Math 261: Homework 2 solutions

Ch.1 22 We are told that \(y_0 \neq 0\) and

\[|y - y_0| < \min \left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right).\]

In other words we know both that \(|y - y_0| < \frac{|y_0|}{2}\) and that \(|y - y_0| < \frac{\varepsilon|y_0|^2}{2}\).

Let me first show that \(|y| > \frac{|y_0|}{2}\),

which gives in particular that \(y \neq 0\) since \(y_0 \neq 0\). Well, \(|y_0| = |y_0 - y + y| \leq |y_0 - y| + |y|\) by the triangle inequality. By what we’re told \(|y_0 - y| + |y| < \frac{|y_0|}{2} + |y|\). So we’ve shown that \(|y_0| < \frac{|y_0|}{2} + |y|\). Rearranging this inequality gives that \(|y| > \frac{|y_0|}{2}\).

Now we can prove the main thing. We already know

\[|y - y_0| < \frac{\varepsilon|y_0|^2}{2}.\]

I just showed that \(\frac{|y_0|}{2} < |y|\). So we get

\[|y - y_0| < \frac{\varepsilon|y_0|^2}{2} < \varepsilon|y_0||y|.\]

Dividing both sides by \(|y_0||y|\) gives

\[\left|\frac{y - y_0}{yy_0}\right| < \varepsilon.\]

But

\[\left|\frac{y - y_0}{yy_0}\right| = \left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon\]

so its just what we were after!

Ch.2 1 I’ll just do (ii) assuming part (i). So we want to prove

\[1^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = (1 + \cdots + n)^2.\]

First, let me rewrite the right hand side: we know \(1 + \cdots + n = \frac{1}{2}n(n + 1)\).

So we need to prove

\[1^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2.\]
Base case. Check the formula is true for \( n = 1 \). Yes, both left and right hand sides are 1.

Induction step. Assume the formula is true for \( n = k \), i.e.
\[
1^3 + \cdots + k^3 = \frac{1}{4}k^2(k + 1)^2.
\]

Add \((k + 1)^3\) to both sides. Get
\[
1^3 + \cdots + k^3 + (k + 1)^3 = \frac{1}{4}k^2(k + 1)^2 + (k + 1)^3
\]
\[
= \frac{1}{4}(k + 1)^2(k^2 + 4k + 4)
\]
\[
= \frac{1}{4}(k + 1)^2(k + 2)^2
\]
which is exactly what we’re trying to prove with \( n = k + 1 \).

Done by P.M.I.

Ch.2 2 (i) We have that
\[
\sum_{i=1}^{n}(2i - 1) = \sum_{i=1}^{2n} i - \sum_{i=1}^{2n} 2i = \sum_{i=1}^{2n} i - 2 \sum_{i=1}^{n} i
\]
This says: sum of first \( n \) odd numbers is sum of first \( 2n \) numbers minus sum of first \( n \) even numbers. Think about it with some examples \( n = 3, 4, 5, \ldots \).
So
\[
\sum_{i=1}^{n}(2i - 1) = \frac{1}{2}2n(2n + 1) - 2 \cdot \frac{1}{2}n(n + 1)
\]
by stuff we already know. The right hand side simplifies to \( n^2 \), which is the answer.

(ii) This time
\[
\sum_{i=1}^{n}(2i - 1)^2 = \sum_{i=1}^{2n} i^2 - \sum_{i=1}^{2n}(2i)^2 = \sum_{i=1}^{2n} i^2 - 4 \sum_{i=1}^{n} i^2
\]
Plugging in what we know,
\[
\sum_{i=1}^{n}(2i - 1)^2 = \frac{1}{6}2n(2n + 1)(4n + 1) - \frac{4}{6}n(n + 1)(2n + 1).
\]
The right hand side simplifies to
\[
\frac{1}{3}n(2n + 1)(4n + 1 - 2n - 2) = \frac{1}{3}n^2(2n + 1)(2n - 1).
\]
Ch.2 5. Done in class.

Ch.2 6. I’m just going to do (iv). We need to sum things looking like

\[
\frac{2n+1}{n^2(n+1)^2}.
\]

Going to try to use the method of differences, so can I come up with a difference that equals this? Guess:

\[
\frac{1}{n^2} - \frac{1}{(n+1)^2}.
\]

Work it out, its \[
\frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}.
\]

Lucky guess! So:

\[
\sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2} = \sum_{i=1}^{n} \left( \frac{1}{i^2} - \frac{1}{(i+1)^2} \right) = 1 - \frac{1}{(n+1)^2}.
\]

You can simplify this to \[
\frac{n^2 + 2n}{(n+1)^2}
\]
if you want.

Ch.2 12 (a) First part: yes. Proof: use contradiction. Suppose \(c = a + b\) is rational. Then, as \(a\) is rational and rational minus rational is rational, you get that \(b = c - a\) is rational, contradicting the assumption on \(b\).

Second part: not necessarily. For example, \(a = \sqrt{2}, b = -\sqrt{2}\), when \(a + b = 0\) which is rational!

(b) Not necessarily, e.g. \(a = 0, b = \sqrt{2}\).

(c) The fourth root of 2 is such a number.

(d) Yes. Try \(\sqrt{2}\) and \(-\sqrt{2}\).

Ch.2 26

I’ll just explain how to describe an inductive algorithm for moving \(n\) rings from spindle 1 to spindle 3 in \(2^n - 1\) moves. This is half the question: proving this is the minimal number of moves is a little bit harder.

Clearly, we can move one ring in just 1 move. Now suppose that we already have an algorithm for moving \((n-1)\) rings in \(2^{n-1} - 1\) moves. Here is an algorithm to move \(n\) rings in \(2^n - 1\) moves. First: move the top \((n-1)\) rings from spindle 1 to spindle 2 using the algorithm given. Second: move the largest ring from spindle 1 to spindle 3. Third: move the \((n-1)\) rings
from spindle 2 to spindle 3 using the algorithm given. Total number of moves = $2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2n - 1$.

Note this question is an example of a definition by induction. We defined the algorithm by first defining it for $n = 1$, then defining the algorithm for $n \text{ in terms of }$ the inductively defined algorithm involving $n - 1$.

Ch.3 1
(i) $\frac{1+x}{2+x}$.
(ii) $\frac{x}{1+x}$.
(iii) $\frac{1}{1+cx}$.
(iv) $\frac{1}{1+x+y}$.
(v) $\frac{1}{1+x} + \frac{1}{1+y} = \frac{2+x+y}{(1+x)(1+y)}$.
(vi) Need $\frac{1}{1+cx} = \frac{1}{1+x}$, so $1 + x = 1 + cx$ which is always true if $x = 0$ regardless of what $c$ is. So the answer is: ALL $c$.

(vii) Okay, so regardless of what $c$ is, $x = 0$ is one solution. To get another non-zero solution, we need that $(c - 1)x = 0$, so for $x \neq 0$ we must have that $c - 1 = 0$ so $c = 1$. So the answer is: $c = 1$ only.

Ch.3 3
(i) $[-1, 1]$.
(ii) $[-1, 1]$.
(iii) All $x \in \mathbb{R}$ except $x = 1, 2$.
(iv) $\{\pm 1\}$.
(v) Need $x \leq 1, x \geq 2$. So the domain is $\emptyset$, the empty set.

Ch.3 4
(i) $(2^y)^2 = 2^{2y} = 4^y$.
(ii) $\sin^2 y$.
(iii) $4\sin x + \sin(2t)$.
(iv) $\sin(t^3)$.