(1) Define the derivative $f'(a)$. Calculate (from the definition) the derivative of $f(x) = 1/x$.

(2) Let

$$f(x) = \begin{cases} 
  x^2 \sin(1/x) & x \neq 0 \\
  0 & x = 0.
\end{cases}$$

Find $f'(0)$. 

Show all your work! There are 15 problems at 10 points each.
3) Calculate the derivatives of the functions below. You may use that the derivative of $\sin(x)$ is $\cos(x)$.

(a) $(x^2 + x)^{30}(x^3 - x)^{40}$

(b) $\sin(x^2 + \sin(x^2 + \sin(x)))$

(c) $\frac{x^4 + x^2}{\sin(x)}$

(d) $(x^2 + x^{-2})^3$

4) Suppose $f : [0, 1] \to [0, 1]$ is a continuous function defined on the closed interval $[0, 1]$. Prove $f(x) = x$ for some $x \in [0, 1]$. 
(5) Prove that if \( f'(a) \) exists, then \( f \) is continuous at \( a \).

(6) Use the **Chain rule** and the **Product rule** to prove the **Quotient rule**: [If \( f'(a), g'(a) \) exists and \( g(a) \neq 0 \) then

\[
(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.
\]
(7) Find a pair of successive integers so that \(4x^3 - 3x^4 + 1\) has a zero between them. State the theorem that you are using.

(8) Prove by induction that

\[
1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.
\]
(9) Find the following limits. In case the limits are $\infty$ or $-\infty$, indicate.

(a) \[ \lim_{x \to 0} \frac{x^2 + x^3}{x} \]

(b) \[ \lim_{x \to 0} \frac{x}{x^2 + x} \]

(c) \[ \lim_{x \to \infty} \frac{x^2 + 3x^3}{5x^3 + x \sin(x) + 2} \]

(d) \[ \lim_{x \to \infty} \sqrt{x^2 + 9x} - \sqrt{x^2 + x} \]

(10) Find an example of two functions $f$ and $g$, neither of which is continuous on all of $\mathbb{R}$ but such that their composite $f \circ g$ is continuous on all of $\mathbb{R}$.
(11) Give an example of a function continuous on all of \( \mathbb{R} \) and differentiable at every point except at integers. A careful graph is sufficient. Give a graph of the derivative of the function you produced.

(12) Give an example of a function that is continuous on \((a, b)\), and bounded above on \((a, b)\) but so that it does not have a maximum value on \((a, b)\). Give the supremum of the values of the function on \((a, b)\).

(13) Suppose that \( f \) and \( g \) are even functions. Prove that \( f \cdot g \) is an even function. Suppose that \( f \) and \( g \) are odd functions. Prove that \( f \cdot g \) is even.
(14) Give a direct proof, using $\varepsilon$ and $\delta$ that $\lim_{x\to 4} \sqrt{x} = 2$.

(15) Answer true or false for each of the below. Supply a short justification if possible.
(a) If $(f + g)'(a)$ exists, then $f'(a)$ and $g'(a)$ exist.
(b) If $f$ is continuous at $a$ then $f$ is differentiable at $a$.
(c) If $f$ is even and $g$ is odd, then $f \cdot g$ is odd.
(d) If $f$ is continuous and bounded above, then $f$ has a maximum value.
(e) If a set $A$ is bounded above, then it has a maximum element.
(f) If $f(x)$ is a polynomial, then $f(x) = 0$ for some $x$.
(g) If $A \subseteq \mathbb{Q}$ has an upper bound, then $\sup(A)$ may not be in $\mathbb{Q}$.
(h) If $f(x)$ is an odd degree polynomial, then $f(x) = 0$ for some $x$. 