Exercise sheet 2

Let $F$ be an infinite field.

1. Consider the algebra $S_{n,1}$. Write down Schur’s product rule for multiplication of $\xi_{i,j}$ and $\xi_{h,l}$ in this special case. Deduce that $S_{n,1}$ is isomorphic to the algebra of all $n \times n$ matrices over $F$.

2. Let $S^\#$ be the subspace of $S_{n,r}$ spanned by $\xi_{i,j}$ for all $i, j \in I(n,r)$ with $i_1 \leq j_1, \ldots, i_r \leq j_r$. Prove directly using Schur’s product rule that $S^\#$ is a subalgebra of $S_{n,r}$.

3. Prove that $\dim S_{n,r} = \binom{n^2 + r - 1}{r}$.

4. Let $w$ be the permutation $(1 \ n)(2 \ n-1)(3 \ n-2) \cdots \in \Sigma_n$. Calculate the length of $w$ as defined in the reading. Convince yourself that this is the unique longest element of $\Sigma_n$.

5. This question depends on reading the definition of the dominance order on row standard tableaux, as defined in the reading. List all row standard $\lambda$-tableaux of weight $\mu$, for $\lambda = (3, 2)$ and $\mu = (2, 1, 2)$. Then work out the diagram for the dominance ordering on the resulting tableaux.

6. The goal in this question is to construct in general an embedding of the algebra $S_{n,r}$ into a matrix algebra.

   (i) Let $F_n = F\langle x_{i,j} \mid 1 \leq i, j \leq n \rangle$ denote the free non-commutative $F$-algebra in $n^2$ indeterminates. Prove that the unique algebra maps defined by

   $$\Delta : F_n \to F_n \otimes F_n, x_{i,j} \mapsto \sum_{k=1}^{n} x_{i,k} \otimes x_{k,j}$$

   and

   $$\varepsilon : F_n \to F, x_{i,j} \mapsto \delta_{i,j}$$

   make $F_n$ into a well-defined bialgebra.

   (ii) Let $K_n$ be the ideal of $F_n$ generated by \{ $x_{i,j} - x_{j,i} \mid 1 \leq i \neq j \leq n$ \}. Prove that $K_n$ is a coideal, hence that the quotient algebra $F_n/K_n$ is a bialgebra.

   (iii) Prove that $F_n/K_n \cong A_n$ as bialgebras.

   (iv) Let $F_{n,r}$ and $K_{n,r}$ denote the degree $r$ parts of each, so that $F_{n,r}/K_{n,r} \cong A_{n,r}$ as coalgebras by (iii). Explain why the dual algebra $A_{n,r}^*$ is a naturally embedded subalgebra of the dual algebra $F_{n,r}^*$.

   (v) Compute the product rule between basis elements of the algebra $F_{n,r}^*$, for the basis of $F_{n,r}$ dual to the basis of $F_{n,r}$ consisting of all monomials. Deduce that $F_{n,r}^*$ is isomorphic to the $F$ algebra of all $N \times N$ matrices over $F$, where $N = n^r$.

   (vi) Combining (iv) and (v), we have an embedding of $S_{n,r}$ into the full matrix algebra. This is usually called the natural representation of $S_{n,r}$.

7. Let $V$ be the natural $GL_n(F)$-module $F^n$ of column vectors.

   (i) Decompose $V$ as a direct sum of weight spaces. Since $V$ is polynomial of degree 1, the weights appearing should be in the set $\Lambda(n, 1)$. 

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(ii) Now do the same as (i) for the $r$-fold tensor power $V^\otimes r$ of $V$, $GL_n(F)$ acting in the usual way on the tensor product.

(iii) The tensor space $V^\otimes r$ is a polynomial degree $r$ representation of $GL_n(F)$. Write down the corresponding structure map of $V^\otimes r$ regarded as a right $A_{n,r}$-comodule.

(iv) Hence determine a formula for the explicit action of a basis element $\xi_{ij}$ of $S_{n,r}$ on a basis of $V^\otimes r$.

(v) Actually, $V^\otimes r$ is the natural representation in 3(vi) in a different realization.