Math 261: Homework 3 solutions

Ch. 3

5. (i) $P \circ s$. (ii) $s \circ P$. (iii) $s \circ S$. (iv) $S \circ s$. (v) $P \circ P$. (vi) $s \circ (P + P \circ S)$. (vii) $s \circ s \circ P \circ P \circ s$. (viii) $P \circ S \circ s + s \circ S + S \circ s \circ (S + s)$.

6. (a) $f_i(x) = \prod_{1 \leq j \leq n, j \neq i} \frac{(x - x_j)}{(x_i - x_j)}$.

(b) $f = \sum_{i=1}^{n} a_i f_i$

8. If $f(f(x)) = x$ then the domains of $f(f(x))$ and of $x$ must certainly be the same. If $c \neq 0$ then $x = -d/c$ is not in the domain of $f(x)$, hence its not in the domain of $f(f(x))$ either. So if $c \neq 0$ there’s no way $f(f(x)) = x$ for all $x$. This shows that $c = 0$. Hence $f(x) = ax/d + b/d$ (and $d \neq 0$ for sure or it wouldn’t make sense). Now let’s expand the equation $f(f(x)) = x$:

$$a(ax/d + b/d)/d + b/d = x$$

Hence

$$a^2x/d^2 + (a/d + 1)b/d = x$$

Hence

$$a^2x + (a + d)b = d^2x.$$

Hence $(a^2 - d^2)x + (a + d)b = 0$. If this is to be true for all $x$, the coefficients $a^2 - d^2$ and $(a + d)b$ must both be zero. Hence $a^2 = d^2$ and either $b = 0$ or $a + d = 0$. Hence either $b = c = 0$ and $a = \pm d \neq 0$ OR $c = 0, a = -d \neq 0$.

Here’s my final answer: either $a = d \neq 0, b = c = 0$ or $a = -d \neq 0, c = 0$.

13. (a) Let $E(x) = \frac{1}{2}(f(x) + f(-x))$ and $O(x) = \frac{1}{2}(f(x) - f(-x))$. Then $E(x) = E(-x)$ and $O(x) = -O(-x)$. So $E$ is even and $O$ is odd, and:

$$f(x) = E(x) + O(x).$$

(b) Suppose $f(x) = E(x) + O(x)$ where $E$ is even and $O$ is odd. Then,

$$f(x) = E(x) + O(x),$$

and

$$f(-x) = E(-x) + O(-x) = E(x) - O(x)$$

using that $E$ is even and $O$ is odd. Adding gives

$$2E(x) = f(x) + f(-x),$$
so \( E(x) = \frac{1}{2}(f(x) + f(-x)) \). Similarly, subtracting the equations gives the formula for \( O(x) \) in (i).

17. Just follow the steps! We’re assuming \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), and \( f \) is not always zero.

(a) Since \( f \) is not always zero we can find \( x \) with \( f(x) \neq 0 \). Then, \( f(x) = f(x1) = f(x)f(1) \). Since \( f(x) \neq 0 \) we can cancel to get \( f(1) = 1 \). A similar argument using addition shows \( f(0) = 0 \).

(b) Consider \( f(a) = f(\frac{a}{b}, b) = f(\frac{a}{b})f(b) \). So if I could show that \( f(a) = a \) for \( a \) an integer, this would give \( f(a/b) = a/b \), i.e. \( f(x) = x \) for any rational \( x \).

Well, \( f(1) = 1 \) by (a), so \( f(2) = f(1) + f(1) = 1 + 1 = 2 \). And so on, get \( f(n) = n \) for \( n \) positive. Now \( 0 = f(0) = f(n-n) = f(n) + f(-n) \). So \( f(-n) = -n \) so \( f(n) = n \) for all \( n \in \mathbb{Z} \).

(c) Suppose \( x > 0 \). Write \( x = y^2 \) for some \( y \), i.e. \( y = \sqrt{x} \) (don’t worry that we don’t know this is possible strictly!). Then, \( f(x) = f(y^2) = f(y)f(y) = f(y)^2 \geq 0 \). But it can’t be zero: if \( f(y) = 0 \) then we’d get \( 1 = f(1) = f(y, \frac{1}{y}) = f(y)f(1/y) = 0 \), a contradiction. So indeed \( f(x) > 0 \).

(d) Now if \( x > y \) then \( f(x) - f(y) = f(x-y) > 0 \) by (c), so \( f(x) > f(y) \).

(e) At last take any \( x \) and suppose for a contradiction that \( f(x) \neq x \). Say \( x < f(x) \). Pick a rational number \( y \) lying between \( x \) and \( f(x) \). So \( f(y) = y \) by (b). But \( f \) preserves inequalities by (d), so \( x < y \) implies \( f(x) < f(y) = y \), while \( y < f(x) \) by choice of \( y \), which is a contradiction. So must have that \( f(x) = x \) FOR ALL \( x \).

Ch. 4
1(iii) \((a - \epsilon, a + \epsilon)\).
(iv) \((-\sqrt{3/2}, -\sqrt{1/2}) \cup (\sqrt{1/2}, \sqrt{3/2})\).
(v) \([-2, 2]\).
4(i) A diamond passing through \((1, 0),(0, 1),(-1, 0),(0,-1)\).
(ii) Draw the line \( y = x-1 \) but only in the northeast quadrant. Now reflect in \( x \) and \( y \) axes so you get something in all four quadrants.
(iii) A cross passing through \((1, 1)\) and going in directions NE, NW, SW, SE.
(iv) Same as (iii).
(v) The origin only!
(vi) Either \( x = 0 \) or \( y = 0 \) so this is the \( x \) and the \( y \) axis in a cross.
(vii) You can write this as
\[
(x - 1)^2 + y^2 = 5
\]
when you complete the square. So its a circle origin \((1, 0)\) radius \( \sqrt{5} \).
(viii) Either $x = y$ or $x = -y$. So its these two diagonal lines forming a cross.

8(a). We may as well move $f$ down by $b$ and $g$ down by $c$, since such translation will not change the angle between the lines. So we just need to consider $f(x) = mx$ and $g(x) = nx$. Now take the triangle as in the hint, passing through 0, (1, $m$) and (1, $n$). The squares of the lengths of the sides are $(1 + m^2), (1 + n^2)$ and $(m - n)^2$. So according to pythagoras, for the triangle to be right angled, we need that $1 + m^2 + 1 + n^2 = (m - n)^2 = m^2 - 2mn + n^2$. So we need that $mn = -1$.

10(i) You know what $f(x) = x$ looks like and $f(x) = 1/x$. To add them together, for large $x$ it’ll look essentially like $f(x) = x$. For small $x$, it’ll look essentially like $f(x) = 1/x$. Then in the middle, say $0.75 \leq x \leq 5$, it’ll gradually curve from one shape to the other.

(iii) This is similar, except of course its symmetric in the $y$-axis this time. The gradual curve from one shape to the other will occur much more quickly this time, too...

14(i) Moved up by $c$.
(ii) Moved left by $c$.
(iii) $y$-axis scaled by $c$ (if $c < 0$ the graph gets turned upsidedown).
(iv) $x$-axis scaled by $c$ (if $c < 0$ the graph gets flipped in the $y$-axis).