ring with $1 \neq 0$. Show that there exists an epimorphism from $R[x]$ onto $R$. Deduce that $R[x]$ is a PID if and only if $R$ is a field.