Math 647: Exercise sheet 4

Due F Oct 29.

I. Look over §§6–8 from chapter two of Hungerford.
II. Hungerford §II.5. 6, 11, 13.
III. If $G$ is a group, its commutator subgroup $G'$ is defined as the subgroup generated by the elements $\{ghg^{-1}h^{-1} | g, h \in G\}$.
   (i) Show that $G'$ can be characterized as the unique smallest normal subgroup of $G$ such that the factor group $G/G'$ is Abelian.
   (ii) If $G$ is a non-Abelian group of order $p^3$, prove that $Z(G) = G'$.
   (iii) Calculate $G'$ in case $G = S_n$ for all $n \geq 2$.
IV. We have shown in class that there are no simple groups of order $p^k$ or $pq$, where $p, q$ distinct primes. (You should remember how). Now use the Sylow theorems to prove:
   (i) There are no simple groups of order $4p$ for $p$ an odd prime.
   (ii) There are no simple groups of order $pqr$ where $p, q, r$ are distinct odd primes.
   (iii) There are no simple groups of order $2pq$ where $p, q$ are distinct odd primes.
   (iv) Now prove that there are no non-Abelian simple groups of order less than 60.
   (v) We have shown that $A_5$ is a non-Abelian simple group of order 60. Prove that any simple group of order 60 is isomorphic to $A_5$. (Hint: show that a simple group of order 60 has exactly 5 Sylow 2-subgroups.)
V. Hungerford §II.6. 4.
VI. Hungerford §II.7. 8.
VII. Let $G_n$ be the group with generators $\{s_1, s_2, \ldots, s_{n-1}\}$ subject to the relations $s_i^2 = e, s_is_j = s_js_i$ for $|i - j| > 1$ and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$. Let $S_n$ denote the symmetric group, and $t_i$ denote the basic transposition $(i \ i+1)$ in $S_n$.
   (i) Prove that the $t_i$ satisfy the same relations as the $s_i$.
   (ii) Embed $S_{n-1}$ into $S_n$ as the subgroup consisting of all permutations fixing $n$. Prove that $\{1, t_{n-1}, t_{n-2}t_{n-1}, \ldots, t_1t_2 \ldots t_{n-1}\}$ is a set of $S_n/S_{n-1}$-coset representatives.
   (iii) By considering the subgroup $G_{n-1}$ of $G_n$ generated by $s_1, \ldots, s_{n-2}$ only and using induction, prove that $G_n \cong S_n$. 

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