Math 648 Final

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. For any $(S, R)$-bimodule $P$, $(P\otimes_R?, \text{Hom}_S(P, ?))$ is an adjoint pair of functors. Explain carefully what this sentence means.

Part II. True or False. Justify your answers briefly.

1. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ as an abelian group.
2. If $M$ is a free $R$-module and $X \subseteq M$ is a minimal spanning set, then $X$ is a basis for $M$.
3. If $G$ is a finite abelian group and $\mathbb{Z}(p)$ is the localization of $\mathbb{Z}$ at the prime ideal $(p)$, then $\mathbb{Z}(p) \otimes_{\mathbb{Z}} G$ is a $p$-group.
4. Any $R$-module of finite length can be decomposed into a direct sum of cyclic submodules.
5. For any ring $R$, the map $f \mapsto f(1_R)$ is a ring isomorphism $\text{End}_R(RR) \cong R$.

Part III. Longer problems.

1. Use Zorn’s lemma to prove that any submodule of a semisimple module has a complement.
2. Let $R$ be an integral domain and let $S$ be the ring of upper triangular $3 \times 3$ matrices with entries in $R$. Find $n \geq 1$ and a collection of orthogonal, primitive idempotents $e_1, \ldots, e_n \in S$ such that $e_1 + \cdots + e_n = 1_S$.
3. Let $f : V \to V$ be an endomorphism of an $n$ dimensional real vector space such that the endomorphism $\text{id}_C \otimes f$ of the complex vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ is diagonalizable. Prove that there exists a basis for $V$ with respect to which the matrix of the linear transformation $f$ is a block diagonal matrix, with diagonal blocks either being $1 \times 1$ matrices of the form $[\lambda]$ for $\lambda \in \mathbb{R}$ or being $2 \times 2$ matrices of the form $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ for $r > 0$ and $0 < \theta < \pi$. When are two such block diagonal matrices similar?
4. Prove that the multiplicative group $C_p^\infty$ consisting of all $p^n$th roots of 1 in $\mathbb{C}^\times$ for all $n \geq 0$ is an injective abelian group.