Math 647: Group theory review questions

These questions are intended to give you an idea of the sort of thing I’m likely to ask on the midterm. I will hand out solutions in class before the midterm – these problems will not be graded!

1. Define the following:
   (i) solvable group;
   (ii) free group on the set $X$;
   (iii) normal subgroup;
   (iv) faithful action.

2. State the following theorems:
   (i) Jordan-Hölder theorem;
   (ii) 1st Sylow theorem;
   (iii) Cauchy’s theorem;
   (iv) Lattice isomorphism theorem.

3. True or False? Prove or give a counterexample.
   (i) For any groups $G, H$, $\text{Aut}(G \times H) \cong \text{Aut} G \times \text{Aut} H$.
   (ii) If $G/Z(G)$ is solvable, then $G$ is solvable.
   (iii) Every group of order $p^2q$ ($p, q$ primes) is nilpotent.
   (iv) If $H, K$ are two subgroups of $G$ of index 2, then $H = K$.

4. Let $p, q$ be distinct primes. How many elements of order $p$ are there in the following groups:
   (i) $\mathbb{Z}_{pq}$;
   (ii) $\mathbb{Z}_p \times \mathbb{Z}_p$;
   (iii) $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$;
   (iv) $\mathbb{Z}_{p^\infty}$.

5. Prove there is no simple group of order 640.

6. Let $|G| = p^n$, $p$ prime. Prove that $Z(G) \neq 1$.

7. Let $p, q$ be distinct odd primes. Prove that any group of order $p^2q^2$ is solvable.

8. Prove that $D_n/Z(D_n) \cong D_{n/2}$ if $n$ is even. What if $n$ is odd?

9. Let $X \subseteq Y$. Let $F$ be free on $X$, $G$ be free on $Y$. Prove that $F$ is isomorphic to the subgroup of $G$ generated by $X$.

10. Calculate $|G|$ if $G = \langle a, b, c | a^2 = b^3 = e, ab = c, bc = cb \rangle$.

11. Let $G$ be the group of all rotational symmetries of a regular tetrahedron. Prove $G \cong A_4$.

12. Find all composition series with no trivial factors for the following groups:
   (i) $D_3$;
   (ii) $A_4$;
   (iii) $S_3 \times \mathbb{Z}_2$;
   (iv) $D_6$.