Exercise sheet 3

These mindless questions are intended to consolidate the confusing combinatorial definitions in the reading!!

1. Let $(i,j) = ((2,2,3,4,1),(1,1,3,3,4)) \in I^2(n,r)$. Write down:
   (i) the row standard tableau $a$ corresponding to $(i,j)$ under our fixed bijection $I^2(n,r) \to T^+_{n,r}$;
   (ii) the shape $\lambda$ of $a$;
   (iii) the weight $\mu$ of $i$;
   (iv) the row standard $\mu$-tableau $b$ corresponding to the double index $(j,i)$;
   (v) the row standard tableau corresponding to the double index $(iw,jw)$ if $w = (123) \in S_5$.

2. Let $n = 2, r = 3$ and list the sets $T^+_{n,r}$ and $T^{++}_{n,r}$. Hence verify explicitly in this example that
   \[ |T^+_{n,r}| = |\{(a,b) \in T^{++}_{n,r} \times T^{++}_{n,r} | a \text{ and } b \text{ have the same shape}\}|. \]

3. The weight of a tableau $a$ is defined to be the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda(n,r)$ where $\lambda_i$ is the number of times the number $i$ appears in $a$. List all standard $(3,2,1)$-tableaux of weight $(2,2,2)$.

4. Take $\lambda \in \Lambda^+(n,r)$. Prove that the only standard $\lambda$-tableau of weight $\lambda$ is the tableau $1_\lambda$.

5. Let $a$ be a standard $\lambda$-tableau of weight $\mu$. Prove from the definitions that $\lambda \geq \mu$ in the dominance order.

6. Prove that $S^\#_{n,r}$ has basis consisting of all $\phi_a$ for all $a \in T^+_{n,r}$ such that every entry in row $i$ of $a$ is $\leq i$. What is the analogous condition on $\phi_a$ to lie in $S^\#_{n,r}$?

7. Let $\Theta_{n,r}$ denote the set of all $n \times n$-matrices with non-negative integer entries summing to $r$.
   (i) Convince yourself that the map $f : T^+_{n,r} \to \Theta_{n,r}, a \mapsto (a_{i,j})$ where $a_{i,j}$ is the number of entries equal to $i$ in the $j$th row of $a$ equal to $j$, is a bijection.
   (ii) Show that $a \geq b$ on the dominance order on $T^+_{n,r}$ if and only if
   \[ \sum_{i=1}^s \sum_{j=1}^t a_{ij} \geq \sum_{i=1}^s \sum_{j=1}^t b_{ij} \]
   for all $s, t = 1, \ldots, n$.
   (iii) Interpret the shape (respectively, the weight) of a tableau $a$ as the ‘column sum’ (respectively, the ‘row sum’) of the matrix $f(a)$.

8. List the distinguished $S_n/S_\lambda$-coset representatives explicitly for $n = 6, \lambda = (3,2,1)$. Then determine the Bruhat ordering on the resulting coset representatives.