Math 261: Midterm practise questions

1. (i) Say precisely what it means for the limit of $f(x)$ as $x$ approaches $a$ to be $l$ (that is, $\lim_{x \to a} f(x) = l$).
   (ii) Prove, using the definition you provided above, that if
   
   $$ f(x) = \begin{cases} 
   -3x + 6 & x \neq 3 \\
   0 & x = 3 
   \end{cases} $$

   then
   
   $$ \lim_{x \to 3} f(x) = -3 $$

   (iii) Prove, using the definition you provided above, that if $f(x) = x^2$ then
   
   $$ \lim_{x \to 5} f(x) = 25. $$

   **Solution.**

   (i) For all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$.
   (ii) Take $\epsilon > 0$. Set $\delta = \epsilon/3$. Then, let $0 < |x - 3| < \delta$. In other words, $3 - \epsilon/3 < x < 3 + \epsilon/3$ and $x \neq 3$. Then, $-3(3 + \epsilon/3) + 6 < -3x + 6 < -3(3-\epsilon/3) + 6$ so $-3-\epsilon < -3x + 6 < -3+\epsilon$. This says that $|f(x) - (-3)| < \epsilon$ which is what we wanted.
   (iii) Take $\epsilon > 0$. Set $\delta = \min(\epsilon/11, 1)$. Then, if $0 < |x - 5| < \delta$, we have that $f(x) - 25 = x^2 - 25 = (x - 5)(x + 5) = (x - 5)(x - 5 + 10)$. So,
   
   $$ |f(x) - 25| \leq |x - 5|(|x - 5| + 10) < \delta(\delta + 10) \leq \frac{\epsilon}{11}(1 + 10) = \epsilon. $$

   This is what we needed.

   2. (i) Define what it means to say the function $f(x)$ is continuous at $a$.
   (ii) Prove (using $\epsilon$ and $\delta$) that the function
   
   $$ f(x) = \begin{cases} 
   x & x \in \mathbb{Q} \\
   0 & \text{else} 
   \end{cases} $$

   is continuous at 0.

   **Solution.**

   (i) It means $\lim_{x \to a} f(x) = f(a)$.
   (ii) Take $\epsilon > 0$. Set $\delta = \epsilon$. Then, for $|x| < \delta$, we have that $|f(x)| = |x| < \delta$ for $x \in \mathbb{Q}$ or $|f(x)| = 0$ otherwise. Either way, $|f(x)| < \epsilon$. This proves that $\lim_{x \to 0} f(x) = 0$. But $f(0) = 0$ by definition. Hence, $f$ is continuous at 0.
3. Answer the following true or false. If you can, give a very short justification for your answer.

(i) \( f(x + y) = f(x) + f(y) \).
(ii) \( [(g + h) \circ f](x) = g(f(x)) + h(f(x)) \).
(iii) \( |x + y| \leq |x| + |y| \).
(iv) \( |x - y| \leq |x| - |y| \).
(v) \( |x| - |y| \leq |x - y| \).
(vi) If \( f(a) = f(b) \) then \( a = b \).

Solution.

(i) False. Consider \( f(x) = x^2 \). Then, \( f(4) = 16 \neq f(2) + f(2) \).
(ii) True. By the definitions, \( [(g + h) \circ f](x) = (g + h)(f(x)) = g(f(x)) + h(f(x)) \).
(iii) True. We proved this in class.
(iv) False. Take \( x = 2, y = 3 \). Then LHS = 1, RHS = -1.
(v) True. Set \( x' = x + y, y' = y \) in (iii), so \( x = x' - y' \). Then, (iii) reads \( |x'| \leq |x' - y'| + |y'| \). Rearranging gives \( |x'| - |y'| \leq |x' - y'| \).
(vi) False. Take \( f(x) = x^2 \). Here, \( f(1) = f(-1) \), but \( 1 \neq -1 \).

4. Find the following limits.

(i) \( \lim_{x \to \infty} \frac{2x^2 - \cos(x)}{x^2 + x + 1} \)

(ii) \( \lim_{x \to a} \frac{x^3 - a^3}{x - a} \).

(iii) \( \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \)

(iv) \( \lim_{x \to \infty} (\sqrt{x^2 + 2x} - x) \).

Solution.

(i) \( \lim_{x \to \infty} \frac{2x^2 - \cos(x)}{x^2 + x + 1} = \lim_{x \to \infty} \frac{2x^2}{x^2 + x + 1} - \lim_{x \to \infty} \frac{\cos(x)}{x^2 + x + 1} \)

\( = \lim_{x \to \infty} \frac{2(x^2 + x + 1) - 2x - 2}{x^2 + x + 1} - 0 = 2 - \lim_{x \to \infty} \frac{2x + 2}{x^2 + x + 1} = 2 - 0 = 2. \)

(ii) \( \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2)}{x - a} = \lim_{x \to a} (x^2 + ax + a^2) = 3a^2. \)
(iii) 
\[
\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{1}{\lim_{x \to 4} \sqrt{x} + 2} = \frac{1}{4}
\]
(iv) 
\[
\lim_{x \to \infty} (\sqrt{x^2 + 2x} - x) = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 2x} - x)(\sqrt{x^2 + 2x} + x)}{\sqrt{x^2 + 2x} + x} = \lim_{x \to \infty} \frac{2}{\sqrt{x^2 + 2x}} = \lim_{x \to \infty} \frac{2}{\sqrt{1 + 2/x} + 1} = \frac{2}{1 + 1} = 1.
\]

5. Find the following limits.

(i) 
\[
\lim_{x \to \infty} \frac{3x^2 - x \sin(x) + x^2 \sin(x)}{x^2 + 4x + 3}
\]

(ii) 
\[
\lim_{x \to a} \frac{1}{x} - \frac{1}{a},
\]

(iii) 
\[
\lim_{x \to 4} \frac{x^2 - 16}{x - 4}
\]

(iv) 
\[
\lim_{x \to \infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 + 3x})
\]

**Solution.**

(i) 
\[
\lim_{x \to \infty} \frac{3x^2 - x \sin(x) + x^2 \sin(x)}{x^2 + 4x + 3} = \lim_{x \to \infty} \frac{(x^2 + 4x + 3)(3 + \sin x) - (4x + 3)(3 + \sin x) - x \sin x}{x^2 + 4x + 3} = \lim_{x \to \infty} (3 + \sin x) - \lim_{x \to \infty} \frac{(4x + 3)(3 + \sin x) - x \sin x}{x^2 + 4x + 3} = \lim_{x \to \infty} (3 + \sin x) - 0
\]

which does not exist.

(ii) 
\[
\lim_{x \to a} \frac{1}{x} - \frac{1}{a} = \lim_{x \to a} \frac{a - x}{ax} = -\lim_{x \to a} \frac{1}{ax} = -\frac{1}{a^2}.
\]
(iii) 
\[ \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} \lim_{x \to 4} (x + 4) = 8. \]

(iv) 
\[ \lim_{x \to \infty} \frac{-x}{\sqrt{x^2 + 2x} + \sqrt{x^2 + 3x}} = -\lim_{x \to \infty} \frac{1}{\sqrt{1 + 2/x} + \sqrt{1 + 3/x}} = -\frac{1}{2}. \]

6. Suppose that \( f(x) > 7 \) for all \( x \) and that \( \lim_{x \to 11} f(x) = \ell \). Prove that \( \ell \geq 7 \). Is it possible that \( \ell = 7 \)?

**Solution.**
Suppose for a contradiction that \( \ell < 7 \). Let \( \epsilon = 7 - \ell > 0 \). By the definition of limit, there exists \( \delta > 0 \) such that \( 0 < |x - 11| < \delta \) implies \( |f(x) - \ell| < \epsilon \). But that means that for such an \( x \), \( f(x) < \ell + \epsilon = 7 \) which contradicts the assumption that \( f(x) > 7 \) for all \( x \).

For the second part its possible. For example you could have
\[ f(x) = \begin{cases} 7 + |x - 11| & x \neq 11, \\ 8 & x = 11. \end{cases} \]

7. Find a pair of successive integers so that \( 4x^3 - 3x^4 + 1 \) has a zero between them. State the theorem that you are using.

**Solution.**
We use the Intermediate Value Theorem, which says that if \( f \) is continuous on \([a, b]\) and \( f(a) > 0, f(b) < 0 \) then there is an \( x \in [a, b] \) so that \( f(x) = 0 \).

In our case \( f(1) = 2 > 0 \) and \( f(2) = -15 < 0 \). So there is a 0 between 1 and 2.

8. Give an example of a function that is continuous on \((a, b)\), and bounded above on \((a, b)\) but so that it does not have a maximum value on \((a, b)\). Give the supremum of the values of the function on \((a, b)\).

**Solution.**
One example is
\[ (0, 1) = (a, b), f(x) = x. \]

Then the supremum of the values is 1, but \( f(x) \) is never 1 on \((0, 1)\).