ORLIK-SOLOMON ALGEBRAS IN ALGEBRA AND TOPOLOGY

SERGEY YUZVINSKY

1. Introduction

The main character of this survey is the Orlik-Solomon (OS) algebra of an arrangement of hyperplanes. These algebras appeared first from theorems by Arnold, Brieskorn, and Orlik-Solomon as the cohomology algebras of the complements to complex hyperplane arrangements. Then they were shown to play an important part in theory of multivariable hypergeometric functions, conformal field theory, cohomology of the Milnor fibers of nonisolated singularities, and Alexander invariants of projective curves. Results about OS algebras intertwine usually several areas of mathematics such as pure algebra, combinatorics, topology, differential and algebraic geometries.

This survey is an attempt to study the OS algebras as such, leaving applications aside. Most results in it were obtained after the book [50] appeared although for convenience of the reader we include relevant definitions and certain basic facts from this book also. When we do the latter we are trying to give the most direct and short proofs using sometimes fact and notions that were understood or discovered later.

The contents of the survey is as follows. In section 2, we give the main definitions and basic properties of the OS algebras. Some proofs of these facts are new. We emphasize the use of Gröbner bases for the OS ideals. In section 3, we give the interpretation of an OS algebra as the local (Whitney) homology of the intersection poset. The new element here with respect to [50] is our use of the relative atomic complex of the poset. We define a structure of a differential graded algebra on it that makes the above interpretation more natural and transparent. In section 4, we prove a combination of the Brieskorn and Orlik-Solomon theorems interpreting the OS algebra as the cohomology algebra of the hyperplane complement. The proof follows [50] with some simplifications.

In section 5, we employ a relatively new point of view from [21] considering an OS algebra A as a module over an exterior algebra E. The main results here are about the annihilator I^0 of the OS ideal that is isomorphic to the dual module of A. We find explicitly a Gröbner basis of the ideal I^0 and

Date: October 26, 2000.

1991 *Mathematics Subject Classification.* Primary: 52C35, 05B35; Secondary: 16E05, 15A75.

prove that its free resolution is linear. This allows us to introduce the module F over a polynomial ring, corresponding to A via the Bernstein-Gel'fand-Gel'fand correspondence. We find certain invariants of F and the singular variety of A.

In section 6, an OS algebra A is treated as a graded finitely generated algebra and we study its Koszul property. The main result here is from [53] asserting that A is Koszul if and only if the hyperplane complement is a $K[\pi, 1]$-space (for complex arrangements). For being Koszul, an algebra must be quadratic, so we start with the problem of quadraticity of A. Falk proved in [26] that the combinatorial condition of being line-closed is necessary for quadraticity. He has also conjectured that this condition is sufficient. A stronger necessary condition, disproving this conjecture, appeared in [20]. It uses the ideal I^0 from section 5. We define all the conditions and in particular give a new proof of Falk's theorem.

In section 7, we view an OS algebra A as a cochain complex under multiplication by an element a of degree 1 and study its cohomology $H^*(A, a)$. When a is in sufficiently general position $H^*(A, a)$ can be described completely - see Theorem 7.5 ([67, 59]). This gives some upper bounds on the dimensions of vanishing cohomology for arbitrary a and ranks of nonvanishing cohomology. To get more information about $H^*(A, a)$ we study the resonance varieties introduced in [25]. The best known of them is R^1_m that is the set of a such that $\dim H^1(A, a) \geq m$. We describe the irreducible components of R^1_m by explicit equations and deduce their properties using mainly results from [46]. For some classes of arrangements, the condition of the existence of essential components of R^1_m is very restricting and allows one to classify these arrangements. This theme is continued in section 8 where the cohomolgy with local coefficients are used to get more information about $H^1(A, a)$ and R^1_m.

Every section from 5 to 8 ends with a discussion of several open questions.

The author is indebted to V. Buchshtaber and A. Vershik without whose encouragement this survey would have never been written.

2. Orlik-Solomon ideals of exterior algebras

In this section, the pure algebraic definition of the OS ideal (and OS algebra) of an arrangement is given. We prove some simple properties of the OS ideals. The results can be found in the book [50] with references to the original papers and examples. The order of the material is different from [50] and some proofs are shorter and more direct.

2.1. **Combinatorics of an arrangement.** The notation we introduce here will be used throughout the paper. We denote by $\mathcal{A}$ an arrangement in an affine space $V \approx K^\ell$ for a fixed arbitrary field K and $\ell \geq 1$. This means that $\mathcal{A}$ is a finite set of affine hyperplanes of V. The cardinality of $\mathcal{A}$ will be usually denoted by n. Very often we will fix an arbitrary linear order on $\mathcal{A}$, i.e., put $\mathcal{A} = (H_1, \dots, H_n)$.

Any time when it is convenient, we fix a linear basis $(x_1, \dots, x_\ell)$ of V^* and identify V with K^ℓ using the dual basis in V. Then in order to define a hyperplane H of K^ℓ it suffices to fix a degree 1 polynomial $\alpha_H \in S = K[x_1, \dots, x_\ell]$ such that H is the zero locus of α_H. This polynomial is uniquely defined up to multiplication by a nonzero element from K. We will abbreviate α_{H_i} by α_i.

In order to define the Orlik-Solomon (abbreviated as OS) algebra of $\mathcal{A}$ we do not need to know the hyperplanes; it suffices to know the combinatorics of $\mathcal{A}$, i.e., its *intersection poset* $L(\mathcal{A})$. Elements of this poset $L = L(\mathcal{A})$ are all subspaces of V that are intersections of some hyperplanes from $\mathcal{A}$ (including V itself as the intersection of the empty set of hyperplanes). The partial order on L is the reverse inclusion of subspaces. In particular L always has the unique minimal element V but has the unique maximal element $\bigcap_{i=1}^n H_i$ if and only if this intersection is nonempty. In the latter case we can use a translation and always assume that this intersection contains 0 (equivalently, all α_H are homogeneous). In the this case $\mathcal{A}$ is called *central*. If the above intersection is 0 $\mathcal{A}$ is called *essential*.

The poset L is far from arbitrary. If $\mathcal{A}$ is central then L is a lattice whose atoms (i.e., the elements succeeding the minimal element) are the hyperplanes themselves. It also has the following extra properties:

(i) it is *atomic*, i.e., its every element is the join (the least upper bound) of some atoms;

(ii) it is *ranked*, i.e., every nonrefinable chain $(V < X_1 < \cdots < X_r = X)$ from V to a fixed $X \in L$ has the same number of elements, namely the codimension of X (following lattice theory we call this number the rank of X and denote it by $\mathrm{rk}X$);

(iii) for every $X, Y \in L$ the following *semimodular inequality* hold

$$\mathrm{rk}X + \mathrm{rk}Y \geq \mathrm{rk}(X \vee Y) + \mathrm{rk}(X \wedge Y)$$

where the symbols $\vee$ and $\wedge$ are used respectively for the operations of join and meet (i.e., the greatest lower bound).

Lattices satisfying the above properties are called *geometric*. The rank $\mathrm{rk}L$ of a geometric lattice L is the maximal rank of its elements. Clearly $\mathrm{rk}L \leq \ell$and $\mathrm{rk}L = \ell$ if and only if $\bigcap_i H_i = 0$. The arrangements satisfying the latter condition are called *essential*. Notice that any nonempty closed interval of a geometric lattice, i.e., the subposet of elements lying inclusively between two fixed ones, is again a geometric lattice.

Among many properties of geometric lattices we need to record a property of its homology. For an arbitrary poset, all its flags (i.e., linearly ordered subsets) form a simplicial complex on the set of all its elements. The homotopy invariants of this complex are attributed to the poset itself. If the poset is a lattice then by its homotopy invariants one usually means those of its subposet with the maximal and minimal elements deleted. In this sense the following result is a part of the celebrated Folkman's theorem [31]: the reduced homology (with integer coefficients) of a geometric lattice vanishes except perhaps in the highest dimension (equal to $\mathrm{rk}L-2$).

In order to understand the structure of the intersection poset $L(\mathcal{A})$ if $\mathcal{A}$ is not central we apply the simple *coning* construction. If $\mathcal{A}$ is an arbitrary (not necessarily central) arrangement of n hyperplanes in K^ℓ then by embedding K^ℓ into the projective space $\mathbb{P}K^\ell$ and adjoining to $\mathcal{A}$ the hyperplane $H_0 = \mathbb{P}K^\ell \setminus K^\ell$ we obtain an arrangement of $n+1$ projective hyperplanes in $\mathbb{P}K^\ell$. The cone $c\mathcal{A}$ over this arrangement in $K^{\ell+1}$ is called the *cone over* $\mathcal{A}$. It is a central arrangement of $n+1$ hyperplanes. Notice also that every nonempty central arrangement $\mathcal{B}$ in K^ℓ can be viewed as the cone over an arrangement $\mathcal{A}$. Indeed fix an $H \in \mathcal{A}$ and a linear functional α_H such that $H = \ker \alpha_H$. Then we can define the affine hyperplane $\bar{H}$ of K^ℓ by the equation $\alpha_H = 1$ and the arrangement $d\mathcal{A} = \{H' \cap \bar{H} | H' \in \mathcal{A} \setminus \{H\}\}$ in $\bar{H}$. It is easy to see that $c(d\mathcal{A})$ is linearly isomorphic to $\mathcal{A}$. The arrangement $d\mathcal{A}$ is called *a deconing* of $\mathcal{A}$.

Now the poset $L(\mathcal{A})$ is the subposet of the (geometric) lattice $L(c\mathcal{A})$. It can be defined as containing all the atoms but H_0 and being closed with respect to join. If we substitute here an arbitrary geometric lattice for $L(c\mathcal{A})$ and its arbitrary atom for H_0 we obtain a definition of *geometric semilattice.* An axiomatic definition can be found in [66]. The most important for us property of a geometric semilattice is that its each closed interval is a geometric lattice whence Folkman's theorem holds for it.

2.2. **Definition of OS algebras.** Let $\mathcal{A} = (H_1, \dots, H_n)$ be an arrangement and k a commutative ring. Let E be the exterior algebra over k with generators $e_1, \dots, e_n$ of degree 1. Notice that the idexes define a bijection from $\mathcal{A}$ to the generating set. Sometimes we will denote the generator corresponding to $H \in \mathcal{A}$ by e_H. The algebra E is graded via $E = \oplus_{p=1}^n E_p$ where $E_1 = \oplus_{j=1}^n k e_j$ and $E_p = \bigwedge^p E_1$. For every k, the k-module E_j is free and has the distinguished basis consisting of monomials $e_S = e_{i_1} \cdots e_{i_p}$ where $S = \{i_1, \dots, i_p\}$ is running through all the subsets of $[n] = \{1, 2, \dots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$. The graded algebra E is a (commutative) DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions $\partial e_i = 1$ for every $i = 1, \dots, n$ and the graded Leibniz formula. Then for every $S \subset [n]$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j}$$

where S_j is the complement in S to its jth element.

It is important to notice that ∂e_S is a pure (or decomposable) element of E for every S. Indeed if again $S = \{i_1, \dots, i_p\} \subset [n]$ then the straightforward check shows that

$$\partial e_S = (e_{i_2} - e_{i_1})(e_{i_3} - e_{i_1}) \cdots (e_{i_p} - e_{i_1}). \tag{2.1}$$

For every $S \subset [n]$, put $\cap S = \bigcap_{i \in S} H_i$ and call S *dependent* if $\cap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i | i \in S\}$ is linearly dependent.

Definition 2.1. *The OS ideal of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by all e_S with $\cap S = \emptyset$ and all ∂e_S with S dependent. The algebra $A = A(\mathcal{A}) = E/I(\mathcal{A})$ is called the OS algebra of $\mathcal{A}$.*

We denote the canonical projection $E \to A$ by π. Since π induces a k-linear isomorphism of E_1 and A_1 we will identify these k-modules.

Notice that for any nonempty $S \subset [n]$ and $i \in S$ one has $e_i \partial e_S = \pm e_S$ whence $I(\mathcal{A})$ contains e_S for every dependent set S. This implies that $A(\mathcal{A})$ is generated as a k-module by the emages of e_S such that $\cap S \neq \emptyset$ and S is independent.

In Definition 2.1, the set of generators can be made much smaller. The minimal by inclusion dependent sets of $[n]$ are called *circuits*.

Lemma 2.2. *An OS ideal is generated by all e_S with $\cap S = \emptyset$ and ∂e_T for the circuits T.*

Proof. Let S be a dependent set and $T \subset S$ is a circuit. Then $e_S = \pm e_T e_{S \setminus T}$. Using the (graded) Leibniz formula for ∂ we have

$$\partial e_S = \pm \partial e_T e_{S \setminus T} \pm e_T \partial e_{S \setminus T}.$$

The result follows. □

Clearly I is a homogeneous ideal of E whence A is a graded algebra. Write $A = \oplus A_p$ where $A_p = E_p / I \cap E_p$. Since every set of more than $\mathrm{rk}L$ elements is dependent, the maximal degree of elements of A is $\mathrm{rk}L$. Notice that factoring V by $\bigcap_i H_i$ makes $\mathcal{A}$ essential and does not change A. Thus, unless we are dealing with subarrangements, we will always assume that $\mathcal{A}$ is essential, i.e., $\mathrm{rk}L = \ell$.

2.3. **Grading by the intersection lattice.** In this subsection we discuss a finer grading on A. The finest grading on E is the grading by the Boolean poset of all subsets of $[n]$. In general, the OS ideal is not homogeneous in this grading. Consider the coarser grading on E assigning to each e_S ($S \subset [n]$) the set $\cap S$. This is a grading by the lattice $\bar{L}(\mathcal{A})$ where $\bar{L}(\mathcal{A}) = L(\mathcal{A})$ if $\mathcal{A}$ is central and otherwise it is $L(\mathcal{A})$ completed by the unique maximal element $\emptyset$. We write $E = \oplus_{X \in \bar{L}(\mathcal{A})} E_X$ where E_X is spanned by e_S with $\cap S = X$. Notice that this grading is in general incomparable with the standard grading by $|S|$.

Proposition 2.3. *The OS algebra $A = A(\mathcal{A})$ is graded by the poset $L(\mathcal{A})$ and this grading is finer than the standard grading $A = \oplus_{p=0}^{n} A_p$.*

Proof. Clearly any e_S is homogeneous. If T is a circuit in $[n]$ and $\cap T = X$ then $\cap T_i = X$ for every $i \in T$. Thus $\partial e_T \in E_X$. Using Lemma 2.2, this shows that $I = \oplus_{X \in \bar{L}(\mathcal{A})} I_X$ where $I_X = I(\mathcal{A}) \cap E_X$. Besides if $\mathcal{A}$ is noncentral then $I_\emptyset$ contains all the e_S with $\cap S = \emptyset$ whence coincides with $E_\emptyset$. Thus $A_\emptyset = 0$ and A is graded by $L(\mathcal{A})$.

The second statement follows from the fact that A is generated as k-module by the images of e_S with $\cap S \neq \emptyset$ and S is independent. For such an S we have $\operatorname{rk} \cap S = |S|$. This shows that

$$A_p = \oplus_{X \in L(\mathcal{A}), \operatorname{rk} X = p} A_X$$

and completes the proof. □

2.4. **Filtration by $L(\mathcal{A})$.** The above grading of A by $L(\mathcal{A})$ induces a filtration of A that can be defined independently. Among subarangements of an arrangement $\mathcal{A}$ there are the ones corresponding to elements of $L(\mathcal{A})$. For every $X \in L(\mathcal{A})$ we put $\mathcal{A}_X = \{H \in \mathcal{A} | X \subset H\}$. We also put $[n]_X = \{i \in [n] | H_i \in \mathcal{A}_X\}$. Notice that all the subarrangements $\mathcal{A}_X$ are central. They can be completely characterized also by the property of being *closed*. This means that with several hyperplanes they contain all hyperplanes dependent of them.

We want to show that the graded algebras $A(\mathcal{A}_X)$ form a filtration of $A(\mathcal{A})$. First denote by $E(X)$ the subalgebra of E generated by all the e_i with $H_i \in \mathcal{A}_X$ and notice that $A(\mathcal{A}_X) = E(X)/I(\mathcal{A}_X)$. Clearly $I(\mathcal{A}_X) \subset I(\mathcal{A}) \cap E(X)$. The key observation is that it is in fact equality.

Lemma 2.4. *For every $X \in L(\mathcal{A})$*

$$I(\mathcal{A}_X) = I(\mathcal{A}) \cap E(X).$$

Proof. Fix $a \in I(\mathcal{A}) \cap E(X)$. Using definition of $I(\mathcal{A})$ we can write

$$a = \sum_{S,T} c_{S,T} e_T \partial e_S + \sum_{T,U} d_{T,U} e_T e_U \tag{2.2}$$

where $c_{S,T}, d_{U,T} \in k$, and $S, T, U \subset [n]$ such that S is a circuit and $\cap U = \emptyset$. Since S is a circuit we have either $S \subset \mathcal{A}_X$ or $S_i \not\subset \mathcal{A}_X$ for every $i \in S$. Besides $U \not\subset \mathcal{A}_X$ for every U. Thus a has the summand $a' = \sum_{S,T \subset \mathcal{A}_X} c_{S,T} e_T \partial e_S \in E(X)$ and all basic elements of $a - a'$ are in $\oplus_{Y \neq X} E(Y)$. Since $a \in E(X)$ we have $a = a'$. Clearly $a' \in I(\mathcal{A}_X)$ and the result follows. □

The previous lemma immediately implies the following.

Proposition 2.5. *The natural homomorphism $i : A(\mathcal{A}_X) \to A(\mathcal{A})$ is injective.*

Corollary 2.6. *The correspondence $X \mapsto A(\mathcal{A}_X)$ $(X \in L(\mathcal{A}))$ defines a monotone map of $L(\mathcal{A})$ to the poset of graded subalgebras of A ordered by inclusion, i.e., a filtration of A.*

The following observation is straightforward.

Proposition 2.7. *The filtration $\{A(\mathcal{A}_X)\}$ is induced by the grading $A = \oplus_{Y\in L(\mathcal{A})} A_Y(\mathcal{A})$. More precisely, $A_Y(\mathcal{A}_X) = A_Y(\mathcal{A})$ for every $X, Y \in L(\mathcal{A})$ such that $X \subset Y$, whence $A(\mathcal{A}_X) = \oplus_{Y\leq X} A_Y(\mathcal{A})$.*

If an arrangement $\mathcal{A}$ is central then A receives another piece of structure from E. In this case I is generated over k by $\{e_T \partial e_S\}$ where T is arbitrary and S is dependent. Using the Leibniz rule we have

$$\partial(e_T \partial e_S) = \partial e_T \partial e_S$$

whence $I(\mathcal{A})$ is invariant with respect to ∂. Thus it induces a differential

$$\partial_A : A \to A$$

that defines the structure of commutative DGA on A. The relations of ∂_A with the gradings and filtration of A are as follows:

1. $\partial_A(A_X) \subset A_{r-1}$ for every $X \in L(\mathcal{A})$ of rank r;
2. $\partial_A(A(\mathcal{A}_X) \subset A(\mathcal{A}_X)$ for every $X \in L(\mathcal{A})$.

The proof is straightforward.

Finally if the arrangement is not empty, the restriction of ∂_A to any $A(\mathcal{A}_X)$ is acyclic. In particular it is true for $A = A(\mathcal{A}) = A(\mathcal{A}_U)$ where U is the unique maximal element of $L(\mathcal{A})$. Indeed the multiplication by any e_i with $H_i \in \mathcal{A}_X$ is clearly a homotopy between the identity and 0 maps.

2.5. **Gröbner basis.** Recall that we fix an arbitrary linear order on an arrangement $\mathcal{A}$. This order induces the *deg-lex linear order* on the set of all standard monomials e_S of E: if $S = (i_1, \dots , i_p)$ and $T = (j_1, \dots , j_q)$ (where $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$) then $e_S < e_T$ if either $p < q$ or $p = q$ and there exists m, $1 \leq m \leq p$, such that $i_r = j_r$ for $r < m$ and $i_m < j_m$. The basis of E consisting of standard monomials is multiplicative up to $\pm$ (i.e., the product of two standard monomials is either 0 or a standard monomial perhaps with minus) and the deg-lex order is multiplicative (i.e., invariant under multiplication by monomials) in the same sense. Thus we can apply theory of Gröbner bases to the ideal $I = I(\mathcal{A})$.

The main goal of this subsection is to prove the following theorem.

Theorem 2.8. *Let $B = \{\partial e_S | S$ is circuit $\} \cup \{e_T | T$ is minimal with $\cap T = \emptyset\} \subset E$. Then B is a Gröbner basis of I.*

In order to prove this theorem we first define combinatorially the initial monomials of elements from B, i.e., their largest monomials in the deg-lex order. Since e_T is a monomial itself its initial monomial coincides with it. The initial monomial of ∂e_S is e_{S_1} (recall that S_1 is obtained from S by deleting its smallest element). Thus these initial monomials correspond to the *broken circuits*, i.e., independent subsets S of $[n]$ such that $S \cup \{i\}$ is a circuit and $i < j$ for all $j \in S$.

Denote by $In(B)$ and $In(I)$ the sets of the initial monomials of elements of B and I respectively. Then the statement of Theorem 2.8 means that

$$\langle In(B) \rangle = In(I)$$

where the left hand side is the set of all monomials divisible by some monomials from In(B). Now denote by $C = C(\mathcal{A})$ (resp. $\bar{C}$) the natural linear complement to $\langle In(B) \rangle$ (resp. $In(I)$) namely the free k-module spanned by all the monomials of E not divisible by any element of $In(B)$ (resp. not in $In(I)$). The monomials from C are called **nbc**-monomials and the respective subsets of $[n]$ (i.e., subsets containing any circuits) are called **nbc**-sets. Clearly

$$E = I \oplus \bar{C}$$

(as k-modules) whence the restriction of π to $\bar{C}$ is an linear isomorphism $\bar{C} \to A$. Since $\langle In(B) \rangle \subset In(I)$ it is always true that $\pi(C) = A$ and Theorem 2.8 is equivalent to

$$\ker \pi|_C = 0$$

i.e., **nbc**-monomials are independent in A.

First notice that C is graduated both by $[n]$ and $L(\mathcal{A})$. Indeed since it is spanned by monomials it is naturally graduated by $[n]$ and $\bar{L}(\mathcal{A})$. Besides for any monomial $e_S \in C$ we have $\cap S \neq \emptyset$ whence the second grading is really by $L(\mathcal{A})$. Moreover S is independent, i.e., $|S| = \mathrm{rk} \cap S$ whence the latter grading is finer than the former, i.e., $C_p = \oplus_{Y \in L, \mathrm{rk} Y = p} C_Y$ for every p, $0 \leq p \leq n$.

The second observation is that $e_S \in C$ implies $e_{S_i} \in C$ for every $i \in S$ whence $\partial(C) \subset C$. By the same reason we have

$$\partial(C_X) \subset \oplus_{Y \subset X, \mathrm{rk} Y = \mathrm{rk} X - 1} C_Y.$$

Lemma 2.9. *For every $X \in L(\mathcal{A}) \setminus \{V\}$ the restriction of ∂ to C_X is injective.*

Proof. Let i be the minimal element of $[n]_X$. It follows from definition of C_X that $i \in S$ for every $e_S \in C_X$ whence $e_i C_X = 0$. On the other hand, $\partial(e_i e_S) = e_S - e_i \partial e_S$ for every subset S of $[n]$ whence the similar formula holds if we substitute an arbitrary $c \in C_X$ for e_S. We have $e_i \partial c = c - \partial(e_i c) = c$ and the result follows. $\square$

Now we are ready to prove the result equivalent to Theorem 2.8.

Theorem 2.10. *For every arrangement $\mathcal{A}$ the restriction of π to C is injective.*

Proof. Since π is homogeneous with respect to the grading by $L(\mathcal{A})$ it suffices to prove that π restricted to C_X is injective for every $X \in L(\mathcal{A})$. Use induction on $\mathrm{rk}X$. If $\mathrm{rk}X = 0$, i.e., $X = V$ then $C_X = k = A_X$ and π restricted to C_X is the identity map. Suppose $\mathrm{rk}X = r > 0$. Consider the commutative diagram

$$\begin{array}{ccc} C_X & \longrightarrow & C_{r-1} \\ \downarrow & & \downarrow \\ A_X & \longrightarrow & A_{r-1} \end{array}$$

where the vertical arrows are restrictions of π and horizontal ones are restrictions of ∂. By Lemma 2.9, ∂ is injective on C_X. Also the restriction of π is injective on C_{r-1} by the inductive hypothesis. Thus π is injective on C_X which completes the proof. □

Corollary 2.11. *All the k-modules A and A_X $(X \in L(\mathcal{A}))$ are free.*

2.6. **Deletion-restriction exact sequence.** In this subsection, we construct a short exact sequence for OS algebras that is very useful for proving properties of these algebras by induction on n.

Let $\mathcal{A}$ be a nonempty arrangement and fix a hyperplane H' from it. Then we obtain two more arrangements: $\mathcal{A}' = \mathcal{A} \setminus \{H'\}$ (obtained by the *deletion* of H' from $\mathcal{A}$) and $\mathcal{A}'' = \{H \cap H' | H \in \mathcal{A}'\}$ (obtained by the *restriction* of $\mathcal{A}$ to H'). Let us emphasize that $\mathcal{A}''$ is an arrangement of hyperplanes in the space H'. Also while the cardinality of $\mathcal{A}'$ is always $n-1$, the cardinality of $\mathcal{A}''$ is at most $n-1$ and may be smaller. For any object O attached to an arrangement we will put $O = O(\mathcal{A})$, $O' = O(\mathcal{A}')$, and $O'' = O(\mathcal{A}'')$.

Since the results we are after do not depend on the choice of order on the arrangements we make that choice in the most convenient way for our goal. First of all, H' will be the last element of $\mathcal{A}$, i.e., $H' = H_n$. Second, let $\lambda : \mathcal{A}' \to \mathcal{A}''$ be defined by $\lambda(H) = H \cap H_n$. Then we choose linear orders on $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ so that the order on $\mathcal{A}'$ is induced by the order on $\mathcal{A}$ (i.e., $\mathcal{A}' = (H_1, \cdots, H_{n-1})$) and $\lambda(H)$ precedes $\lambda(\bar{H})$ only if H precedes $\bar{H}$ for every $H, \bar{H} \in \mathcal{A}'$.

Now extend λ to ordered subsets of $\mathcal{A}'$ via $\lambda(H_{i_1}, \dots, H_{i_p}) = (\lambda(H_{i_1}), \dots, \lambda(H_{i_p}))$. Notice that the image is a sequence of elements from $\mathcal{A}''$ perhaps with repetitions but always in nondecreasing order. The extended λ can be also considered as the map $S \mapsto \lambda(S)$ from the set of subsets of $[n-1]$ to the nondecreasing subsequences of $[n'']$ where $n'' = |\mathcal{A}''|$.

Now we can consider the natural sequence of k-linear maps:

$$0 \longrightarrow E' \xrightarrow[i]{} E \xrightarrow[j]{} E'' \longrightarrow 0 \tag{2.3}$$

where i is the natural embedding and j is defined via $j(e_S) = e_{\lambda(S\setminus\{n\})}$ if $n \in S$ and $j(e_S) = 0$ otherwise. Here for every subsequence T of n'' we denote by e''_T the product of generators e''_k of E'' taken in the order on T. Notice that although the sequence (2.3) is clearly exact in the terms E' and E'' and $ji = 0$ it does not have to be exact in E.

Lemma 2.12. *(1) $i(I') \subset I$; (2) $j(I) \subset I''$.*

Proof. (1) is immediate so let us prove (2). Let $S \subset \mathcal{A}$ with $n \in S$ and put $S' = S \setminus \{n\}$. Notice that $\cap\lambda(S') = \cap S$. If $\cap S = \emptyset$ then $\cap\lambda(S') = \emptyset$ whence $j(e_S) \in I''$. If S is dependent then $\cap\lambda(S') \neq \emptyset$. Besides a functional $\alpha_{H\cap H_n}$ is the restriction of α_H to H_n (for $H \in \mathcal{A}'$) whence $\lambda(S')$ is dependent. Thus we have $j(\partial e_S) = j(\partial e_{S'} e_n) = j((\partial e_{S'})e_n \pm e_{S'}) = \partial(j(e_S)) \in I''$ that completes the proof. $\square$

Corollary 2.13. *The sequence* (2.3) *generates the sequence of k-maps*

$$0 \longrightarrow A' \xrightarrow[\bar{i}]{} A \xrightarrow[\bar{j}]{} A'' \longrightarrow 0. \tag{2.4}$$

Our goal is to prove that the sequence 2.4 is exact. For that we use the **nbc**-bases and Theorem 2.10.

Lemma 2.14. *For the embedding $i : E' \to E$ we have $i(C') \subset C$.*

Proof. This follows immediately from the fact that H_n is the last in the ordering of $\mathcal{A}$. $\square$

Lemma 2.15. *For the surjection $j : E \to E''$ we have $j(C) = C''$.*

Proof. First it is clear that j maps the set of e_S with $n \in S$ onto the canonical basis of E''. Now it is left to notice that, because of the special choice of orderings, a set $S \subset [n]$ satisfies the conditions $\cap S \neq \emptyset$ and S is **nbc** if and only if $\lambda(S \setminus \{n\})$ satisfies these conditions. The result follows. $\square$

Proposition 2.16. *The sequence*

$$0 \longrightarrow C' \xrightarrow[i'']{} C \xrightarrow[j'']{} C'' \longrightarrow 0$$

where i'' and j'' are the restrictions of i and j respectively is exact.

Proof. Because of Lemmas 2.14 and 2.15, it suffices to prove that $\ker(j'') \subset \mathrm{im}(i'')$. Since $i(E')$ is generated by e_S with $n \notin S$ it suffices to prove that no nonzero combination of e_S with $n \in S$ belongs to $\ker(i) \cap C$. Since $\lambda(S \setminus \{n\})$ is **nbc** if S is, it suffices to prove that λ is injective on the collection of all sets $S \subset [n-1]$ such that $S \cup \{n\}$ is **nbc**. Indeed suppose $\lambda(S_1) = \lambda(S_2)$ where $S_i \subset [n-1]$. Then there are $m \in S_1$ and $p \in S_2$ so that $H_m \cap H_n = H_p \cap H_n$, i.e., the set $\{m, p, n\}$ is dependent. Without loss of generality we can assume that $m < p$ which makes $\{p, n\}$ a broken circuit. Thus $S_2 \cup \{n\}$ is not **nbc** which is a contradiction. This completes the proof. □

Now Proposition 2.16 and Theorem 2.10 imply the main result of this subsection.

Corollary 2.17. *The sequence* (2.4) *is exact.*

2.7. **Central-affine exact sequence.** In this subsection we construct another exact sequence relating the OS algebras of an arrangement $\mathcal{A}$ and its cone $c\mathcal{A}$.

Let $\mathcal{A} = (H_1, \dots, H_n)$ and recall that $c\mathcal{A}$ consists of cones cH_i over H_i ($i = 1, \dots, n$) and the extra hyperplane H_0. In order to consider the OS algebras of these two arrangements at the same time we need to deal with two exterior algebras E generated by $e_1, \dots, e_n$ and cE generated by $e_0, e_1, \dots, e_n$. Clearly $E = cE/(e_0)$. Denote by E' the subalgebra of cE generated by all the differences $e_i - e_j$ and by I' the ideal of E' generated by ∂e_S for all dependent subsets S of $\{0, 1, \dots, n\}$. It follows from (2.1) that $\partial e_S \in E'$ whence the definition of I' indeed makes sense. Notice that E is isomorphic to E' under the isomorphism defined by $e_i \mapsto e_i - e_0$ ($i = 1, 2, \dots, n$).

If B is a factor of an exterior algebra over a graded ideal we put $B\langle e_0\rangle = B \otimes_k \Lambda(e_0)$ where $\Lambda(e_0)$ is the exterior algebra with the single generator e_0 of degree 1 and the tensor product is taken in the category of skew-commutative algebras. The following proposition is straightforward.

Proposition 2.18. *The OS ideal of $c\mathcal{A}$ is $cI = I'(cE)$ and the OS algebra $cA = cE/cI$ of $c\mathcal{A}$ is naturally isomorphic to $(E'/I')\langle e_0\rangle$. Furthermore the OS algebra A of $\mathcal{A}$ is $cE/(cI + (e_0)) \cong E'/I'$. In particular $cA = A\langle e_0\rangle$.*

Applying this proposition to the homogeneous components we obtain the following split exact sequence.

Corollary 2.19. *For each p we have the decomposition*

$$cA_p = A_p \oplus A_{p-1}$$

where the embedding of the first summand is induced by the identical map on e_S ($S \subset [n]$) and that of the second summand is induced by $e_S \mapsto e_0 e_S$.

3. Homological interpretation of OS algebras

In this section we give the interpretation of the OS-algebra as the local (Whitney) homology of the intersection poset. The main result is in [50]. We introduce a product on the relative atomic complex which lifts the interpretation to the chain complex level and makes it more transparent.

3.1. **Relative atomic complex of a lattice.** The first homological interpretation of the OS algebra A of an arrangement $\mathcal{A}$ comes from homology of the intersection poset $L = L(\mathcal{A})$. In this subsection, we review certain facts about homology of lattices. We want to accommodate both central and non-central arrangements. Because of that we use the letter L in this subsection for an arbitrary upper semilattice. This class of posets consists of lattices and lattices stripped of their maximal elements. The unique minimal and maximal (if it exists) elements of L are denoted by $\check{0}$ and $\check{1}$ respectively. The set of atoms of L is denoted by $\mathcal{A} = \mathcal{A}(L)$ and we put $n = |\mathcal{A}|$. An arbitrary linear order is fixed on $\mathcal{A}$ which allows us to identify $\mathcal{A}$ with $[n]$. A set $\sigma \subset [n]$ is called *bounded* if it has an upper bound in L whence its least upper bound $\bigvee \sigma$ exists in L. Otherwise it is called unbounded.

All homologies are taken with coefficients in a fixed commutative ring k. As above $H_*(L)$ denotes the homology of the flag complex of $L \setminus \{\check{0}, \check{1}\}$. There exist a few smaller complexes homotopic to the flag complex. We recall one of them, namely the simplicial atomic complex $\Delta = \Delta(L)$. It is the complex on $[n]$ whose simplexes are all subsets σ that are bounded in $L \setminus \{\check{1}\}$.

Even more important for us is the *relative atomic complex* $D = D(L)$. This is the chain complex that is the free module over a commutative ring k with a basis consisting of all bounded in L sets $\sigma \subset [n]$. The grading on D assigns degree $|\sigma|$ to σ. The differential d on D is defined as follows. If $\sigma = \{i_1, \ldots, i_p\}$ and $\sigma_j = \sigma \setminus \{i_j\}$ then

$$d(\sigma) = \sum_{\{j \in \sigma | \bigvee \sigma_j = \bigvee \sigma\}} (-1)^{j-1} \sigma_j.$$

We will use the following two properties of the complex D.

1. D is the direct sum of the complexes $D(X)$ $(X \in L)$ where $D(X)$ is spanned by subsets $\sigma \subset [n]$ such that $\bigvee \sigma = X$.

This is immediate.

2. For every p and $X \in L$ the group $H_p(D(X))$ is naturally isomorphic to $\tilde{H}_{p-2}(L_X)$ where $L_X = \{Y \in L | Y \leq X\}$. Notice that L_X is always a lattice.

Indeed consider the exact sequences of chain complexes

$$0 \to D'(X) \to D''(X) \to D(X)_{(+1)} \to 0$$

where $D'(X)$ is the chain complex of the atomic complex of L_X and $D''(X)$ is the chain complex of the simplex with $\mathcal{A}(L_X)$ as the set of vertices (this exact sequence justifies the name of $D(X)$). The claimed property follows from the respective long exact sequence for homology using the acyclicity of $D''(X)$.

3. If L_X is a geometric lattice for every $X \in L$ then $H_p(D(X)) = 0$ for all $p \neq \mathrm{rk} X$.

This follows immediately from (2) and Folkman's theorem.

The last property allows us, in the case where L_X is a geometric lattice, to exhibit nice generators of $H_*(D(X))$ as a k-module. We call a set $\sigma \subset [n]$ *independent* if it is bounded and $\bigvee \sigma_i < \bigvee \sigma$ for every $i \in \sigma$. Clearly if $\mathcal{A}$ is a hyperplane arrangement and $L = L(\mathcal{A})$ this definition coincides with the definition in Section 1.

In general case, each independent set σ of atoms is a cycle in D whence it defines a homology class ζ_σ in $H_{|\sigma|}(D(X))$ where $X = \bigvee \sigma$. Besides if L is a geometric lattice then $\mathrm{rk}(\bigvee \sigma) \leq |\sigma|$. Thus in this case, each set σ of atoms such that $\bigvee \sigma = X$ and $|\sigma| = \mathrm{rk} X$ is independent. This implies that $D_{\mathrm{rk} X}(X)$ consists of cycles only whence $H_*(D)$ is generated by $\{\zeta_\sigma | \sigma \text{ is independent}\}$.

3.2. **Multiplicative structure on the relative atomic complex.** Here we consider only the case where each L_X is geometric. We introduce a bilinear (over k) multiplicative structure on D that converts it into a differential graded algebra (DGA). This structure is defined via the product of generators. Let σ and τ be bounded sets of atoms. Then

(3.1)
$$\sigma \cdot \tau = \begin{cases} 0, & \text{if } \sigma \cup \tau \text{ is unbounded,} \\ 0, & \text{otherwise and } \mathrm{rk}(\bigvee(\sigma \cup \tau)) \neq \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau)), \\ \epsilon(\sigma,\tau)\sigma \cup \tau, & \text{otherwise.} \end{cases}$$

where $\epsilon(\sigma,\tau)$ is the sign of the permutation of $\sigma \cup \tau$ putting all elements of τ after elements of σ and preserving fixed orders inside these sets (the *shuffle* of σ and τ).

Proposition 3.1. *The multiplication defined in* (3.1) *converts D to a (commutative) DGA.*

Proof. It suffices to check the Leibniz rule since other defining properties of DGA can be checked straightforwardly. Explicitly we need to check that

$$d(\sigma \cdot \tau) = d(\sigma) \cdot \tau + (-1)^{|\sigma|} \sigma \cdot d(\tau) \tag{3.2}$$

for all bounded sets σ and τ of atoms. We consider three cases.

(i) $\sigma \cup \tau$ is unbounded. Then for every σ_i such that $\bigvee \sigma_i = \bigvee \sigma$ the set $\sigma_i \cup \tau$ is unbounded also whence $d(\sigma) \cdot \tau = 0$. Similarly $\sigma \cdot d(\tau) = 0$, i.e., both sides of (3.2) vanish.

(ii) $\sigma \cup \tau$ is bounded and $\mathrm{rk}(\bigvee(\sigma \cup \tau)) \neq \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau))$. Then $\sigma \cdot \tau = 0$. On the other hand, for every σ_i such that $\bigvee(\sigma_i) = \bigvee(\sigma)$

$$\begin{aligned} \mathrm{rk}(\bigvee(\sigma_i \cup \tau)) \leq \mathrm{rk} \bigvee((\sigma \cup \tau)) &< \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau)) \\ &= \mathrm{rk}(\bigvee(\sigma_i)) + \mathrm{rk}(\bigvee(\tau)) \end{aligned} \tag{3.3}$$

whence $d(\sigma)\cdot\tau = 0$. Similarly $\sigma \cdot d(\tau) = 0$ and again both sides of (3.2) vanish.

(iii) $\mathrm{rk}(\bigvee(\sigma\cup\tau)) = \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau))$. First notice that in this case $\sigma\cap\tau = \emptyset$. Indeed $p \in \sigma\cap\tau$ would imply

$$\mathrm{rk}(\bigvee(\sigma\cup\tau)) \le \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau)) - 1 < \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau)).$$

Furthermore, both sides of (3.2) are combinations of basic elements $\omega_i = (\sigma\cup\tau)_i$. Suppose that the ith element of ω is the jth element of σ (in particular it is not in τ). Then the coefficient of ω_i in the left hand side is

$$(-1)^{i-1}\epsilon(\sigma,\tau) \tag{3.4}$$

and that in the right hand side is

$$(-1)^{j-1}\epsilon(\sigma_j,\tau). \tag{3.5}$$

We have

$$\epsilon(\sigma,\tau) = \epsilon(\sigma_i,\tau)(-1)^{i-j} \tag{3.6}$$

which implies that the coefficients of ω_i are equal in this case. The case where the ith element of ω is in τ can be done similarly. □

From the definition of the multiplication we obtain the following.

Proposition 3.2. *Every independent set σ of atoms is the product of its all one-element subsets (taken in the natural order). In particular the algebra $H_*(D)$ is generated by $\zeta_i = \zeta_{\{i\}}$ ($i \in [n]$).*

Proof. It is clear that any subset of σ is again independent whence we can use induction on $|\sigma|$. In other words it suffices to prove that $\{i\}\sigma_1 = \sigma$ where i is the first element of σ. But $\mathrm{rk}(\bigvee\sigma) = |\sigma|$ and $\mathrm{rk}(\bigvee\sigma_1) = |\sigma_1|$ whence $\mathrm{rk}(\bigvee\sigma) = \mathrm{rk}(\bigvee\{i\}) + \mathrm{rk}(\bigvee\sigma_1)$ and the result follows. □

It is easy to see that if a set σ of atoms is unbounded or "dependent" (i.e., bounded and not independent) then the product (in any order) of its all one-element subsets is 0.

3.3. **An interpretation of OS algebras.**

Here we return to the arrangement case, that is $\mathcal{A}$ is an arrangement and $L = L(\mathcal{A})$ is its intersection poset. Recall that although L is not a lattice if $\mathcal{A}$ is not central, every L_X is a geometric lattice.

Theorem 3.3. *The correspondence $e_i \mapsto \zeta_i$ ($i = 1, 2, \dots, n$) generates an isomorphism of algebras $A(\mathcal{A}) \to H_*(D)$.*

Proof. Denote by E (as before) the exterior algebra generated by all e_i. Since ζ_i are homology classes of degree 1 generating $H_*(D)$, the correspondence $e_i \mapsto \zeta_i$ generates a surjective algebra homomorphism $\phi : E \to H_*(D)$. If $S \subset [n]$ is either unbounded or dependent then $\phi(S) = 0$ whence for all such sets $e_S \in \ker\phi$.

Now consider a combination (over k) of some e_S with independent sets S. Such a combination belongs to $\ker\phi$ if and only if it is the sum of elements whose images under ϕ are boundaries in some $D(X)$. The latter elements have form ∂e_S where S is bounded and $\operatorname{rk}\bigcap S = |S| - 1$. The last equality means that $e_S = \pm e_T e_U$ where T is independent and U is a circuit. Write $\partial e_S = (\partial e_T)e_U \pm e_T(\partial e_U)$. Since the first summand is the sum of e_P for dependent subsets P of $[n]$, it lies in $\ker\phi$. Thus the second summand lies in $\ker\phi$ and $\ker\phi$ is generated over k by e_S with unbounded S and $e_T(\partial e_U)$ with circuits U. That means that $\ker\phi = I(\mathcal{A})$ and completes the proof. $\square$

Theorem 3.3 allows us to give a combinatorial formula for the Hilbert series $H(A,t) = \sum_{p\geq 0} \dim_R A_p t^p$ of A.

Corollary 3.4.

$$H(A,t) = \sum_{p\geq 0} \sum_{X\in L, \operatorname{rk} X = p} |\mu(X)| t^p$$

where $\mu(X) = \mu(V,X)$ is value of the Möbius function of L on X.

This follows immediately from Theorem 3.3, property (2) of the complex D, and the second part of Folkman's theorem claiming that if L is a geometric lattice then $\dim_R H_*(L) = |\mu(\check{0},\check{1})|$.

The right hand side of the above equality is the *Poincarè polynomial* of L and is denoted by $P(L,t)$.

Example 3.5. *Let $\mathcal{A}$ be the arrangement in K^3 given by the linear functionals $x, y, z, x-y, x-z, y-z$. If $K = \mathbb{R}$ then $\mathcal{A}$ can be obtained from the arrangement of reflection hyperplanes of the Coxeter group of type* A_3 *by factoring out the mutual intersection of these hyperplanes. The arrangement $\mathcal{A}$ is the first nontrivial one in the series of essential arrangements whose complements for $K = \mathbb{C}$ are classifying spaces of braid groups. By this reason, all these arrangements are called braid arrangements.*

The intersection lattice L of $\mathcal{A}$ has 6 atoms 1,2,3,4,5,6 that correspond to the hyperplanes in the order they are written. The rank two elements can be identified with the sets of atoms below them. In this sense they are $\{1,2,4\}, \{1,3,5\}, \{2,3,6\}, \{4,5,6\}, \{1,6\}, \{2,5\}, \{3,4\}$. *The Möbius function is easy to compute. In particular we obtain*

$$H(A,t) = P(L,t) = 1 + 6t + 11t^2 + 6t^3.$$

An interesting observation is that $H(A,t) = (1+t)(1+2t)(1+3t)$. This property has deep generalizations which can be found in [50], *section 4.6.*

4. Cohomology algebra of hyperplane complement

Here we give another homological interpretation of $A(\mathcal{A})$ when $\mathcal{A}$ is a complex arrangement. This is a combination of the celebrated Brieskorn and Orlik-Solomon theorems ([10, 52]). The main claim is that $A(\mathcal{A})$ is naturally isomorphic to the cohomology ring of the complement M to $\mathcal{A}$. We give the shortest known to us proof of this claim.

Throughout this section $K = \mathbb{C}$ and $k = \mathbb{Z}$. All cohomologies are also taken with coefficients in $\mathbb{Z}$.

4.1. **Topological induction.** Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be the deletion-restriction triple (see subsection 2.6) defined by $H_n \in \mathcal{A}$. Denote the respective complements by M', M, and M''. Notice that while $\mathcal{A}$ and $\mathcal{A}'$ are arrangements in V, $\mathcal{A}''$ is an arrangement in H_n whence $M'' = H_n \setminus \bigcup_{i \neq n}(H_i \cap H_n)$. The goal of this subsection is to exhibit an exact sequence connecting cohomology of M', M, and M''. All cohomologies in this section are taken with the integer coefficients.

Since $M'' = M' \cap H_n$, it makes sense to consider a tubular neighborhood N of M'' in M' and the fiber bundle $N \to M''$ with the fiber homeomorphic to $\mathbb{C}$. Then the restriction of the fibration to $N_0 = N \setminus M''$ is the fiber bundle with the fiber $\mathbb{C}^*$. Since both of these bundles are restrictions of trivial bundles over H_n they are trivial, i.e., up to a homeomorphism $(N, N_0) = M'' \times (\mathbb{C}, \mathbb{C}^*)$. Thus $H^*(N, N_0) = H^*(M'') \otimes H^*(\mathbb{C}, \mathbb{C}^*)$. Denote by t the generator of $H^2(\mathbb{C}, \mathbb{C}^*) \approx \mathbb{Z}$ corresponding to the natural orientation of $\mathbb{C}$. Then multiplication (on the left) by t defines a group isomorphism

$$\tau : H^*(M'') \to H^*(N, N_0)(2) \tag{4.1}$$

increasing degrees by 2. (It is just a specification of the Thom isomorphism to our simple situation).

Since $M = M' \setminus M''$ consider the excision of $U = M' \setminus N$ from (M', M). Using the excision isomorphism $\alpha : H^*(M', M) \to H^*(N, N_0)$ we obtain from (4.1) the group isomorphism

$$\alpha^{-1}\tau : H^*(M'') \to H^*(M', M)(2). \tag{4.2}$$

Substituting the isomorphism (4.2) into the exact sequence of the pair (M, M') we have the exact sequence

(4.3)

$$\cdots \longrightarrow H^p(M') \xrightarrow{i^*} H^p(M) \xrightarrow{\tau^{-1}\alpha\delta} H^{p-1}(M'') \xrightarrow{j^*} H^{p+1}(M') \cdots$$

where $i : M \to M'$ and $j : (M', \emptyset) \to (M', M)$ are natural embeddings and $\delta : H^p(M) \to H^{p+1}(M', M)$ is the connecting homomorphism. In the rest of this section we put $\delta' = \tau^{-1}\alpha\delta$.

4.2. **Basic cohomology classes.** If H is a hyperplane of V then $V \setminus H$ is homotopy equivalent to $\mathbb{C}^*$ (via projection along H) whence $H^1(V \setminus H)$ is generated by a canonical element β_H (corresponding to the natural orientation of $\mathbb{C}^*$). If $H = H_m \in \mathcal{A}$ and $i_{H_m} : M \to V \setminus H_m$ is the natural inclusion then we put $\beta_m = i^*_{H_m}(\beta_{H_m}) \in H^1(M)$. We call elements β_m the *basic cohomology classes* of $H^*(M)$. Substituting $\mathcal{A}'$ and $\mathcal{A}''$ for $\mathcal{A}$ we obtain the basic cohomology classes $\beta'_m \in H^1(M')$ and $\beta''_m \in H^1(M'')$ $(m = 1, \dots, n-1)$.

The naturality of the basic cohomology classes immediately implies the following properties for every p and $m = 1, \dots, n-1$.

(i) $i^*(\beta'_m) = \beta_m$.
(ii) $\delta(\beta_m) = 0$.
(iii) $\delta(a\beta_m) = \delta(a)\beta'_m$ for every $a \in H^*(M)$.
(iv) $\tau^{-1}(\alpha(\beta'_m)t) = \beta''_m$.
(v) $\alpha(\delta(\beta_n)) = t$.

The properties (i)-(v) immediately imply the following.

Proposition 4.1. *In the sequence* (4.3) *we have*

$$i^*(\beta'_{m_1} \cdots \beta'_{m_p}) = \beta_{m_1} \cdots \beta_{m_p},$$

$$\delta'(\beta_{m_1} \cdots \beta_{m_p}\beta_n) = \beta''_{m_1} \cdots \beta''_{m_p},$$

and

$$\delta'(\beta_{m_1} \cdots \beta_{m_p}) = 0$$

for every $1 \le m_1, \dots, m_p \le n-1$.

In order to find other properties of products of β_i it is convenient to represent these classes by differential forms (using the de Rham theorem). The canonical generator of $H^1(\mathbb{C}^*)$ can be represented by the form $\frac{1}{2\pi\sqrt{-1}}\frac{dz}{z}$. Pulling this form back to M we represent β_m by the form

$$\omega_m = \frac{1}{2\pi\sqrt{-i}}\frac{d\alpha_m}{\alpha_m}$$

where α_m is (as above) a linear polynomial on V whose zero locus is H_m.

Now for every $S = \{i_1, \dots, i_p\} \subset [n]$ $(i_1 < \cdots < i_p)$ put $\omega_S = \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ and $\partial\omega_S = \sum_{j=1}^p (-1)^{j-1}\omega_{S_j}$ where $S_j = S \setminus \{i_j\}$. Then we have the following.

Proposition 4.2. *(i) If* $\cap S = \emptyset$ *then* $\omega_S = 0$.
(ii) If S *is dependent then* $\partial\omega_S = 0$.

Proof. (i) If $\cap S = \emptyset$ then there exist $c_j \in \mathbb{C}$ $(j = 1, \dots, p)$ such that $\sum_j c_j\alpha_{i_j} = 1$. Differentiating we have $\sum_j c_j d\alpha_{i_j} = 0$ whence $\omega_S = 0$.

(ii) Now assume that S is dependent. Without loss of generality we can assume that $\alpha_{i_p} = \sum_{j<p} c_j\alpha_{i_j}$ for some $c_j \in \mathbb{C}$. This implies $\omega_{i_p} =$

$\sum_{j<p}\frac{c_j\alpha_{i_j}}{\alpha_{i_p}}\omega_{i_j}$. Using this we have

$$\partial\omega_S = \frac{1}{\alpha_{i_p}}(\sum_{j<p}(-1)^{j-1}\omega_{i_1}\wedge\cdots\wedge\widehat{\omega_{i_j}}\wedge\cdots\wedge\omega_{i_{p-1}})\wedge c_j\alpha_{i_j}\omega_{i_j}$$

$$+(-1)^{p-1}\omega_{S_p} = (-1)^p[\frac{1}{\alpha_{i_p}}\sum_{j<p}c_j\alpha_{i_j} - 1]\omega_{S_p} = 0$$

which completes the proof. □

Applying the de Rham map to the relations of forms in Proposition 4.2 we obtain similar relations for the cohomology classes. For a set S as above put $\beta_S = \beta_{i_1}\cdots\beta_{i_p}$ and define $\partial\beta_S$ similarly to $\partial\omega_S$.

Corollary 4.3. *(i) If $\cap S = \emptyset$ then $\beta_S = 0$.*
(ii) If S is dependent then $\partial\beta_S = 0$.

4.3. **Cohomology of M.** Since the ring $H^*(M)$ is graded commutative and $\beta_m \in H^1(M)$ $(m = 1,\dots,n)$, the assignment $e_m \mapsto \beta_m$ defines a ring homomorphism $\phi : E \to H^*(M)$.

Theorem 4.4. *The homomorphism ϕ induces an isomorphism $\psi : A \to H^*(M)$.*

Proof. First of all, Corollary 4.3 implies that $\phi(I) = 0$ whence ϕ induces a ring homomorphism $\psi : A \to H^*(M)$. To prove that ψ is an isomorphism we use induction on $|\mathcal{A}|$. If $\mathcal{A} = \emptyset$ then both rings are equal to $\mathbb{Z}$ and the result is trivial. Suppose $\mathcal{A} \neq \emptyset$ and consider the deletion-restriction triple $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ defined by H_n. Using the notation from subsection 2.6 for the OS algebras of these arrangements we have the diagram for every p:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "\psi'"] & A \arrow[r] \arrow[d, "\psi"] & A'' \arrow[r] \arrow[d, "\psi''"] & 0 \\
0 \arrow[r] & H^p(M') \arrow[r, "i^*"] & H^p(M) \arrow[r, "\delta'"] & H^{p+1}(M'') \arrow[r] & 0.
\end{tikzcd} \tag{4.4}$$

Proposition 4.1 implies that the diagram is commutative. By the inductive hypothesis, ψ' and ψ'' are isomorphisms. We also know from (2.4) that the first row is exact. This immediately implies that δ' is surjective for every p whence i^* is injective. This means that the second row is exact and ψ is an isomorphism from the 5-lemma. □

The following corollaries are immediate.

Corollary 4.5. *The de Rham homomorphism defines an isomorphism between the ring of differential forms generated by ω_i $(i = 1,\dots,n)$ and $H^*(M)$.*

Corollary 4.6. *The Poincarè polynomials of M and L coincide (cf. Corollary 3.4).*

5. OS algebras as modules over exterior algebras

In this section we study an OS algebra A as a finite module over the exterior algebra E (via the canonical projection $\pi : E \to A$). We compute certain invariants of this module and define the module over a polynomial ring corresponding to A via the Bernstein-Gel'fand-Gel'fand (BGG) correspondence. The first explicit treatment of the OS-algebra as a module over exterior algebra appeared in [21] where the main results of 5.3-5.5 were found. The main results of 5.2 were obtained in [20].

5.1. **Dual modules.** Let A be an OS algebra considered as a module over the exterior algebra E. First notice that according to Proposition 2.18, the structure of the OS algebra of an arbitrary arrangement $\mathcal{A}$ is defined by the structure of that algebra for the cone over $\mathcal{A}$. Hence we reduce our considerations to *central* and as usual *essential* arrangements.

In order to study the E-module A it is convenient to use the dual module A^*. Let us discuss the definitions and specify the grading on A^*.

According to the general definition of the dual module over a ring we put

$$A^* = \mathrm{Hom}_E(A, E).$$

According to the usual agreement about the degree of homogeneous homomorphisms of graded modules, the degree of $\phi : A \to E$ is d if $\phi(A_p) \subset E_{p+d}$ for every p.

Any homomorphism $\phi : A \to E$ is defined by $\phi(1)$ and the necessary and sufficient condition on $\phi(1)$ is $I\phi(1) = 0$. The set of such elements forms the ideal I^0 that is the annihilator of I. Clearly the correspondence $\phi \mapsto \phi(1)$ defines an isomorphism of graded E-modules A^* and I^0 (in particular A^* is graded in nonnegative degrees). Moreover since a basis in E_1 is fixed the linear space E_n can be identified with k_n, that is the trivial E-module whose grading is concentrated in degree n. Then the multiplication in E defines for every p ($p = 1, \dots, n$) a nondegenerate pairing $E_p \times E_{n-p} \to k_n$. In turn, these pairings define an isomorphism of E-modules $\bar{A} = \mathrm{Hom}_k(A, k_n)$ and I^0 that sends linear mappings of degree p to elements of degree $n - p$. In particular we have $\dim I^0_p = \dim A_{n-p}$ and $(I^0)^0 = I$. Notice that with our definition of the gradings we have the following graded E- modules isomorphic

$$A^* \approx I^0 \approx \bar{A}.$$

This implies in particular the following equality for the Hilbert series:

$$H(A^*, t) = t^n H(A, 1/t).$$

5.2. **Pure ideals.** As before we fix a linear order on $\mathcal{A}$ and write elements of $\mathcal{A}$ as H_i, $i \in [n]$. Recall that an element $a \in E$ is called *pure* (decomposable) if it is the product of elements of degree 1. Unless $a \in E_1$ the linear factors of a are not uniquely defined. We call an ideal J of E *pure* if it is generated by a set of pure elements. In particular, the OS ideal I is pure.

In general if J is a pure ideal, its annihilator J^0 does not have to be pure. An example will be given in the next subsection. Our immediate goal is to show that the annihilator I^0 is pure and find its pure generators.

Extending definition 2.64 from [50] we say that a partition Π of $[n]$ is p-*independent* for some p, $3 \le p \le \ell + 1$, if any choice of one element from each of not more than p arbitrary elements of Π forms an independent set. It is clear that $(\ell+1)$-independent partition can have at most ℓ elements. Following [50], we call $(\ell+1)$-independent partition *independent.*

For any partition Π of $[n]$, if $P \in \Pi$ denote by i_P the smallest element of P and put

$$z(\Pi) = \prod_{P\in\Pi}\prod_{i\in P\setminus\{i_P\}}(e_i - e_{i_P}). \tag{5.1}$$

Here (and below) we use the symbol $\prod$ for the product of the factors taken in the induced order from $[n]$ on both Π and P for $P \in \Pi$. Clearly $z(\Pi) \in E_{n-|\Pi|}$.

Now denote by $J(p)$ the ideal of E generated by $\bigcup_{j\le p} I_j$. In particular $J(\ell) = I$. Notice that each ideal $J(p)$ is homogeneous.

Lemma 5.1. *If a partition Π of $[n]$ is p-independent then*

$$z(\Pi) \in (J(p-1)^0)_{n-|\Pi|}.$$

Proof. Let $S \subset [n]$ be a dependent set and $|S| \le p$. It suffices to prove that $z = z(\Pi)$ annihilates ∂e_S. Since Π is p-independent there exists at least one $P \in \Pi$ such that $P \cap S$ contains at least two elements, say it contains r and s. Then both z and ∂e_S are divisible by $e_r - e_s$ whence $z\partial e_S = 0$. This completes the proof. □

For a time being we reduce our considerations to the OS ideal $I = J(\ell)$. It turns out that it suffices to consider only independent partitions of the following special type. Let $F = (X_0 = V < X_1 < X_2 < \cdots < X_\ell)$ be a maximal flag of elements of L. To every X_p associate the subset $[n]_p = [n]_{X_p}$ (see subsection 2.4) and put $S_i = [n]_i \setminus [n]_{i-1}$ $(i = 1, 2, \dots, \ell)$. Then denote by $\Pi(F)$ the partition $\{S_1, \dots, S_\ell\}$ of $[n]$.

Lemma 5.2. *For every maximal flag F the partition $\Pi(F)$ is independent.*

Proof. In the above notation for F, fix $i_j \in S_j$ for every j, $1 \leq j \leq \ell$. It suffices to prove that the set $T_p = \{i_1, \dots, i_p\}$ is independent for every p, $1 \leq p \leq \ell$. Suppose that T_p is dependent for some p and choose p the minimal such. Then the intersection of the respective hyperplanes is X_{p-1} which contradicts the choice of i_p. The contradiction completes the proof. □

To simplify the notation put $z(F) = z(\Pi(F))$ for every maximal flag F. The two previous lemmas imply that $z(F) \in I^0$. Now we reduce the class of the partitions further. Recall that an independent set $T \subset [n]$ is called **nbc** if it does not contain a broken circuit. For every p denote by $\mathbf{nbc}_p$ the set of **nbc**-ssubsets of $[n]$ of cardinality p. If $T = \{i_1, \dots, i_p\}$ where $i_1 < i_2 < \cdots < i_p$ then $T \in \mathbf{nbc}_p$ if and only if $i_r = \min cl(\{i_r, \dots, i_p\})$ for each r, $1 \leq r \leq p$. Here by $cl(S)$ where $S \subset [n]$ we mean the smallest closed set $[n]_X$ ($X \in L$) containing S (cf. subsection 2.4). We will also write $X = \bigwedge S$.

Now fix $T = (i_1 < i_2 < \cdots < i_\ell) \in \mathbf{nbc}_\ell$ and define the maximal flag $F(T) = (X_0 = V < X_1 < \cdots < X_\ell)$ via $X_p = \bigwedge(\{i_{\ell-p+1}, \dots, i_\ell\})$. To simplify the notation put $z(T) = z(F(T))$.

Lemma 5.3. *For each p, fix the lexicographic ordering on the subsets of a cardinality p of $[n]$ and the respective ordering on the monomials in E_p. Then for each $T \in \mathbf{nbc}_\ell$ the initial monomial of $z(T)$ is $e_{\bar{T}}$ where $\bar{T} = [n] \setminus T$.*

Proof. Put $\Pi = \Pi(F(T))$. By definition of $z(T)$ and the order on the monomials, the initial monomial of $z(T)$ is the product of the initial monomials of $z_P(T) = \prod_{i \in P \setminus \{i_P\}} (e_i - e_{i_P})$, one for each $P \in \Pi$. By definition of i_P, the initial monomial of $z_P(T)$ is $e_{P \setminus \{i_P\}}$. Since T is **nbc** we have $T = \{i_P | P \in \Pi\}$ which completes the proof. □

Put $Z = \{z(T) | T \in \mathbf{nbc}_\ell\}$. The above lemmas imply the main result of this subsection

Theorem 5.4. *The set Z is a Gröbner basis of I^0 (with respect to the deg-lex monomial ordering).*

Proof. To prove the assertion we need to show that $In(I^0) = \langle In(Z) \rangle$ (see subsection 2.5). Since $Z \subset I^0$ it suffices to prove the inclusion

$$In(I^0) \subset \langle In(Z) \rangle, \tag{5.2}$$

the opposite inclusion being obvious. It is clear also that

$$In(I^0) \subset \overline{In(I)} \tag{5.3}$$

where $\overline{In(I)}$ is the set of monomials annihilating every element of $In(I)$. Recall that $In(I)$ consists of all the monomials divisible by some monomials e_S with S being a broken circuit. Thus $\overline{In(I)}$ consists of the monomials

e_T corresponding to the sets T transversal to all the broken circuits, i.e., intersecting each one of them nontrivially (we will call this set **tbc**). It is easy to see that the collection of **tbc**-sets consists of the complements to **nbc**-sets. In particular, the inclusion minimal **tbc**-sets have $n-\ell$ elements. Thus

$$\overline{In(I)} = \langle In(Z)\rangle. \tag{5.4}$$

Now (5.3) and (5.4) imply (5.2) which concludes the proof. □

Corollary 5.5. *The set Z is a generating set of I^0. In particular I^0 is a pure ideal generated in degree $n-\ell$.*

The first assertion is routine for Gröbner bases theory and can be proved by induction on the initial monomials of elements from I^0. The second one follows.

Corollary 5.5 implies a result for the OS ideal itself that may have an independent use. For each $T \in \mathbf{nbc}_\ell$ denote by I_T the linear ideal (i.e., generated in degree 1) generated by the factors of $z(T)$ (for instance, by $(e_i - e_{i_P})$ for all $P \in \Pi(T)$ and $i \in P \setminus \{i_P\}$). It is a particular case of a well known property of exterior algebras that $(I_T)^0 = Ez(T)$ where the right hand side is the principal ideal generated by $z(T)$.

Corollary 5.6. *The OS ideal I is the intersection of linear ideals. More precisely*

$$I = \bigcap_{T\in\mathbf{nbc}_\ell} I_T.$$

Proof. Corollary 5.5 gives $I^0 = \sum_T Ez(T)$. Thus we have

$$I = (I^0)^0 = (\sum_T Ez(T))^0 = \bigcap_T (Ez(T))^0 = \bigcap_T I_T.$$

□

Theorem 5.4 allows us to exhibit a basis for each $(I^0)_p$ (as a free k-module). For that we need to define elements $z(T) \in I^0$ for $T \in \mathbf{nbc}_p$ $(1 \le p \le \ell)$. The flag $F(T) = (X_0 < X_1 < \cdots < X_p)$ is defined exactly like for $p=\ell$ but it is not maximal anymore. More precisely, its maximal element X_p is not the maximal element of L. The sets $S_i = [n]_{X_i} \setminus [n]_{X_{i-1}}$ $(i = 1, \ldots, p)$ form a partition $\Pi = \Pi(F(T))$ of $[n]_p = [n]_{X_p}$ that is still independent. Extend the definition of z from (5.1) by setting

$$\overline{z(T)} = z(\Pi)e_{[n]\setminus[n]_p} \tag{5.5}$$

Notice that $\overline{z(T)} = z(T)$ if $p = \ell$.

The properties of $z(T)$ for $|T| = \ell$ can be generalized to an arbitrary p.

Lemma 5.7. *For each* **nbc***-set T, the initial monomial of $\overline{z(T)}$ is $e_{\bar{T}}$.*

The proof is similar to proof of Lemma 5.3.

Lemma 5.8. *For each $T \in \mathbf{nbc}_p$ we have*

$$\overline{z(T)} \in (I^0)_{n-p}.$$

Proof. Extend $F(T)$ to a maximal flag $\tilde{F} = (X_0 < \cdots X_p < X_{p+1} < \cdots < X_\ell)$. By Lemmas 5.2 and 5.1, $z(\tilde{F}) \in I^0$. Let us simplify notation slightly. First we extend the above definition of S_i to $i = p+1, \ldots, \ell$. Then we put $e(m) = e_{j_m}$ where $j_m = i_{S_m}$ $(m = p+1, p+2, \ldots, \ell)$. A straightforward check shows that $\overline{z(T)} = \pm z(\tilde{F})e(p+1)e(p+2)\cdots e(\ell)$. Thus $\overline{z(T)} \in I^0$ which completes the proof. □

Denote by Z_p the set of $\overline{z(T)}$ for all $T \in \mathbf{nbc}_p$.

Proposition 5.9. *For every p, the set Z_p form a basis of $(I^0)_{n-p}$.*

Proof. Since the k-module $(I^0)_{n-p}$ is isomorphic to $(E/I)_p$, we see that the cardinality of Z_p is correct. Thus we have to prove only that Z_p generates $(I^0)_{n-p}$.

Let $e \in (I^0)_{n-p} \setminus \{0\}$. By Theorem 5.4, $In(e) = \pm e_S e_{\bar{T}}$ for $T \in \mathbf{nbc}_\ell$. Since the set $U = [n] \setminus (S \cup \bar{T})$ is again $\mathbf{nbc}$, we can consider $\overline{z(U)}$. Since $In(\overline{z(U)}) = e_U$, we have $e + c\overline{z_U} \in (I^0)_{n-p}$ for some $c \in k$ and $In(e + cz_U) < In(e)$. Now the proof can be finished by induction in $In(e)$. □

Example 5.10. *Consider Example 3.5 using the same order of hyperplanes. Then*

$$Z = Z_3 = \{e_6 - e_2)(e_4 - e_1)(e_5 - e_1), (e_3 - e_1)(e_4 - e_1)(e_5 - e_1),$$

$$(e_3 - e_2)(e_4 - e_1)(e_5 - e_1), (e_3 - e_2)(e_5 - e_2)(e_6 - e_2),$$

$$(e_6 - e_4)(e_2 - e_1)(e_3 - e_1), (e_5 - e_4)(e_2 - e_1)(e_3 - e_1)\}.$$

Also $|Z_4| = \dim A_2 = 11, |Z_5| = \dim A_1 = 6$, *and* $|Z_6| = \dim A_0 = 1$ *(consisting of $e_1 \cdots e_6$). For instance for $\{1,2\}, \{4,6\} \in \mathbf{nbc}_2$ we have $\overline{z(\{1,2\}} = (e_4 - e_1)e_3e_5e_6$ and $\overline{z(\{4,6\}} = (e_5 - e_4)e_1e_2e_3$.*

Remarks

1. In fact, the proof of Proposition 5.9 contains the proof that Z_p is a Gröbner basis of the ideal $\bigcup_{r \geq n-p}(I^0)_r$.

2. It is not hard to prove that the correspondence $F \mapsto \overline{z(F)}$ induces a map $F\ell_p \to (I^0)_{n-p}$ where $F\ell$ is the graded flag-module introduced by Schechtman and Varchenko in [58]. Then Proposition 5.9 can be deduced from the results of [58].

5.3. **Minimal free resolution of I^0.** In this subsection we again assume that the arrangement $\mathcal{A}$ is central although the main results easily generalize to an arbitrary case using Proposition 2.18. **Also for the rest of the section, we assume that k is a field.** Knowing a generating set of I^0, the next natural problem is to construct the minimal free resolution of I^0 (as an E-module). Although this problem seems tedious (cf. [20]) we can proof an important property of this resolution.

By a free resolution of a graded E-module M we will always mean its graded free resolution, that is an exact sequence

$$\ldots \to R_2 \to R_1 \to R_0 \to M \to 0$$

in which all R_i are free graded E modules and all the maps are homogeneous of degree 0. Such a resolution is called *linear* if M is generated in a single degree, say s, and each R_i is generated in degree $s+i$. The latter condition is equivalent to saying that every map $R_p \to R_{p-1}$ in the resolution can be given by a matrix with all the entries from E_1 (which justifies the term linear). Notice that a linear resolution is automatically minimal.

Theorem 5.11. *A minimal free resolution of I^0 is linear. The Betti numbers of I^0, i.e., the ranks β_i of the terms of the resolution can be computed from the formula*

$$\sum_{i=0}^{\infty} \beta_i t^i = (-t)^{\ell} \frac{H(A,-1/t)}{(1-t)^n}.$$

Proof. We know already from Corollary 5.5 that I^0 is generated in degree $n-\ell$.

As before, we use the deg-lex order on the monomials of E and denote by $In(I)$ the initial ideal of I which is the ideal generated by the monomials e_S where S is running through the broken circuits. It follows from the general theory of Gröbner bases that $In(I)$ is a flat degeneration of I. In our situation we can give an explicit version of the degeneration. Let q be an indeterminate and consider the algebra $\tilde{E} = k[q] \otimes_k E = E[q]$ and its ideal $I(q)$ generated by the elements

$$\partial_q e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j} q^{i_j - i_1}$$

for every dependent subset $S = \{i_1 < i_2 < \cdots < i_p\}$ of $[n]$ (recall that $S_j = S \setminus \{i_j\}$). Then for every $\lambda \in k$ we can identify $\tilde{E}/(q-\lambda) = E$. If $\lambda \neq 0$ then $I(\lambda) = (I(q) + (q-\lambda))/(q-\lambda)$ is an ideal of E isomorphic to I under the linear automorphism of E mapping e_i to $\lambda^i e_i$. Otherwise $I(0) = (I(q) + (q))/(q) = In(I)$. Notice that the algebra $\tilde{E}/I(q)$ is free (**nbc**- monomials form a basis) and thus flat over $k[q]$. Similarly, the $k[q]$-module $\overline{A(q)} = \mathrm{Hom}_{k[q]}(\tilde{E}/I(q), k[q])$ is flat and $\overline{A(\lambda)} = \overline{A(q)}/(q-\lambda)\overline{A(q)}$ is isomorphic to $\overline{A}$ for $\lambda \neq 0$ whereas $\overline{A(0)} \cong \mathrm{Hom}_k(E/In(I), k)$.

Since the E-modules A and I^0 are dual to each other and E is a Gorenstein ring, the dual to a minimal free resolution of I^0 is a minimal injective resolutions of A. Also if a free module is generated in a degree r its dual is generated in the degree $-r$. Hence the first statement of the theorem is equivalent to $\mathrm{Tor}_p^E(A,k)_m = 0$ for $p > 0$ and $m \neq -n + \ell - p$. (Here Tor_p^E is the pth derived functor of the tensor product on the category of *graded* E-modules.) Since these linear spaces are homologies of a complex of linear spaces, the vanishing of any one of them is an open condition in the flat family above. Hence it suffices to show that the minmal injective resolution of the special fiber of the family, $A(0) = E/\mathrm{In}(I)$ is linear. This in turn is equivalent to the linearity of the minimal free resolutions of the ideal $(\mathrm{In}(I))^0$ that we denote by J.

The ideal J is generated by the monomials whose supports are **tbc**-sets of length $n-\ell$, i.e., complements to elements of $\mathbf{nbc}_\ell$. Denote by ${}_SJ$ the ideal of $S = k[e_1, \ldots, e_n]$ generated by the same (square-free) monomials. According to [5] (see also [21], Section 5), a minimal free resolution of J is linear if and only if the same is true about ${}_SJ$.

By definition of J the ideal ${}_SJ$ is the Stanley-Reisner ideal for the simplicial complex Δ on $[n]$ whose simplexes are sets not transversal to broken circuits, i.e., whose complements contain broken circuits. In other words, Δ is the Alexander dual to the simplicial complex Δ' on $[n]$ whose simplexes are precisely the **nbc**-sets. The latter complex has been well studied. In particular it is known (see [55]) that Δ' is shellable that is a sufficient condition for it to be Cohen-Macaulay (see [62]). On the other hand, due to [19], the free resolution of the Stanley-Reisner ideal of a simplicial complex is linear if and only if the Alexander dual to this complex is Cohen-Macaulay. Application of this result to ${}_SJ$ completes the proof of the first statement.

In order to prove the second statement write a minimal resolution as

$$\cdots \to E^{\beta-2}(\ell-n-2) \to E^{\beta_1}(\ell-n-1) \to E^{\beta_0}(\ell-n) \to I^0 \to 0.$$

Then from the exactness we have for the Hilbert series

$$H(I^0,t) = \sum_p (-1)^p H(E^{\beta_p}(\ell-p),t) = \sum_p (-1)^p \beta_p t^{-\ell+n+p}(1+t)^n.$$

On the other hand, we know from subsection 5.1 that

$$H(I^0,t) = H(A^*,t) = t^n H(A,1/t).$$

Comparing the last two equalities we obtain the desired formula. □

Example 5.12. *An arrangement $\mathcal{A}$ in K^ℓ is called generic if it is central and no set of ℓ or fewer hyperplanes of $\mathcal{A}$ is dependent. An arrangement is in general position if every set of $\ell+1$ or fewer hyperplanes meet transversely (in particular, the mutual intersection of any set of more than ℓ hyperplanes is empty). It is easy to see that an arrangement $\mathcal{A}$ is in general position if and only if $c\mathcal{A}$ is generic.*

In the general position case it follows immediately from the definition that I is the $(\ell+1)$st *power* $\mathbf{m}^{\ell+1}$ *of the maximal ideal* $\mathbf{m}$ *of* E*. It follows from this and Proposition 2.18 that in the generic case* I *is generated by the* ℓth *power of the maximal ideal of the subalgebra* E' *(generated by the differences* $e_i - e_j$*).*

If $\mathcal{A}$ *is in general position whence* $I = \mathbf{m}^{\ell+1}$*, then* $I^0 = \mathbf{m}^{n-\ell}$*. An explicit computation of the resolution of any power of* $\mathbf{m}$ *is given in terms of Schur functors in* [22]*. In particular the resolution is linear.*

5.4. **Singular varieties of OS algebras.** The acting of an element $x \in E_1$ defines on a graded E-module M the structure of a cochain complex (M, x). We say that x is *singular* on M if the cohomology of this complex does not vanish. In other words x is singular if the set of elements of M annihilated by x is not the same as xM. The set $\mathrm{sing}(M)$ of singular elements is an algebraic subset of E_1 called the *singular variety* of M. It is known (see [5]) that the dimension of $\mathrm{sing}(M)$ is the *complexity* of M, defined as the exponent of growth of the Betti numbers of M. This complexity plays, for modules over an exterior algebra, a role analogous to that of the projective dimension for modules over a polynomial ring. In this subsection we will compute the singular variety of the OS algebra A of an arrangement $\mathcal{A}$. We consider arbitrary arrangements in this section, not assuming anymore that they are central. It follows at once from the definition that the singular variety of a module M is the same as that of $\mathrm{Hom}_k(M, k)$, in particular $\mathrm{sing}(A^*) = \mathrm{sing}(A)$.

In order to state our results about $\mathrm{sing}(A)$ we need to introduce the product of arrangements. The *product* $\mathcal{A}_1 \times \mathcal{A}_2$ of arrangements $\mathcal{A}_i$ in K^{ℓ_i} is the arrangement $\mathcal{A}$ in $K^{\ell_1+\ell_2}$ consisting of the hyperplanes $H \times K^{\ell_2}$ for $H \in \mathcal{A}_1$ and the hyperplanes $K^{\ell_1} \times H$ for $H \in \mathcal{A}_2$. An arrangement is *irreducible* if it is not affine isomorphic to the product of two nonempty arrangements. Any arrangement can be represented as the product of irreducible arrangements. This representation is essentially unique (up to taking product with empty arrangements that does not change OS algebras). It is easy to express the OS algebra of the product in terms of the OS algebras of the factors.

Proposition 5.13. *The OS algebra of* $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ *is given by* $A(\mathcal{A}) = A(\mathcal{A}_1) \otimes_k A(\mathcal{A}_2)$*, where the tensor product is taken in the category of graded skew-commutative* k*-algebras. Moreover* $\mathrm{sing}(A(\mathcal{A})) = \mathrm{sing}(A(\mathcal{A}_1)) \times \mathrm{sing}(A(\mathcal{A}_2))$*.*

Proof. Denote by E, $E(1)$, and $E(2)$ the exterior algebras corresponding to $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ respectively. Clearly the set $[n]$ of indexes of $\mathcal{A}$ can be identified with the union of $[n]_1$ and $[n]_2$, the sets of indexes for $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus $E_1 = E(1)_1 \oplus E(2)_1$ whence $E = E(1) \otimes E(2)$ as skew-commutative algebras. Also a minimal dependent set in $[n]$, or a minimal set with empty intersection, lies either in $[n]_1$ or in $[n]_2$. Thus the defining relation set for $A(\mathcal{A})$ is the union of these sets for $A(\mathcal{A}_1)$ and $A(\mathcal{A}_2)$. This implies the first assertion.

To prove the second assertion fix $x = (x_1, x_2) \in E_1$ where $x_i \in E(i)_1$. Then

$$(A(\mathcal{A}), x) = (A(\mathcal{A}_1), x_1) \otimes (A(\mathcal{A}_2), x_2)$$

as cochain complexes. Now we can apply the Künneth formula which implies the second statement. $\square$

Now we need to learn more about relations among arrangements and their cones. We use the notation from Proposition 2.18: $c\mathcal{A}$ is the cone of an arrangement $\mathcal{A}$, cE is the exterior algebra corresponding to $c\mathcal{A}$, and cA is the OS algebra of $c\mathcal{A}$. Recall that $cA = A\langle e_0 \rangle$ where e_0 is the extra generator of cA corresponding to the extra hyperplane H_0. For $a \in A_1$ put $\bar{a} = \bar{a}_0 e_0 + a$ where $\bar{a}_0 = -\sum_{i=1}^n a_i$. Notice that $\sum_{i=0}^n \bar{a}_i = 0$ which means that $\bar{a} \in E'$. In fact, $\bar{a}$ is the image of a under the isomorphism $E \to E'$ from subsection 2.7.

Represent every $x \in cA_p$ as $x = x_0 e_0 + x_1$ with $x_0 \in A_{p-1}$ and $x_1 \in A_p$. Then the direct computation shows that

$$\bar{a}x = (a x_0 + \bar{a}_0 x_1) e_0 + a x_1.$$

This implies that we have the exact sequence of the complexes

$$0 \to (A, a)(1) \to (cA, \bar{a}) \to (A, a) \to 0 \tag{5.6}$$

that induces the long exact sequence in cohomology:

(5.7)

$$\cdots \longrightarrow H^{p-1}(A, a) \longrightarrow H^p(cA, \bar{a}) >>> H^p(A, a) \xrightarrow{\delta} H^p(A, a) \longrightarrow \cdots.$$

Lemma 5.14. *For every p, the connecting homomorphism δ in (5.7) vanishes. In other words, for every p we have the short exact sequence*

(5.8)

$$0 \longrightarrow H^{p-1}(A, a) \longrightarrow H^p(cA, \bar{a}) >>> H^p(A, a) \longrightarrow 0. \longrightarrow \cdots.$$

Proof. Take $x \in A_p$ such that $ax = 0$. Choose $x + \partial x e_0$ as a representative of x in cA_p (here $\partial = \partial_{cA}$ is defined in cA since $c\mathcal{A}$ is central). Then, since $\partial a = -\bar{a}_0$, we have

$$\bar{a}(x + \partial x e_0) = \bar{a}_0 x e_0 + a(\partial x e_0) = -\partial(ax) e_0 = 0$$

. This implies the assertion. $\square$

We will also need a result essentially from [16]. It says that for irreducible central $\mathcal{A}$

$$\bar{\chi}(\mathcal{A}) = H(A, t)/(1+t)_{|t=-1} \neq 0 \tag{5.9}$$

(see also [57], Sect. 2). The number $\bar{\chi}(\mathcal{A})$ is called *the beta-invariant of* $\mathcal{A}$. Notice that it is the Euler characteristic of the OS algebra of a deconing of $\mathcal{A}$.

Now we can find the singular varieties for irreducible arrangements.

Theorem 5.15. *Let $\mathcal{A}$ be an arrangement without central factors (for example, irreducible).*
(a) If $\mathcal{A}$ is noncentral then $\operatorname{sing}(A) = E_1$.
(b) If $\mathcal{A}$ is central then $\operatorname{sing}(A)$ *is the hyperplane E_1' of E_1 spanned by the elements $e_i - e_j$.*

Proof. If $e \in E_1 \setminus \operatorname{sing}(A)$ then A is a free module over the subring $k[e]/e^2$. It follows that the Hilbert series $1 + t$ of $k[e]/e^2$ divides the Hilbert series $H(A, t)$ of A.

If $\mathcal{A}$ is noncentral without central factors then the cone $c\mathcal{A}$ is irreducible. Applying (5.9) to $c\mathcal{A}$ we see that $1 + t$ does not divide $H(A, t)$. Statement (a) follows from that and the previous paragraph.

If now $\mathcal{A}$ is central without central factors then it is irreducible. Represent it as the cone $c(d\mathcal{A})$ over a deconing $d\mathcal{A}$. Then $d\mathcal{A}$ does not have any central factors whence $\operatorname{sing}(A(d\mathcal{A})) = E(d\mathcal{A})_1$. The exact sequence (5.8) implies that $\operatorname{sing}(A(\mathcal{A})) = \operatorname{sing}(A(d\mathcal{A}))$ under the linear isomorphism $E(d\mathcal{A})_1 \to E_1' \subset E_1$. This implies (b). □

From 5.15 and 5.13 we get the general case.

Corollary 5.16. *The variety* $\operatorname{sing}(A)$ *for any arrangement $\mathcal{A}$ is a linear subspace of E_1 of codimension equal to the number of central factors in an irreducible decomposition of $\mathcal{A}$. Using the coordinates of vectors in the canonical basis $(e_H)_{H\in\mathcal{A}}$ of E_1, sing(A) is given by the system of equations*

$$\sum_{H\in\mathcal{A}_j} x_H = 0$$

one for each central factor $\mathcal{A}_j$.

Example 5.17. *The noncentral arrangement $\mathcal{A} = \{x, y, x+1, y+1\}$ is not irreducible but does not have central factors. Thus* $\operatorname{sing}(A) = E_1$.

5.5. **Module $F(\mathcal{A})$.** Let $W = A_1^*$ be the dual vector space to $A_1 = E_1$, and let $S = \operatorname{Sym}(W)$ be the symmetric algebra of W, i.e., a polynomial ring over k.

In this subsection we study the module over S corresponding to A via the Bernstein-Gel'fand-Gel'fand (BGG) correspondence. As usually stated, the BGG correspondence is an isomorphism between the derived category of bounded complexes of coherent sheaves on the projectivization $\mathbb{P}(W)$ and the derived category of bounded complexes of finitely generated graded modules over $E = \wedge E_1$. But if one examines the proof one can extract a functor $\mathbb{R}$ from the category of graded modules over S to the category of linear free complexes over E, and also a functor $\mathbb{L}$ from the category of graded E-modules to the category of free linear complexes over S. These functors are equivalences of categories (see [22], Proposition 2.1).

Starting with a graded E-module P the corresponding complex $\mathbb{L}(P)$ over S is

$$\cdots \to S \otimes P_i \to S \otimes P_{i+1} \to \cdots$$

with the differential $1 \otimes p \mapsto \sum x_i \otimes e_i p$, where (x_i) and (e_i) are dual bases of W and E_1. Starting with a graded S-module M the corresponding complex $\mathbb{R}(M)$ over E is

$$\cdots \to \mathrm{Hom}_k(E, M_i) \to \mathrm{Hom}_k(E, M_{i+1}) \to \cdots$$

with a differential defined similarly.

Since minimal injective resolutions of A are linear we may define $F(\mathcal{A})$ to be the graded S-module that is mapped by $\mathbb{R}$ to an injective resolution of A. Then we have.

Theorem 5.18. *If M is a graded S-module and P is a finitely generated graded E-module, then $\mathbb{L}(P)$ is a free resolution of M if and only if $\mathbb{R}(M)$ is an injective resolution of P.*

This theorem allows us to derive some basic properties of $F(\mathcal{A})$.

Corollary 5.19. *$F(\mathcal{A})$ is generated over S in degree ℓ and has the linear free resolution*

$$\mathbb{L}(A): \quad 0 \to S \otimes_k A_0 \to \cdots \to S \otimes_k A_\ell \to F(\mathcal{A}) \to 0$$

(this resolution is often called the Aomoto complex). In particular, we have the following.
(a) $F(\mathcal{A})$ has projective dimension ℓ and $\mathrm{Ext}^\ell_S(F(\mathcal{A}), S) = k$.
(b) The support of $F(\mathcal{A})$ is a linear space whose codimension is the number of central arrangements in an irreducible decomposition of $\mathcal{A}$.
(c) The Hilbert series of $F(\mathcal{A})$ is

$$H(F(\mathcal{A}), t) = \sum_{i=0}^{\infty} \dim_k(F(\mathcal{A})_i) t^i = (-1)^\ell \frac{\chi(\mathcal{A}, t)}{(1-t)^n}.$$

Proof. Since the injective resolution of A is linear, Theorem 5.18 implies that the Aomoto complex is a (linear) free resolution of $F(\mathcal{A})$, proving the first statement and computing the projective dimension.

(a): Since $A_i = E_i$ for $i = 0, 1$, the two nonzero left-hand terms of the resolution above are the same as those in $\mathbb{L}(E)$, the Koszul complex. This allows us to compute the Ext in part (a).

(b):As it is shown in [5] the singular variety of an E-module A is the support of the $\mathrm{Ext}^*_E(k, k)$-module $\mathrm{Ext}^*_E(A, k)$, which is the same (since E is Gorenstein) as the support of the module $\mathrm{Ext}^*_E(k, A)$. It is well-known (and in fact equivalent to E being Koszul - see the next section) that $\mathrm{Ext}^*_E(k, k) = S$. By [22], this is the module $\mathrm{Ext}^*_E(k, A) = F(\mathcal{A})$.

(c): Knowing the free resolution of $F(\mathcal{A})$ allows us to compute its Hilbert series, just as in the proof of Theorem 5.11. ∎

Example 5.20. *If $\mathcal{A}$ is a general position arrangement of n hyperplanes in K^ℓ, then A is $E/\mathbf{m}^{\ell+1}$, so the free resolution of $F(\mathcal{A})$ is a truncation of the Koszul complex, and $F(\mathcal{A})$ is isomorphic to the $(n-\ell)$*th *syzygy module of the trivial S-module k.*

5.6. **Problems.** 1. Find any information about the minimal free resolutions of A (as a module over E), for instance, the Betti numbers of A or linear strands in a resolution.

At the present, the answers are known only for general position or generic central arrangements, see Example 5.12.

2. Construct a minimal free resolution of I^0 (equivalently, a minimal injective resolution of A) explicitly.

In [20], this problem is solved for the first two terms of the resolution, i.e., a presentation of the E-module I^0 is given.

6. Koszul OS algebras

In this section we study the OS algebras from the point of view of theory of graded connected finitely generated algebras. An interesting and well-studied class of those algebras is the Koszul algebras. The Koszul property of OS algebras was first studied in [60] where the main result of 6.3 was proved (in a more general situation). Here we give a much shorter proof using Gröbner bases and results of [11].

The main result of this section is in 6.4-6.6. It claims that A is Koszul if and only if M is rational $K[\pi, 1]$ (for $K = \mathbb{C}$). This result is from [53]. The study of arrangements with rational $K[\pi, 1]$ complements has a relatively long history. Let us mention the papers [2, 40, 41, 42, 27, 24, 36] devoted to this property and close questions. More detailed references can be found in [29].

Before approaching the Koszul condition, we study the necessary condition for that - the quadraticity of A. This condition was first studied by Falk in [26] where combinatorial notions of taut and line-closed lattices (or rather matroids) were used as a necessary conditions. Here we follow [20] where a stronger necessary condition was found, disproving a conjecture of Falk (see subsection 6.1).

We assume in this section that $\mathcal{A}$ is central although most of the definitions and results can be generalized to arbitrary arrangements.

6.1. Quadratic OS algebras.

Definition 6.1. *Let R be a graded algebra $R = \sum_{p\geq 0} R_p$ over k. We always assume that R is connected (i.e., $R_0 = k$) and generated by R_1. Then R can be naturally represented as the factor of the tensor algebra $T(R_1)$ of R_1 by a homogeneous ideal $J = \sum_{p\geq 2} J_p$. Algebra R is called quadratic if J is generated by J_2.*

In other words, R is quadratic if all its relations follow from those of degree 2. Notice that if $R = N/I$ where N is a quadratic algebra and I its homogeneous ideal then R is quadratic if and only if I is generated by I_2. In particular the OS algebra $A = E/I$ is quadratic if and only if the OS ideal I is generated by I_2.

During the last several years, attempts were made to understand the quadraticity of the OS algebras directly from the lattice L, without defining A first. These attempts led to several necessary conditions that we will discuss (going from the weakest to the strongest).

I. Taut lattices. If L and L' are geometric lattices then a map $f : L' \to L$ is called *projection* if it is surjective, bijective on atoms, and preserves the least upper bounds (that is $f(X \vee Y) = f(X) \vee f(Y)$ for all $X, Y \in L'$). Using f one can always identify the sets of atoms of L' and L. If there exists a projection from L' to L then L is called a *quotient* of L'. For instance, every atomic lattice is a quotient of the Boolean lattice with the same set of atoms. It is a simple exercise that $\mathrm{rk}L' \geq \mathrm{rk}L$ and if $\mathrm{rk}L' = \mathrm{rk}L$ then $L' = L$ (and $f = \mathrm{id}_L$).

Recall that the set of atoms of a geometric lattice L is the set of elements of rank 1. Denote by $L[p]$ the set of elements of L of rank less than or equal to p.

Definition 6.2. *A geometric lattice is called taut if it is not a quotient of any lattice $L' \neq L$ with $L'[2] = L[2]$.*

If a geometric lattice L is represented as the intersection lattice of a central arrangement $\mathcal{A}$ then there is a simple condition necessary for the tautness of L. If one fixes linear functionals whose kernels are the hyperplanes from $\mathcal{A}$ then any linear dependence among these functionals must follow from linear dependences of length 3. Arrangements with this property are called *formal.*

Example 6.3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be (given by the linear functionals in $\mathbb{R}^3$)*

$$\mathcal{A}_1 = \{x, y, z, x+y+z, 2x+y+z, 2x+3y+z, 2x+3y+4z, 3x+5z, 3x+4y+5z\},$$

$$\mathcal{A}_2 = \{x, y, z, x+y+z, 2x+y+z, 2x+3y+z, 2x+3y+4z, x+3z, x+2y+3z\}.$$

The straightforward check shows that their intersection lattices are isomorphic, $\mathcal{A}_1$ is formal, and $\mathcal{A}_2$ is not formal. Thus their common lattice L

is not taut. And indeed it is the quotient of the intersection lattice L' of the arrangement

$$\mathcal{A}_3 = \{x, y, z, u, x+u, x+2y+u, x+2y+3z+u, x+3z, x+2y+3z\}$$

in $\mathbb{R}^4$ under the unique projection that is the bijection on the atoms substituting $x+y+z$ for u. The lattice L' is taut (see below).

A natural generalization of the tautness is the notion of local tautness. A geometric lattice L is *locally taut* if for every $X \in L$ the lattice $L_X = \{Y \in L | Y \leq X\}$ is taut.

II. Line-closed lattices. Let again L be a geometric lattice with the set of atoms $\mathcal{A}$. Recall some notions we used in Section 2. A set $\mathcal{B}$ of atoms is dependent if $\text{rk}(\bigvee \mathcal{B}) < |\mathcal{B}|$. The inclusion smallest dependent sets are circuits and a set $\mathcal{C}$ of atoms is closed if there is no circuit $\mathcal{B}$ such that $|\mathcal{C} \cap \mathcal{B}| = |\mathcal{B}| - 1$. Equivalently $\mathcal{C}$ is closed if there exists $X \in L$ such that $\mathcal{C} = \mathcal{A}_X = \{H \in \mathcal{A} | H \leq X\}$.

If L is the intersection lattice of an arrangement $\mathcal{A}$ then the closed subarrangements are those whose sets of linear functionals are closed with respect to linear relations inside all linear functionals of hyperplanes of $\mathcal{A}$. Viewing the linear functionals as vectors in the dual space, one can put more relaxed natural conditions on their sets. For instance, one can require a set to include all the functionals lying on the line passing through two elements of this set. Adjusting this property to arbitrary geometric lattices we obtain the following.

Definition 6.4. *A subset $\mathcal{B}$ of the set $\mathcal{A}$ of atoms of a geometric lattice L is called line-closed if for every $X \in L$ with* $\text{rk}X = 2$ *such that $|\mathcal{B} \cap \mathcal{A}_X| \geq 2$ we have $\mathcal{A}_X \subset \mathcal{B}$. The lattice L is called line-closed if the only line-closed sets of atoms are the closed ones.*

The property of a lattice to be line-closed is stronger than the local tautness.

Theorem 6.5. *If a geometric lattice L is line-closed then it is locally taut.*

Proof. Suppose L is not locally taut, i.e., there is $X \in L$, a geometric lattice $L' \neq L_X$, and a projection $f : L' \to L_X$ that is identity on $L'[2] = L_X[2]$. If f were bijective then it would have been the identity map which is a contradiction. Hence there exists $Y \in L_X$ such that $|f^{-1}(Y)| > 1$. By definition of a projection we have $U = \bigvee(f^{-1}(Y)) \in f^{-1}(Y)$. Denote the sets of atoms of L by $\mathcal{A}$ and notice that $\mathcal{A}_X$ is the set of atoms of L'. Now choose $Z \in f^{-1}(Y)$ different from U and put $\mathcal{B} = (\mathcal{A}_X)_Z = \mathcal{A}_Z$. The set $\mathcal{B}$ is closed in L' whence it is line-closed. Since $L'[2] = L_X[2]$ we have $\mathcal{B}$ is still line-closed in L_X. However $\mathcal{B}$ cannot be closed in L_X since $\bigvee(\mathcal{B}) = Y$ in this lattice but there exists the lager set $(\mathcal{A}_X)_U = \mathcal{A}_U$ with $\bigvee(\mathcal{A}_U) = Y$ in L_X. This implies that the latter lattice is not line-closed. The contradiction completes the proof. $\square$

The converse of Theorem 6.5 is false.

Example 6.6. *Define the arrangement $\mathcal{A} = \{x, y, z, x+y, x+z, y+z\}$. The subarrangement $\mathcal{B} = \{x+y, x+z, y+z\}$ is line-closed but not closed. Thus the lattice L of $\mathcal{A}$ is not line-closed.*

We want to prove that L is locally taut. Since all proper sublattices L_X ($X \in L$) have ranks less than 3 they are trivially taut. Thus it suffices to check that L is taut. Suppose L' is a geometric lattice with $L'[2] = L[2]$ and $f : L' \to L$ a projection. Notice that the minimal line-closed set of atoms containing x, y, and z is $\mathcal{A}$ in L whence it is same in L'. Thus the minimal closed set of atoms in L' containing those three atoms is again $\mathcal{A}$. This implies that $\mathrm{rk} L' = 3 = \mathrm{rk} L$ *whence $L' = L$.*

III. 3-independent lattices. We use the independent partitions from subsection 5.2 to define yet another property of a geometric lattice. Let L be a geometric lattice of rank ℓ with the set of atoms $\mathcal{A}$ and $|\mathcal{A}| = n$. As before, we fix a linear order on $\mathcal{A}$ and use the notions of p-independent partitions of $[n]$ from subsection 5.2. Then we can make the following definition.

Definition 6.7. *Let $3 \leq p \leq \ell$. The lattice L is p-independent if any p-independent partition of $[n]$ is independent.*

The condition to be 3-independent is stronger than the line-closeness.

Theorem 6.8. *If L is 3-independent than it is line-closed.*

Proof. Suppose L is not line-closed. To prove the theorem it suffices to exhibit a 3-independent partition having more than ℓ elements (whence not independent).

Let S be a line-closed subset of $[n]$ that is not closed. Let T be the closure of S, i.e., $T = [n]_X$ where $X = \bigvee(S)$. Include X into maximal flag F of $L_{\geq X} = \{Y \in L | Y \geq X\}$. Fixing notation, $F = (X_r = X < X_{r+1} < \cdots < X_\ell)$ where $r = \mathrm{rk} X$. Also choose an increasing sequence (S_i) ($i = 0, 1, \ldots, r$) of subsets of S such that $\mathrm{rk} \bigvee(S_i) = i$. We have $S_r = S$. Now we can define the partition Π that consists of $\ell + 1$ elements, namely $S_p \setminus S_{p-1}$ for $p = 1, 2, \ldots, r$, $S' = T \setminus S$, and $\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}$ for $i = r+1, \ldots, \ell$.

We claim that Π is 3-independent. Let P consist of 3 elements, one from each of 3 distinct elements of Π. If at least one of these elements does not belong to T then the independence of P is obvious (cf. Lemma 5.1). The same is true if at least two of the elements belong to S_{r-1}. Thus the only interesting case is when $P = \{a, b, c\}$ where $a \in S_{r-1}, b \in Y_r = S \setminus S_{r-1}$, and $c \in S' = T \setminus S$. In this case, we have $|P \cap S| = 2$ and if P is dependent this contradicts the condition that S is line-closed. Thus Π is 3-independent that concludes the proof. □

Again we can show that the new condition is strictly stronger than the previous one, i.e., the converse of Theorem 6.8 is false.

Example 6.9. *Let L be the intersection lattice of the following arrangement in $\mathbb{R}^3$*

$$\mathcal{A} = \{x, y, x-y, x+y, z, x+y+z, 2x+y+z, 3y+z\}.$$

Enumerating the atoms (i.e., hyperplanes) in the order they are given we have the following collection of dependent sets of rank 2:

$$\{\{1,2,3,4\}, \{1,6,7\}, \{2,5,8\}, \{3,7,8\}, \{4,5,6\}\}.$$

Picturing each hyperplane as a point and drawing a line through dependent sets one obtains the **matroid picture** *of this arrangement shown in Figure 1(a).*

There are 48 independent sets of cardinality 3. Checking directly all of them one can see that the line-closure of each of them is $[8]$. This implies that L is line-closed. On the other hand, consider the partition

$$\Pi = \{\{1,3,7\}, \{2,4,5\}, \{6\}, \{8\}\}.$$

Since none of the dependent 3-sets intersects non-trivially with more than two elements of Π we see that Π is 3-independent. Since it has 4 elements it is not independent. Hence L is not 3-independent.

Now we relate these conditions on geometric lattices to the quadraticity of the OS algebras.

Theorem 6.10. *If A is quadratic then L is 3-independent.*

Proof. Recall that I is the OS ideal and $J(p)$ is the ideal generated by I_j for $j \le p$ $(p = 2, 3, \dots, \ell)$. The condition of the theorem means that $J(2) = I$ whence $(J(2))^0 = I^0$. Let Π be a 3-independent partition of $[n]$. By Lemma 5.1, $z(\Pi) \in (J(2))^0$ whence $z(\Pi) \in I^0$.

Assume that Π is not independent, i.e., there exists a dependent subset $S \subset [n]$ such that $|S \cap P| \le 1$ for every $P \in \Pi$. To find a contradiction it suffices to show that $z(\Pi)\partial e_S \neq 0$. Indeed we have equality (up to a sign)

$$z(\Pi)\partial e_S = \pm z(\bar{\Pi})$$

where $\bar{\Pi}$ is the partition obtained from Π by gluing together all the elements $P \in \Pi$ such that $S \cap P \neq \emptyset$. Since for every partition ρ we have $z(\rho) \neq 0$ we obtain the contradiction implying the result. $\square$

Theorems 6.10 and 6.8 immediately imply that the other two properties of geometric lattices from this subsection are also necessary for the quadraticity of their OS algebras.

Corollary 6.11. *If A is quadratic then L is line-closed and locally taut.*

Since being 3-independent is a strictly stronger condition than the line-closeness and tautness neither of the latter is sufficient for the quadraticity. We can show that the former condition is not sufficient either.

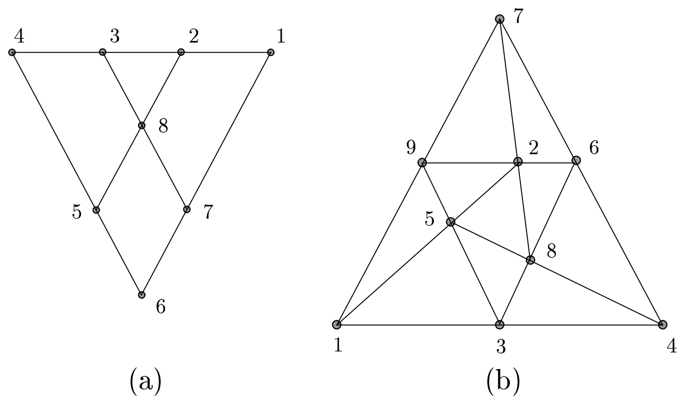


FIGURE 1. the matroid picture of (a) Example 6.9; (b) $(9_3)_2$;

Example 6.12. *This is the arrangement of lines in* $\mathbb{R}^2$ *that appeared in the book* [38] *by Hilbert and Cohn-Vossen, where it was called the* $(9_3)_2$ *configuration. Its matroid picture is shown on Figure 1(b). We put* $J = J(2)$ *for the length of this example.*

That A *is not quadratic can be seen by the following simple computation (that applies in fact to all three* 9_3 *configurations from* [38]*). The ideal* J *is generated by 9 elements from* E_2*. Since each of them is annihilated by a 2-dimensional subspace of* E_1 *we have*

$$\dim J_3 \leq 7 \times 9 = 63. \tag{6.1}$$

On the other hand, since $\mathcal{A}$ *is central the Hilbert series of* A *should be divisible by* $1 + t$ *whence a simple computation gives* $H(A,t) = 1 + 9t + 27t^2 + 19t^3$*. This implies*

$$\dim I_3 = \binom{9}{3} - 19 = 65. \tag{6.2}$$

Thus $J \neq I$*.*

Proving that $\mathcal{A}$ *is 3-independent requires more work. Let us say for convenience that a subset* T *of* [9] represents *a 3-circuit* S *if* $|T \cap S| \geq 2$*. A useful observation about the matroid is that it has an automorphism of order 9 that can be written as a permutation* $\tau = [261594837]$ *in two-line notation. In particular, it is transitive on* [9]*. Now we assert the following for an arbitrary subset* T *of* [9] *(it suffices to consider only the cases where* $1 \in T$*).*

(1) *If* $|T| = 3$ *then it represents at least one circuit.*

Indeed the only elements of [9] *that do not represent a circuit with 1 are 6 and 8 and* $\{6,8\}$ *represents a circuit.*

(2) *If* $|T| = 4$ *then it represents at least 2 and at most 5 circuits.*

There are two cases. Suppose T *contains a circuit* S *(one can assume* $S = \{1,3,4\}$ *or* $S = \{1,2,5\}$*). Then by inspection, any other element of* [9] *represents another circuit with one of the elements of* S. *The first statement follows. Clearly in this case* T *can represent at most 4 circuits.*

Now suppose T *does not contain a circuit. Since two distinct subsets of* T *of cardinality* 3 *cannot represent the same circuit, again the first assertion must hold. On the other hand, the matroid picture viewed as a graph does not have a complete subgraph with* 4 *vertices, and this implies the second assertion.*

(3) *If* $|T| = 5$ *then it represents at most* 7 *circuits. If* $|T| = 6$ *then this number is* 8.

This follows from (1) and the first assertion of (2) by passing to the complements.

Now suppose Π *is a* 3*-independent partition of* [9]. *This means that the elements of* Π *represent each of the* 9 *circuits. Considering all the possibilities for the cardinalities of elements of* Π *if* $|\Pi| = 4$, *i.e.* $\{6,1,1,1\}$, $\{5,2,1,1\}$, $\{4,3,1,1\}$, $\{4,2,2,1\}$, *we deduce from the second assertion of (2) and (3) that the elements of* Π *can represent at most* 8 *elements. Thus* $|\Pi| \leq 3$ *whence* Π *is independent. This shows that* $\mathcal{A}$ *is* 3*-independent.*

In fact using Macaulay 2 [33] *we can obtain a basis of* $(J^0)_5$ *and show that* $(J^0)_5$ *does not contain any pure elements (in particular,* J^0 *is not a pure ideal). To describe this basis let*

$$x = e_1e_2e_3e_6e_8 - e_1e_2e_4e_6e_9 + e_1e_2e_4e_8e_9 - e_1e_2e_6e_8e_9 - e_2e_3e_5e_6e_9,$$

$p = (1-\tau)(1+\tau^3+\tau^6)x$, *and* $q = \tau p$. *Then* p *and* q *form a basis for* $(J^0)_5$.

For any element $r \in E_5$, *let* $\mathcal{S}(r)$ *be the support of* r: *that is, the set of all subsets* $\{i_1, i_2, \ldots, i_5\}$ *that index nonzero monomials* $e_{i_1}\cdots e_{i_5}$ *in* r. *Suppose* r *is pure, that is* $r = a_1a_2\cdots a_5$ *where each* $a_i \in E_1$. *Then the coefficients of the monomials that make up* r *are the minors of a* 9×5 *matrix whose* i*th column is given by* a_i. *In particular, the set* $\mathcal{S}(r)$ *is the set of bases of the matroid on the set* [9] *given by the rows of this matrix.*

Consider a nonzero linear combination $r = \alpha p + \beta q$. *We show that* r *cannot be pure as follows. We can assume that* $\alpha \neq 0$, *by replacing* p *and* q *with* τp *and* τq, *respectively, otherwise. Then put* $B = \{1,2,6,8,9\}$. *By writing out* r *we find* $B \in \mathcal{S}(r)$. *However, neither* $B\cup\{5\}\setminus\{9\}$ *nor* $B\cup\{7\}\setminus\{9\}$ *are in* $\mathcal{S}(r)$. *Thus* $\mathcal{S}(r)$ *does not satisfy the main axiom for the set of bases of a matroid. It follows that* $(J^0)_5$ *contains no pure elements.*

6.2. **Definitions of Koszul algebras, a sufficient condition.** At the beginning A may be an arbitrary positively graded connected finitely generated k-algebra. Let Ext^p_A be the pth derived functor of the *graded* homomorphism functor on the category of graded A-modules. For two such modules M and N and every p, the linear space $\mathrm{Ext}^p_A(M,N)$ is graded, i.e. $\mathrm{Ext}^p_A(M,N) = \oplus_{q\geq 0}(\mathrm{Ext}^p_A(M,N))_q$.

Definition 6.13. *A is Koszul if*

$$(\mathrm{Ext}^p_A(k,k))_q = 0 \tag{6.3}$$

for every $p \neq q$, where $k = {}_Ak$ is the trivial graded left A- module (equal to A/A_+).

In other words, the minimal free resolutions of k (as a graded A-module) are linear.

If A is as above then there exists a canonical graded algebra homomorphism $T(A_1) \to A$ where $T(A_1)$ is the tensor algebra of a linear space A_1. Recall that we call the kernel of this homomorphism the ideal of A.

Definition 6.14. *Let A be a quadratic algebra and J its ideal. Let $J_2^!$ be the annihilator in $A_1^* \otimes A_1^* = (A_1 \otimes A_1)^*$ of the linear space J_2 and let $J^!$ be the ideal of $T(A_1^*)$ generated by $J_2^!$. Then the quadratic algebra $A^! = T(A_1^*)/J^!$ is called the quadratic dual of A.*

Observe that $(A^!)^! = A$. The quadratic dual allows us to give another equivalent definition of Koszul algebras. Namely, A is Koszul if and only if its Koszul complex $\mathcal{K}$ is exact. Here $\mathcal{K} = \oplus_{p \geq -1} \mathcal{K}_p$ where $\mathcal{K}_p = \mathrm{Hom}_k(A_p^!, A)$ for $p \geq 0$, $\mathcal{K}_{-1} = k$, and $d_p : \mathcal{K}_p \to \mathcal{K}_{p-1}$ is defined for $p \geq 1$ via $d_p f(a) = \sum_{m=1}^n f(x_m a) e_m$ where $a \in A_{p-1}^!$, $(e_1, \ldots, e_n)$ is a basis of A_1, and $(x_1, \ldots, x_n)$ is the dual basis of $A_1^!$.

The following statements (for example, see [8]) will be used often without reference.

Theorem 6.15. *(i) If A is Koszul then it is quadratic (in particular generated in degree 1). For any quadratic algebra the condition* (6.3) *holds for $p < 3$ and $q \neq p$.*

(ii) A quadratic algebra A is Koszul if and only if $A^!$ is Koszul.

(iii) If A is Koszul then the following relation between the Hilbert series holds

$$H(A,t) \cdot H(A^!, -t) = 1.$$

The converse of 6.15 (iii) is false (see [54, 56]).

Now we reduce our attention to algebras A that are quotients of an exterior algebra. For them, similarly to modules over commutative algebras, there exists a simple condition in terms of Gröbner bases sufficient for A to be Koszul. Let E be an exterior algebra and $A = E/I$ where I is a homogeneous ideal of E,

Theorem 6.16. *If I has a quadratic (i.e., of degree 2) Gröbner basis then A is Koszul.*

The proof of this theorem reduces to the case where I is a monomial ideal generated in degree 2 similarly to how it was done in the proof of Theorem 5.11. The monomial case can be handled by induction on the dimension of A_1 (see [32] or [60]).

6.3. **Supersolvable arrangements.** In order to give an example of arrangements whose OS algebras are Koszul, we define a class of arrangements interesting on its own rights.

Let $\mathcal{A}$ be a central arrangement in a space V and L its intersection lattice. Recall that L is geometric and in particular ranks of any two $X, Y \in L$ satisfy the semimodular inequality

$$\mathrm{rk}(X) + \mathrm{rk}(Y) \geq \mathrm{rk}(X \vee Y) + \mathrm{rk}(X \wedge Y). \tag{6.4}$$

If (6.4) is equality for a pair $\{X, Y\}$ then this pair is called *modular*. An $X \in L$ is *modular* if it forms a modular pair with every $Y \in L$. In fact one can require less: X is modular if and only if (6.4) is equality for every $Y \in L$ such that $X \wedge Y = V$.

Definition 6.17. *L and $\mathcal{A}$ are called supersolvable if L has a maximal flag of modular elements.*

Example 6.18. *It is easy to find examples of supersolvable arrangements among the arrangements of all reflecting hyperplanes of finite Coxeter groups. In fact all such arrangements for types A and B are supersolvable. The smallest representatives of the series are $\mathcal{A}_1 = \{x, y, x-y, z, x-z, y-z\}$ (see Example 3.5 for details) and $\mathcal{A}_2 = \{x, y, z, x \pm y, x \pm z, y \pm z\}$. The maximal flag $(\{x = 0\} < \{x = 0, y = 0\} < 0)$ is modular for both arrangements.*

We will call a maximal flag of modular elements a *modular flag*. It turns out that the partition $\Pi(F)$ for a single modular flag F provides all the **nbc**-monomial for a special kind of orders.

Fix a maximal flag $F = (X_0 < X_1 < \cdots < X_\ell)$ and put $\mathcal{A}_p = \mathcal{A}_{X_p} \setminus \mathcal{A}_{X_{p-1}}$. A linear order on $\mathcal{A}$ is called *compatible with* F if the index of any element of $\mathcal{A}_p$ is smaller than the index of any element of $\mathcal{A}_{p+1}$ for every p.

Now fix an order compatible with F. As usual it defines a one-to-one correspondence of $\mathcal{A}$ and $[n]$. We define a *partial section* of a partition Π of $[n]$ as a subset T of $[n]$ such that $|T \cap P| \leq 1$ for every $P \in \Pi$. Usually we suppress 'partial'.

Theorem 6.19. *F is modular if and only if the collection of sections of $\Pi = \Pi(F)$ coincides with the collection of* **nbc***-sets.*

Proof. (i) Put $\Pi = \{S_1, S_2, \ldots, S_\ell\}$ and suppose that the flag F is modular. First we want to prove that every **nbc**-set is a section of Π. Let S be an **nbc**-set. Assume that it is not a section, i.e., there is a p and $i \neq j$ such that $i, j \in S \cap S_p$. Put $Y = \bigvee(\{i, j\})$ and observe that $\mathrm{rk}Y = 2$ whence $\mathrm{rk}X_p \geq 2$. Using the modularity of X_{p-1} we have

$$1 \leq \mathrm{rk}(Y \wedge X_{p-1}) = \mathrm{rk}Y + \mathrm{rk}X_{p-1} - \mathrm{rk}(Y \vee X_{p-1}) \leq p+1 - \mathrm{rk}(Y \vee X_p) = 1.$$

This implies that $\mathrm{rk}(Y \wedge X_{p-1}) = 1$, i.e., this is an atom, say H_m, of L. Since $m < i, j$ and $\{m, i, j\}$ is a circuit we have a contradiction with S being an **nbc**-set.

(ii) Now we prove that every section of Π is **nbc**. Assume not and choose an inclusion minimal section T that is not **nbc**. Then there exists m smaller than the minimal element t of T such that $\bar{T} = T \cup \{m\}$ is dependent. Suppose $t \in S_p$. If $m \in S_q$ with $q < p$ then $\bar{T}$ is a section which contradicts the independence of Π. If $m \in S_p$ then using the same computation as in (i) we can find $r \in S_{p-1}$ such that $T \cup \{r\}$ is dependent that is again a contradiction.

(iii) Finally suppose that the collection of sections of Π coincides with the collection of **nbc**-sets. Fix a p $(2 \leq p \leq \ell - 1)$ and suppose $Y \in L$ is such that $X_p \wedge Y = V$. Choose a set S such that $|S \cap S_i| = 1$ for $1 \leq i \leq p$. It is independent by Lemma 5.2. Choose a maximal **nbc**-subset T of $[n]_Y$ (using the induced order on $[n]_Y$). Since $[n]_Y$is closed, T is **nbc** in $\mathcal{A}$ whence by (i) we have $|T \cap S_i| \leq 1$ for all i. By the condition on Y, we have $T \cap S_i = \emptyset$ for $i \leq p$ whence $S \cup T$ is a section of Π and independent. By counting numbers of elements in the independent sets we have

$$\mathrm{rk}(X_p \vee Y) = \mathrm{rk} X_p + \mathrm{rk} Y = \mathrm{rk} X_p + \mathrm{rk} Y - \mathrm{rk}(X_p \wedge Y)$$

whence X_p is modular. This completes the proof. □

Corollary 6.20. *Let $\mathcal{A}$ be a supersolvable arrangement and $F = (X_0 < X_1 < \cdots < X_\ell)$ a modular flag of L. Fix an order on $\mathcal{A}$ compatible with F. Then*

1. *The simplicial* **nbc***-complex of $\mathcal{A}$ is the join of 0-dimensional complexes; moreover $H(A,t) = \prod_{p=1}^{\ell}(1 + b_p t)$ where $b_p = |X_p \setminus X_{p-1}|$.*
2. *The OS ideal I of $\mathcal{A}$ has a quadratic Gröbner basis.*

Proof. (1) follows immediately from the description of the **nbc**-sets in Theorem 6.19. In order to see (2) one observes that any subset of $\mathcal{A}$ that is not **nbc** contains at least two elements from some X_p whence it contains a set of cardinality 2 that is again not **nbc** (by Theorem 6.19). □

Corollary 6.21. *For every supersolvable arrangement the OS algebra is Koszul.*

Example 6.22. *Since the reflection arrangements for Coxeter groups of types A and B are supersolvable their OS algebras are Koszul. It is interesting to remark that for the groups of type D_n $(n \geq 4)$ the OS algebras are not even quadratic.*

It is not hard to prove that if I has a quadratic Gröbner basis with respect to some order then $\mathcal{A}$ is supersolvable (see [11]). On the other hand, no examples of non-supersolvable arrangements whose OS algebras are Koszul are known.

6.4. **1-minimal models of DGA's.** In this subsection we define minimal models of differential graded algebras and holonomy Lie algebras.

First we recall pure algebraic definitions. We suppose for the rest of this section that k is a field of characteristic 0. Let $D = \oplus_{n\geq 0} D_n$ be a connected differential graded algebra (DGA). In difference to Section 2, the differential in this section has degree 1. The DGA homomorphisms are required to be homogeneous of degree zero, multiplicative, and to commute with the differentials. Two DGA's are said to be *quasi-isomorphic* if they can be connected by a sequence of DGA maps inducing isomorphisms in cohomology.

Assuming that D is generated by D_1, the canonical (increasing) filtration of D is defined by $D(0) = k$ and $D(n)$ being the subalgebra of D generated by $D(n-1)$ and $d^{-1}(D(n-1)) \cap D_1$. Recall [9] that D is called *minimal* if $D = \wedge D_1$ (i.e., D is freely generated in degree one as a graded commutative algebra) and $\bigcup_n D(n) = D$. For every DGA A with $H^0(A) = k$ there exists a minimal DGA D as above and a DGA homomorphism $f : D \to A$ such that $f^* : H^*(D) \to H^*(A)$ is an isomorphism in dimension 1 and a monomorphism in dimension 2. The DGA D is unique up to isomorphism ([34], Theorem 12.1). D is called a *1-minimal model* of A.

For the rest of the section we denote by $A = \oplus_{n\geq 0} A_n$ a graded commutative connected algebra. We can view A as a DGA providing it with $d = 0$. The 1-minimal model of A can be realized as dual to an important Lie algebra attached to A. Let $\mathbb{L} = \mathbb{L}(A_1^*)$ be the free Lie algebra on A_1^*, graded by bracket length. Notice that $\mathbb{L}_2$ can be identified with $A_1^* \wedge A_1^*$. The multiplication of A generates a comultiplication $A_2^* \to A_1^* \wedge A_1^*$. Define the *holonomy Lie algebra* $G = G(A)$ of A as the factor of $\mathbb{L}$ by the Lie ideal generated by the image of the comultiplication. G is a graded Lie algebra and we write $G = \oplus_{n\geq 1} G_n$. We also consider the filtration of G by the Lie ideals $\Gamma_n G$ where $\Gamma_n G$ is the nth term of the lower central series of G, i.e. $\Gamma_1 G = G$ and $\Gamma_{n+1} G = [\Gamma_n G, G]$. Notice that the grading of G is closely related to the filtration, namely $\Gamma_n G = \oplus_{p\geq n} G_p$. In particular if we denote by $G(n)$ the nilpotent Lie algebra $G/\Gamma_{n+1} G$ then $G(n) \approx \oplus_{i=1}^n G_i$, as k-linear spaces.

For every n we define the DGA $D(n)$ freely generated in degree one via $D(n) = \wedge(G(n))^*$. The differential d is dual to the Lie bracket $G(n) \wedge G(n) \to G(n)$ on $(G(n))^*$ and is extended to $D(n)$ by the graded Leibniz condition. The natural projections $G(n+1) \to G(n)$ generate the embeddings $D(n) \subset D(n+1)$ of DGA's and we put $D = \bigcup_{n\geq 1} D(n)$. Notice that $D(1) \approx \wedge A_1$ as a graded algebra with differential $d = 0$ whence there is a natural DGA homomorphism $D(1) \to A$.

In order to analyze D deeper we put $V_n = G_n^*$, for every $n \geq 1$. Of course $V_1 = A_1$ and $d = 0$ on V_1. More importantly $D(n) = \wedge(\oplus_{i=1}^n V_i)$ and

$$d(V_n) \subset \sum_{i+j=n} V_i \wedge V_j, \tag{6.5}$$

for $n > 1$, where the symbol $\wedge$ applied to different spaces means the tensor product. Now extend f to a DGA homomorphism $f : D \to A$ by setting $f = 0$ on $\oplus_{n>1} V_n$.

Proposition 6.23. *([48] , Lemma 1.8 (i)) Provided with the above homomorphism f, D is a 1-minimal model of A.*

Note that there is a shift of the degrees: what we call here V_n was called V_{n-1} in [48]. This rescaling has features convenient for our purposes. Put

$$C^p(q) = \oplus_{i_1+\cdots+i_p=q} V_{i_1} \wedge \cdots \wedge V_{i_p} \ , \ C(q) = \oplus_p C^p(q) \ ,$$

and notice that $D = \oplus_q C(q)$ (as a linear space). The following easily seen properties were noticed in the particular case of arrangements in [24] , for a completely different construction of a 1-minimal model.

Proposition 6.24. *(i) The space $C(q)$ is a subcomplex of D for every q, whence $H^*(D) = \oplus_q H^*(C(q))$.*

(ii) The subspace $B = \oplus_q H^q(C(q))$ is the subalgebra of $H^*(D)$ generated by $H^1(D) = A_1$. Moreover this algebra is quadratic.

(iii) The homomorphism $f^* : H^*(D) \to A$ is 0 on $\oplus_{p \neq q} H^p(C(q))$.

(iv) If A is quadratic then $B = A$ and f^* is the projection of $H^*(D)$ to B along $\oplus_{p\neq q} H^p(C(q))$.

Proof. (i) The invariance of $C(q)$ with respect to d follows immediately from (3.1) and implies the first statement. (ii) Denote by J the ideal of $D(1) = \wedge V_1 = \oplus_q C^q(q)$ generated by $d(V_2)$. Then $d(C^{q-1}(q)) = J_q$ whence $B = \oplus_q H^q(C(q)) = \wedge V_1/J$. (iii) It follows immediately from the construction of f. (iv) If A is quadratic then $A = \wedge A_1/I$ where I is a homogeneous ideal of $\wedge A_1$ generated by I_2. It follows from the construction of D that $V_1 = A_1$ and $d(V_2) = I_2$. Thus $B = A$. Using (iii) and descending to cohomology we obtain the second statement. □

6.5. **1-minimal models and Koszul algebras.** Here we prove the main algebraic result of the section - a relation between the 1-minimal model and the property of algebras to be Koszul. We use the universal enveloping algebra U of the Lie algebra G. A more explicit description of U is as follows. Let $T = T(A_1^*)$ be the tensor algebra of the linear space A_1^* and $\tau : A_1^* \otimes A_1^* \to A_1^* \otimes A_1^*$ be the linear map generated by $v \otimes w \mapsto w \otimes v$. Then U is the factor of T by the ideal J generated by $(\mathrm{Im}(1-\tau)) \cap \mathrm{Im}\Delta$, where Δ is the comultiplication $A_2^* \to A_1^* \otimes A_1^*$. This description gives the grading of U (which is induced by the standard grading of T) and shows that U is a quadratic algebra.

Lemma 6.25. *If the algebra A is quadratic then it is the quadratic dual of U, i.e., $A = U^!$.*

Proof. If A is quadratic it is the factor of $\wedge A_1$ by the ideal generated by the kernel of the product $A_1 \wedge A_1 \to A_2$. Equivalently A is the factor of the tensor algebra of A_1 by the ideal generated by the kernel of the product $A_1 \otimes A_1 \to A_2$ and the elements $a + \tau^* a$, $a \in A_1 \otimes A_1$. The result follows since the degree 2 component of this ideal is the annihilator of the degree 2 component of J. □

Remark 6.26. Similarly one can show that in the general case $U^!$ is the factor algebra of $\wedge A_1$ by the ideal generated by the kernel of the multiplication $A_1 \wedge A_1 \to A_2$.

Now we study the Yoneda algebra $\mathrm{Ext} = \oplus_{p,q}(\mathrm{Ext}^p_U(k,k))_q$.

Proposition 6.27. *The bigraded algebra* Ext *coincides with the algebra $H^*(D) = \oplus_p H^p(D)$ where the second (pure) grading is induced on $H^p(D)$ by the grading of D by the subcomplexes $C(q)$.*

Proof. We have $\mathrm{Ext}^p = H^p(G,k) = H^p(\mathrm{Hom}(\wedge G,k))$ (see, for example, Chapter XIII of [12]). The space of n-cochains of the complex $\bar{C} = \mathrm{Hom}(\wedge G,k)$ consists of the skew symmetric multilinear functions of n variables from G with the differential δ defined by

$$(\delta f)(x_1,\dots,x_{n+1}) = \sum_{1\le i<j\le n+1} (-1)^{i+j} f([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}).$$

The functions of pure degree q are those ones that are nonzero only if $\sum_{i=1}^n \deg x_i = q$, where the initial grading on G is used. These functions form a finite dimensional subcomplex $\bar{C}(q)$ of $\bar{C}$ and $\bar{C} = \oplus_q \bar{C}(q)$. Since $\bar{C}(q)$ is finite dimensional we have

$$\bar{C}(q) = (\oplus_{i_1+i_2+\cdots+i_p=q}\, G_{i_1} \wedge G_{i_2} \wedge \cdots \wedge G_{i_p})^* = C(q)$$

as graded linear spaces. Thus one can identify $\bar{C}$ and D as linear spaces. The algebra multiplication on D induces then the standard shuffle multiplication of skew symmetric forms from $\bar{C}$. The restrictions to the algebra generators of the differentials δ and d clearly coincide, up to a sign. Besides both differentials satisfy the graded Leibnitz condition with respect to the algebra multiplication on D and the shuffle multiplication on $\bar{C}$. Thus these differentials coincide up to a sign, which implies the statement. □

Proposition 6.28. *The graded algebra homomorphism $f^* : H^*(D) \to A$ is an isomorphism if and only if A is Koszul.*

Proof. Suppose A is Koszul. Then it is quadratic and U is its quadratic dual whence U is also Koszul. Thus

$$H^p(C(q)) = (\mathrm{Ext}^p_U(k,k))_q = 0 \tag{6.6}$$

for $q \neq p$. Now Proposition 6.24 (iv) implies that f^* is an isomorphism.

Conversely if f^* is an isomorphism, then we have immediately

$$\oplus_{p\neq q} H^p(C(q)) = 0,$$

so U is Koszul. Proposition 6.24 (iii) implies that $A \approx B$ and consequently A is quadratic, by Proposition 6.24 (ii). Thus A is the quadratic dual of U whence A is also Koszul. □

Remark 6.29. Proposition 6.24 also implies that both properties of A from Proposition 6.28 are equivalent to A being quadratic and $H^*(D)$ being generated in degree 1.

6.6. **Formal topological spaces.** In this subsection we interpret the results of the previous subsection for the complement M of a complex central arrangement. The main property of M we need for that is the formality.

A space X (connected and with finite Betti numbers) is called *formal* ([34], p.158) if it has the homotopy type of a simplicial complex Δ and the DeRham DGA $A_{PL}(\Delta)$ of $\mathbb{Q}$-polynomial forms on the complex Δ is quasi-isomorphic to $A = H^*(X, \mathbb{Q})$ endowed with zero differential.

Remark 6.30. *An equivalent definition of the formality of X is that its $\mathbb{Q}_\infty$-completion $\mathbb{Q}(X)$ depends only on A (for example, see* [9] *, 12.2). An enlighting discussion of this definition can be found in* [37]*, p. 235-236.*

It follows immediately from Corollary 4.5 that the complement M of an arbitrary complex hyperplane arrangement is formal. Another class of well-known examples of formal spaces is given by compact Kähler manifolds [18].

If X is formal then a 1-minimal model of A is also a 1-minimal model of $A_{PL}(\Delta)$ and it is called a 1-minimal model of X. The holonomy Lie algebra $G = G(X)$ of A is then the $\mathbb{Q}$-version of the $\mathbb{R}$-algebra defined by Chen in [13] and it is called the *holonomy Lie algebra of X*.

Using Proposition 6.23, the holonomy algebra G of a formal space X has the following interpretation in terms of $\pi_1(X)$. Let Γ_n be the nth term of the lower central series of $\pi = \pi_1(X)$, i.e. $\Gamma_1 = \pi$ and inductively $\Gamma_n = [\Gamma_{n-1}, \pi]$. The graded abelian group $\mathrm{Gr}^*_\Gamma(\pi) = \oplus_{n\geq 1}(\Gamma_n/\Gamma_{n+1})$ has a natural graded $\mathbb{Z}$-Lie algebra structure induced by the group commutator. Then one has a graded Lie algebra isomorphism $G_* \approx \mathrm{Gr}^*_\Gamma(\pi_1(X)) \otimes \mathbb{Q}$, see, for example, [34] , Theorem 12.2. In particular we have $\dim G_n = \phi_n$, where $\phi_n = \mathrm{rank}(\Gamma_n/\Gamma_{n+1})$. Applying the Poincaré-Birkhoff-Witt theorem we obtain the following expression for the Hilbert series of the universal enveloping algebra $U = U(X)$ of G:

$$H(U,t) = \prod_{n\geq 1}(1-t^n)^{-\phi_n}. \tag{6.7}$$

Now we define the main for this section property of X.

Definition 6.31. *The space X is a rational $K[\pi, 1]$ if its 1-minimal model D is a minimal model of X, i.e., $f^* : H^*(D) \to A$ is an isomorphism.*

Remark 6.32. *An equivalent definition that justifies the term better is that the $\mathbb{Q}$-completion $\mathbb{Q}_\infty(X)$ of X is aspheric, i.e., $\pi_n \mathbb{Q}_\infty(X) = 0$ for $n > 1$ (see* [9] *, 12.8 (iii)).*

Applying Proposition 6.28 we obtain the following.

Theorem 6.33. *Suppose X is a formal topological space. Then X is a rational $K[\pi, 1]$ if and only if A is a Koszul algebra.*

Corollary 6.34. *If a formal space X is a rational $K[\pi, 1]$ then the following formula holds*

$$P(X, -t) = H(A, -t) = \prod_{n \geq 1} (1 - t^n)^{\phi_n}$$

where $P(X, t)$ is the Poincaré polynomial of X.

Proof. By Theorem 6.33 , $A = H^*(X, \mathbf{Q})$ is Koszul. Since U is the quadratic dual of A, Theorem 6.15 (iii) implies that

$$H(A, -t) = H(U, t)^{-1}.$$

Then (6.7) completes the proof. □

This formula is often called the LCS (Lower Central Series) formula. It is unknown if the LCS formula can hold for an arrangement such that M is not a rational $K[\pi, 1]$. The following result solves this problem negatively in a low dimensional case.

Proposition 6.35. *If $\ell = 3$ and the LCS formula holds for an arrangement $\mathcal{A}$ then its complement M is a rational $K[\pi, 1]$.*

Before we prove the proposition we reduce it to a perhaps non-central arrangement in a plane by deconing. Fix $H \in \mathcal{A}$ with the equation $\alpha = 0$ and denote by $\mathcal{A}'$ the deconing of $\mathcal{A}$ in the affine hyperplane H' defined by the equation $\alpha = 1$. Let M' be the complement of $\mathcal{A}'$ to $H' \cong \mathbb{C}^{\ell-1}$, $A' = H^*(M', \mathbb{Q})$ the OS algebra of $\mathcal{A}'$, and $U' = U(M')$ its holonomy algebra. Put also $\phi'_n = \operatorname{rank}(\Gamma_n \pi_1(X')/\Gamma_{n+1}\pi_1(X'))$, $n \geq 1$.

We know that M' is formal and $A \cong A'\langle e \rangle$ with e of degree 1 (see Proposition 2.18). Now we can infer the following two equivalences for $\mathcal{A}$ and $\mathcal{A}'$. Denote by $f' : (\wedge V', d') \to (A', 0)$ the 1-minimal model of A'. It follows from the Künneth formula that $f = f' \otimes id : (\wedge V' \otimes \wedge(e), d' \otimes id) \to (A, 0)$ is the 1-minimal model of A. Hence f^* is an isomorphism if and only if $(f')^*$ is an isomorphism. In other words M is a rational $K[\pi, 1]$ if and only if M' is a rational $K[\pi, 1]$. On the other hand, clearly $\phi_k = \phi'_k$, if $k > 1$, and $\phi_1 = \phi'_1 + 1$. Since $H(A, t) = H(A', t)(1 + t)$, it follows that the LCS formula holds for $\mathcal{A}$ if and only if it holds for $\mathcal{A}'$.

Denote the Poincaré polynomial of M' by $H(A',t) = 1 + mt + nt^2$. Then the LCS formula for $\mathcal{A}'$ is

$$H(U',t) = (1 - mt + nt^2)^{-1}.$$

Consequently Proposition 6.35 will follow from the simple lemma below, via Theorem 6.33.

Lemma 6.36. *Let U' be a quadratic algebra with*

$$H(U',t) = (1 - mt + nt^2)^{-1}. \tag{6.8}$$

Then U' is Koszul.

Proof of the lemma. Let I be the ideal of U'. The condition (6.8) immediately implies that $m = \dim U'_1$ and $n = \dim I_2$. The natural complex

$$0 \to U' \otimes I_2 \to U' \otimes U'_1 \to U' \to k \to 0$$

is exact except maybe at $U' \otimes I_2$ for every quadratic algebra. Since all the maps in that complex are of degree 1 we have

$$(\mathrm{Ext}^p_{U'}(k,k))_q = 0 \tag{6.9}$$

for $0 \leq p \leq 2$ and $q \neq p$. The condition (6.8) implies that the above complex is indeed exact whence it is a graded free resolution of k. Thus $\mathrm{Ext}^i_{U'}(k,k) = 0$ for $i \geq 3$, which together with (6.9) implies the result. □

Proof of the proposition. The Koszul duality formula 6.15 (iii) together with (6.8) imply that A' and $(U')^!$ have the same Hilbert series. On the other hand since A' is generated in degree one it is a quotient of $(U')^!$. Consequently $A' \approx (U')^!$ and A' is Koszul. □

In conclusion let us exhibit an example showing that the quadraticity of A is not sufficient for it being Koszul.

Example 6.37. *Consider the following arrangement of rank 3 $\{x, y, x+y, z, x-z, y-z, x+y-2z\}$. Then its OS algebra A is quadratic. One can compute that $H(A',t) = 1 + 6t + 10t^2$. If M is a rational $K[\pi,1]$ then by Corollary 6.34 and (6.7) the series $(1 - 6t + 10t^2)^{-1}$ equals $H(U',t)$ whence all its coefficients are nonnegative. This contradicts the fact that the roots of $1 - 6t + 10t^2$ are not real (cf. the proof of [1], Lemma 3.4).*

6.7. **Problems.** Prove or disprove the following two statements:

1. The OS algebra $A(\mathcal{A})$ is Koszul only if the arrangement $\mathcal{A}$ is supersolvable.

In pure algebraic terms, this statement says that in the class of OS algebras being Koszul is equivalent to having a quadratic Gröbner basis (with respect to the deg-lex order for certain ordering of generators). In general, many series of examples of Koszul algebras without quadratic Gröbner bases are known.

This problem is interesting already for rank 3 arrangements.

2. The LCS formula holds for an arrangement $\mathcal{A}$ only if $A\mathcal{A}$ is Koszul, i.e., the complement M of $\mathcal{A}$ is a rational $K[\pi, 1]$ (cf. Proposition 6.35).

Again in pure algebraic terms, this statement says that for OS algebras the numerical criterion Theorem 6.15(iii) is sufficient for the algebra being Koszul. Recall that counterexamples for more general algebras can be found in [54, 56].

7. Cohomology of OS complexes

In this section we study the OS algebra A of an arbitrary arrangement as a cochain complex under multiplication by an element $a \in A_1$. This complex first appeared in work on hypergeometric functions, mostly by Aomoto in [2, 3] and in the related paper on cohomology of local systems [23]. The algebraic study of this complex (mostly for generic a) was started in [67] where Theorem 7.5 was proved. A part of this theorem can be deduced also from the determinantal formula from [59]. The generic case results give low bounds on dimensions of cohomology for arbitrary a (see 7.3). This was discovered in [21] for a more general situation.

A general approach to the nongeneric case was started in [25] where the resonance varieties R_1^p where introduced. The conjecture by Falk that the irreducible components of R_1^1 are linear was first proved in [14] and [44] independently. This property was generalized to arbitrary R_m^p in [45] and [15]. The other properties of R_m^1 and the results of 8.5 and 8.6 are from [46].

For the most results of this section, the basic ring k has to be a field; so we assume from the beginning that it is so. Recall that in subsection 5.4 we define the cochain complex (A, a) for every $a \in E_1 = A_1$. This complex is

$$0 \to A_0 \to A_1 \to \cdots \to A_\ell \to 0$$

where the differential is multiplication by a. Recall also that $H^*(A, a) = 0$ unless $\sum_{H \in \mathcal{A}_j} a_H = 0$ for every $\mathcal{A}_j$ that is a central factor of $\mathcal{A}$ where $a = \sum_{H \in \mathcal{A}} a_H e_H$.

The main question studied in this section is the dependence of $\dim H^*(A, a)$ on a . Some topological motivation for this will be presented in section 8.

7.1. **A complex of sheaves on L^{op}.** In this subsection we represent the complex (A, a) as the complex of global sections of a complex of sheaves. The topological space on which these sheaves are defined is the poset $L^{\mathrm{op}} = L^{\mathrm{op}}(\mathcal{A})$ of all the nonempty intersections of hyperplanes from $\mathcal{A}$ ordered *by inclusion*. This poset has the unique maximal element, V, but it has a unique minimal element if and only if $\mathcal{A}$ is central. A subset of L^{op} is called *increasing* if with its each element Y it contains all elements of L^{op} larger than Y. Each $X \in L^{\mathrm{op}}$ defines the increasing set $L^X = \{Y \in L^{\mathrm{op}} | Y \geq X\}$ and the subarrangement $\mathcal{A}_X = \{H \in \mathcal{A} | X \subset H\}$ of $\mathcal{A}$. The standard (*order*) topology on $L^{\rm op}$ is given by all the increasing subsets of $L^{\rm op}$. This topology has the minimal base consisting of sets L^X for all $X \in L^{\rm op}$.

Sheaves on $L^{\rm op}$ can be identified with functors on $L^{\rm op}$ viewed as the small category whose objects are elements and morphisms are pairs $(X \le Y)$. If $\mathcal{F}$ is a contravariant functor from L to an additive category then the respective presheaf is defined on the basic open set L^X as $\mathcal{F}(X)$ with the restriction $\mathcal{F}(X) \to \mathcal{F}(Y)$ $(X \le Y)$ given by the morphism $\mathcal{F}(X \le Y)$. Then the sheafification gives for every subset $U \subset L^{\rm op}$ the object of sections on U as

$$\Gamma(U,\mathcal{F}) = \{(a_X) \in \oplus_{X\in U}\mathcal{F}(X) | \mathcal{F}(X \le Y)(a_X) = a_Y,\ X,Y \in U\}.$$

Fix p, $0 \le p \le \ell$, and for each $X \in L^{\rm op}$ put $\mathcal{F}^p(X) = A_p(\mathcal{A}_X)$. Recall from section 2.3 that $A = \oplus_{Z\in L^{\rm op}} A_Z$ where if $\dim Z = \ell - p$ then $A_Z = A_p(\mathcal{A}_Z)$ (cf. Proposition 2.7). The linear spaces A_Z serve as building blocks, namely $A_p(\mathcal{A}_X) = \oplus A_Z$ where the summation is taking over all $Z \in L$ of dimension $\ell - p$ such that $Z \ge X$ (see Proposition 2.7). Thus if $Y \ge X$ then $\mathcal{F}^p(Y)$ is a direct summand of $\mathcal{F}^p(X)$ and there is a projection $\mathcal{F}^p(X) \to \mathcal{F}^p(Y)$. These projections define a contravariant functor on L whence a sheaf $\mathcal{F}^p$ on $L^{\rm op}$. Notice that $\mathcal{F}^p$ is supported on the subposet of $L^{\rm op}$ consisting of all elements of dimension less then or equal to $\ell - p$.

Proposition 7.1. *For every p we have*

$$\Gamma(L^{\rm op},\mathcal{F}^p) = A_p.$$

Proof. Fix p. Since $A_p = \oplus_{\dim Y=\ell-p} A_Y$ and for every $X \in L^{\rm op}$ we have

$$A_p(\mathcal{A}_X) = \oplus_{\dim Y=\ell-p, Y\ge X} A_Y,$$

there is the canonical projection $\rho_X : A_p \to \mathcal{F}^p(X)$. Clearly $\rho = (\rho_X)_X$ is an injective linear map $A_p \to \Gamma(L^{\rm op},\mathcal{F}^p)$. It suffices to check that ρ is surjective. Let $(a_X)_{X\in L^{\rm op}} \in \Gamma(L^{\rm op},\mathcal{F}^p)$. Here $a_X \in A_p(\mathcal{A}_X)$ whence $a_X = (a_{X,Y})_{\dim Y=\ell-p, Y\ge X}$. For every Y from this set the restriction of a_X to Y is $a_{X,Y}$. Thus since (a_X) is a section, $a_{X,Y}$ does not depend on X (for every $X \le Y$) and can be denoted just by $a(Y)$. Now taking $a = (a(Y))_{\dim Y=\ell-p} \in A_p$ we see that $\rho(a) = (a_X)$ that completes the proof. $\square$

Proposition 7.2. *For every $X \in L^{\rm op}$ and $p < \ell - \dim X$ we have*

$$\Gamma(L^X,\mathcal{F}^p) = A_p(\mathcal{A}_X).$$

The proof is similar to the proof of Proposition 7.1.

Proposition 7.3. *For every p, the sheaf $\mathcal{F}^p$ is flasque.*

Proof. Let $\mathcal{U} \subset L^{\rm op}$ be an open (i.e., increasing) nonempty proper subset and $a = (a_X)_{X\in\mathcal{U}} \in \Gamma(\mathcal{U},\mathcal{F}^p)$. Let Z be one of the maximal elements of $L^{\rm op} \setminus \mathcal{U}$. It suffices to prove that a can be extended to Z. Consider the nonempty set $\mathcal{U}_Z = L^Z \cap \mathcal{U}$. If $\dim Z \ge \ell - p$ then the restriction b of a to $\mathcal{U}_Z$ is 0 and we put $a_Z = 0$. If $\dim Z < \ell - p$ then, due to Proposition

7.2, there exists (and unique) an element $c \in \mathcal{F}^p(Z)$ that extends b to Z. Putting $a_Z = c$ we complete the proof. $\square$

Now we fix an $a \in A_1$ and combine the sheaves $\mathcal{F}^p$ in a complex of sheaves. In order to do that, for every $X \in L^{\text{op}}$ put $a(X) = \sum a_i e_i$ where summation is taking over all indexes $i \in [n]_X$. Denote by $d_a(X)$ the left multiplication by $a(X)$. Clearly for every p, the set $(d_a(X))_{X \in L^{\text{op}}}$ is a homomorphism $d_a(p) : \mathcal{F}^p \to \mathcal{F}^{p+1}$ and these homomorphisms form the complex of sheaves

$$\mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^\ell \to 0. \tag{7.1}$$

Moreover $\mathcal{E} = \ker(d_a) \subset \mathcal{F}^0$ is the sheaf whose only nonzero stalk is at V and it equals to k. Thus we can augment the complex (7.1) on the left and obtain the complex

$$0 \to \mathcal{E} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \mathcal{F}^\ell \to 0. \tag{7.2}$$

The following lemma is almost tautological.

Lemma 7.4. *The complex* (7.2) *is exact (and thus a flasque resolution of* $\mathcal{E}$*) if and only if the complex* $(A(\mathcal{A}_X), d_a(X))$ *is acyclic for every* $X \in L^{\text{op}} \setminus \{V\}$.

7.2. **Cohomology for a general position a.**

Theorem 7.5. *(i) Let* $\mathcal{A} = \{H_1, \ldots, H_n\}$ *be an arbitrary arrangement with the intersection poset* L *of rank* ℓ. *Suppose* $a = \sum_{i=1}^n a_i e_i \in A_1$ *satisfies the following condition*

$$\sum_{X \subset H_i} a_i \neq 0 \tag{7.3}$$

for all $X \in L^{\text{op}}$ *such that the arrangement* $\mathcal{A}_X$ *is irreducible. Then* $H^p(A, a) = 0$ *for every* $p < \ell$.

(ii) Let $\mathcal{A}$ *and* L *be as in (i) but* $\mathcal{A}$ *is central. Let* a *satisfy* (7.3) *for* $X \in L^{\text{op}}$ *not equal to the minimal element but* $\sum_{i=1}^n a_i = 0$. *Then* $H^p(A, a) = 0$ *for every* $p < \ell - 1$ *and* $\dim H^{\ell-1}(A, a) = \dim H^\ell(A, a) = \bar{\chi}(\mathcal{A})$ *(recall that* $\bar{\chi}(\mathcal{A})$ *is the beta-invariant of* $\mathcal{A}$ *equal* $(H(A,t)/(1+t))_{|t=-1}$*).*

Proof. First we prove (ii) assuming that (i) holds for arrangements of smaller ranks, the case of rank 1 being trivial. Denote by $\mathcal{A}'$ the deconing of $\mathcal{A}$ with respect to H_1. Also denote by A' the OS algebra of $\mathcal{A}'$ and by L' the intersection poset of $\mathcal{A}'$. Recall that $H(A,t) = (1+t)H(A',t)$. Put $a' = \sum_{i=2}^n a_i e_i$.

Since a satisfies (7.3) for every $X \in L^{\text{op}}$ that is not minimal we have that a' satisfies (7.3) for every $X \in (L')^{\text{op}}$. Thus by the induction hypothesis

$$H^p(A', a') = 0,\ p \neq \ell - 1. \tag{7.4}$$

Since $\bar{\chi}(\mathcal{A})$ is the Euler characteristic of the complex (A', a') we have

$$\dim H^{\ell-1}(A', a') = |\bar{\chi}(\mathcal{A})|. \tag{7.5}$$

Since $a_1 = -\sum_{i=2}^n a_i$ we can use the exact sequence 5.8 that in the present notation is

$$0 \to H^{p-1}(A', a') \to H^p(A, a) \to H^p(A', a') \to 0. \tag{7.6}$$

The result follows from (7.4)-(7.6).

Now we prove (i) assuming that (ii) holds for arrangements of the same and smaller ranks. Consider the complex (7.1) of flasque sheaves on L^{op} and fix $X \in L^{\mathrm{op}} \setminus \{V\}$. Since $\mathcal{A}_X$ is a central arrangement in dimension not greater than ℓ we can use (ii). If (7.3) holds for X then $(A(\mathcal{A}_X), a(X))$ is acyclic by Corollary 5.16. If (7.3) does not hold for X then $\mathcal{A}_X$ is a product of at least two central arrangements. This implies that $1 + t$ divides $H(A', t)$ whence $\bar{\chi}(A(\mathcal{A}_X)) = 0$. Thus (ii) implies that $(A(\mathcal{A}_X), a(X))$ is again acyclic. In both cases Lemma 7.4 implies that (7.2) is a flasque resolution of $\mathcal{E}$. Using also Proposition 7.1 we have

$$H^p(A, a) \cong H^p(L^{\mathrm{op}}, \mathcal{E}) \tag{7.7}$$

for every p.

Computation of $H^p(L^{\mathrm{op}}, \mathcal{E})$ is a routine combinatorial problem. First one can use the exact sequence of sheaves

$$0 \to \mathcal{E} \to \tilde{k} \to \tilde{k}_0 \to 0$$

where $\tilde{k}$ is the locally constant sheaf with the stalks equal k and $\tilde{k}_0$ is its restriction to $L_0 = L^{\mathrm{op}} \setminus \{V\}$. Since $\tilde{k}$ is flasque, the respective cohomology sequence give $H^p(L^{\mathrm{op}}, \mathcal{E}) = \tilde{H}^{p-1}(L_0, \tilde{k}_0)$ for every p. Now since L^{op} is a geometric semilattice, we have $H^p(L^{\mathrm{op}}, \mathcal{E}) = 0$ for $p < \ell - 1$ (see subsection 2.1). This completes the proof. □

Remark 7.6. *The conditions of Theorem 7.5 are not necessary for the vanishing of $H^p = H^p(A, a)$ for $p < \ell$. If the complex* (7.2) *is not exact then it defines cohomology sheaves $\mathcal{H}^p$ (not all vanishing) and the spectral sequence with $E_2^{p,q} = H^p(L^{\mathrm{op}}, \mathcal{H}^q)$ that converges to H^{p+q}. In principal, one can use this spectral sequence, to compute $H^*(A, a)$ in particular cases. We give an example of this in Proposition 7.8.*

The important for applications particular case is given by the class of general position arrangements. In this case, all $\mathcal{A}_X$ for $\mathrm{rk} X > 1$ are reducible and the condition of Theorem 7.5 gives $a_i \neq 0$ for all i. This condition can be significantly relaxed.

Proposition 7.7. *Let $\mathcal{A}$ be a general position arrangement and $a \neq 0$. Then $H^p(A, a) = 0$ for every $p < \ell$.*

Proof. Recall from Example 5.12 that the OS ideal I is the $(\ell + 1)$st power of the maximal ideal $\mathbf{m}$ of E. In particular I is invariant with respect to the natural action of the group $\mathrm{GL}(n, k)$ on E. Thus this group acts on A and the action is transitive on the set $A_1 \setminus \{0\}$.

Now fix $a \in A_1 \setminus \{0\}$ and put $b = \sum_{i=1}^n e_i$. There exists an isomorphism ψ of A sending a to b. Consequently ψ generates an isomorphism $(A, a) \to (A, b)$. According to Theorem 7.5, we have $H^p(A, b) = 0$ for every $p < \ell$. Hence the same is true for (A, a) which completes the proof. $\square$

The other particular case we consider is the case of an arbitrary arrangement but $a = e_1$.

Proposition 7.8. *For any integer p, $H^p(A, e_1) = 0$ if and only if there is no $X \in L$ of dimension $\ell - p$ such that $X \cap H_1 = \emptyset$.*

Proof. Put $L_1 = \{X \in L^{\mathrm{op}} | X \subset H_1\}$ and notice that for $a = e_1$ we have $a(X) = e_1$ if $X \in L_1$ and $a(X) = 0$ otherwise. This and Proposition 5.16 imply that for every q the cohomology sheaf $\mathcal{H}^q$ is 0 on L_1 and, on $L^{\mathrm{op}} \setminus L_1$, coincides with the restriction of $\mathcal{F}^q$ to this set. This allows us to consider the short exact sequence

$$0 \to \mathcal{G}^q \to \mathcal{F}^q \to \mathcal{H}^q \to 0 \tag{7.8}$$

where the sheaf $\mathcal{G}^q$ is 0 on $L^{\mathrm{op}} \setminus L_1$ and coincides with the restriction $\mathcal{F}^q_{L_1}$ of $\mathcal{F}^q$ to L_1. Since $\mathcal{F}^p$ is flasque, $\mathcal{F}^p_{L_1}$ is flasque also (for example, see Proposition 1.6 of [67]). Since L_1 is closed, $H^p(L, \mathcal{G}^q) = H^p(L_1, \mathcal{F}^q_{L_1}) = 0$ for all $p > 0$. Then the cohomology sequence of the sequence (7.8) implies that $H^p(L^{\mathrm{op}}, \mathcal{H}^q) = 0$ for every $p > 0$ whence $E_2^{p,q} = 0$ for every q and $p > 0$. This implies that $H^p(A, e_1) = E_2^{0,p} = \Gamma(L^{\mathrm{op}}, \mathcal{H}^p)$.

Now suppose that there exists a non-zero $s \in \Gamma(L^{\mathrm{op}}, \mathcal{H}^p)$. By definition of $\mathcal{F}^p$, this means that there exists $X \in L$ of dimension $\ell - p$ such that $s(X) \neq 0$. Since besides $s = 0$ on L_1 we have $X \cap H_1 = \emptyset$.

Conversely, suppose that there exists $X \in L$ of dimension $\ell - p$ and $X \cap H_1 = \emptyset$. Choose $a \in A_X \setminus \{0\}$. For every $Y \in L, Y \leq X$, $a \in A_p(\mathcal{A}_Y) = \oplus_{\dim Z = \ell - p, Z \geq Y} A_Z$ since X is among those Z. Then for every $Y \in L^{\mathrm{op}}$ put $s(Y) = a(Y)$ if $Y \leq X$ and $s(Y) = 0$ otherwise. One easily checks that $(s(Y))_{Y \in L} \in \Gamma(L^{\mathrm{op}}, \mathcal{H}^p)$. Since this element is non-zero the proof is complete. $\square$

That proposition could be proved also by using the **nbc**-basis instead of sheaves.

It is instructive and useful for applications to make the isomorphism (7.7) explicit. We do it for the inverse isomorphism which makes formulas nicer.

We are working under assumptions and notation of Theorem 7.5. In particular (7.2) is a flasque resolution of $\mathcal{E}$. Put $\mathcal{E}^0 = \mathcal{E}$ and denote by $\mathcal{E}^p$ the subsheaf of $\mathcal{F}^p$ that is the kernel of the differential of the resolution (i.e., the p^{th} syzygy of the sequence). Then for each p the sequence

$$0 \to \mathcal{E}^p \to \mathcal{F}^p \to \cdots \to \mathcal{F}^\ell \to 0$$

is a flasque resolution of $\mathcal{E}^p$. Thus $H^i(L^{\mathrm{op}}, \mathcal{E}^p) = 0$ for $i \neq \ell - p - 1$ and

$$H^{\ell-p-1}(L^{\mathrm{op}}, \mathcal{E}^p) \cong H^{\ell-1}(L^{\mathrm{op}}, \mathcal{E}). \tag{7.9}$$

The isomorphism (7.9) is the composition of the connecting homomorphisms of the cohomology sequences corresponding to the short exact sequences

$$0 \to \mathcal{E}^i \to \mathcal{F}^i \to \mathcal{E}^{i+1} \to 0. \tag{7.10}$$

The only nontrivial connecting homomorphism for this sequence is

$$\delta : H^{\ell-i-2}(L^{\mathrm{op}}, \mathcal{E}^{i+1}) \cong H^{\ell-i-1}(L^{\mathrm{op}}, \mathcal{E}^i)$$

and by definition we can write symbolically $\delta = d \circ d_a^{-1}$. Here $d_a : \mathcal{F}^i \to \mathcal{F}^{i+1}$ is multiplication by the appropriate restrictions $a(X)$ of a and d is the standard differential $d : C^{\ell-i-2} \to C^{\ell-i-1}$ where by C^* we denote the standard cochain complex on flags of L^{op} with coefficients in $\mathcal{F}^i$. This gives $\delta^{-1} = d_a \circ d^{-1}$.

To describe δ^{-1} explicitly fix a flag $F = (X_1 < X_2 < \cdots < X_{\ell-i} = X)$ such that $\dim X_j = j$ and $b \in A_i(\mathcal{A}_X) = A_i(X)$. This gives us the element $\beta \in C^{\ell-i-1}$ having the only nonzero value $\beta(F) = b$. Then we choose $\beta' \in C^{\ell-i-2}$ such that $d\beta' = \beta$. The simplest choice for β' is defined as follows. Put $F' = (X_1 < \cdots < X_{\ell-i-1} = X')$. Since $A_i(\mathcal{A}_{X'})$ has a direct summand A_X we can view b as an element of $A_i(\mathcal{A}_{X'})$. Then we define β' as having the only nonzero value $\beta'(F') = (-1)^{\ell-i}b$. It is a straightforward check that $d\beta' = \beta$ so we can write $\delta^{-1}(\beta) = a(X)\beta'$.

Now we can come back to the isomorphism $\phi : \tilde{H}^{\ell-1}(L^{\mathrm{op}}, \mathcal{E}) \to H^\ell(A, a)$. Notice that every maximal flag $F = (X_1 < X_2 < \cdots < X_\ell = V)$ of L^{op} defines the cocycle F^* of dimension $\ell - 1$ with value in $\mathcal{E}$ whose value on F is 1 and it vanishes on all the other flags. On the other hand, every element of A_ℓ is a cocycle for multiplication by a. We obtain the formula for ϕ in terms of these cocycles by taking the composition of the inverse connecting homomorphisms for all the short exact sequences (7.10):

$$\phi(F^*) = (-1)^{\ell(\ell+1)/2} a(X_1)a(X_2)\cdots a(X_\ell) \in A_\ell. \tag{7.11}$$

Formula (7.11) can be used to exhibit cocycles forming a basis for $H^\ell(A, a)$. For that one should just choose a basis of $H^\ell(L^{\mathrm{op}}, \mathcal{E})$. If a linear order on $\mathcal{A}$ is fixed than the latter can be defined naturally using a natural basis for homology of a shellable complex. We will just give the answer leaving details to the reader (see [69] and [30]).

Let $S \in \mathrm{nbc}_\ell$. It is called $\beta(\mathbf{nbc})$ if for every $i \in S$ there exits $i' \in [n]$ such that $i' < i$ and $(S \setminus \{i\}) \cup \{i'\}$ is still independent. On the other hand, each subset $S = \{i_1, i_2, \ldots, i_\ell\}$ $(i_1 < i_2 \cdots < i_\ell)$ generates the maximal flag $F(S) = (X_1 < X_2 < \cdots < X_\ell = V)$ of L^{op} where $X_p = \bigcap_{j=\ell-p+1}^{\ell} H_{i_j}$.

Theorem 7.9. *The cocycles* $\{(F(S))^* | S \text{ is } \beta\mathbf{nbc}\}$ *form a basis of* $H^{\ell-1}(L^{\mathrm{op}}, \mathcal{E})$.

Corollary 7.10. *The formula* (7.11) *for F running through $F(S)$ with S* β**nbc** *gives a basis for* $H^\ell(A, a)$.

Notice that the above basis depends on a. For some classes of arrangements one can find a basis of $H^\ell(A,a)$ that does not depend on a (satisfying the conditions of Theorem 7.5). For instance it is true for arrangements in general position to infinity [67] and the ones whose union of hyperplanes forms a normal crossing divisor [65].

7.3. **The highest nonzero cohomology.** Previous results for a general position $a \in \operatorname{sing}(A)$ allows us to get certain low bounds on the dimension of cohomology $H^p(A,a)$ for all a and large p.

First we need to be able to move an arbitrary a in general position in $\operatorname{sing}(A)$. Recall from Theorem 5.15 and Corollary 5.16 that $\operatorname{sing}(A)$ is A_1 if $\mathcal{A}$ does not have any central factors and it is $\Delta = \{a \in A_1 | \sum_i a_i = 0\}$ if $\mathcal{A}$ is central and irreducible.

Lemma 7.11. *Assume that the field k is infinite.*

(i) Suppose $\mathcal{A}$ does not have central factors. Then the set of all a from A_1 satisfying the condition (1) of Theorem 7.5 is open and dense in A_1 in Zariski topology.

(ii) Suppose $\mathcal{A}$ is central and irreducible. Then the set of all $a \in \Delta$ satisfying the condition (ii) of Theorem 7.5 is open and dense in Δ in Zariski topology.

Both statements follow immediately from the fact that the conditions of Theorem 7.5 describe the complements of several hyperplanes in A_1 and Δ respectively. □

Now represent an arrangement $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_r \times \mathcal{B}$$

where $\mathcal{A}_i$ $(i = 1, \ldots, r)$ is an irreducible central arrangement and $\mathcal{B}$ is an arrangement without central factors. Denote by χ_i the beta-invariant $\bar{\chi}(\mathcal{A}_i)$ (see Theorem 7.5) and put $\chi_{r+1} = H(A(\mathcal{B}), -1)$ where $H(A(\mathcal{B}), t)$ is the Hilbert series of $A(\mathcal{B})$.

Theorem 7.12. *Suppose that $a \in \operatorname{sing}(A)$ (i.e., $H^*(A,a) \neq 0$). Then for every p such that $\ell - r \leq p \leq \ell$ we have*

$$\dim H^p(A,a) \geq \prod_{i=1}^{r+1} |\chi_i|.$$

Proof. Recall from Proposition 5.13 that we can identify A_1 with $\oplus_{i=1}^{r+1} A_1(i)$ where $A(i) = A(\mathcal{A}_i)$ for $i = 1, 2, \ldots, r$ and $A(r+1) = A(\mathcal{B})$. Then according to Corollary 5.16, the projection $a(i) = \sum_{H_j \in \mathcal{A}_i} a_j e_j$ of a to every $A_1(i)$ with $i = 1, 2, \ldots, r$ lies in the linear subspace Δ_i defined by the equation

$\sum_{H_j\in\mathcal{A}_i} a_j = 0$. If besides every $a(i)$ $(i = 1,\dots,r)$ satisfies the condition (ii) of Theorem 7.5 then according to that theorem

$$\dim H^{\ell_i}(A(i), a(i)) = \dim H^{\ell_i - 1}(A(i), a(i)) = |\chi_i|.$$

Denote by $a(r+1)$ the projection of a to $A_1(r+1)$. If $a(r+1)$ satisfies the condition (i) of that theorem then

$$\dim H^{\ell_{r+1}}(A_1(r+1), a(r+1)) = |\chi_{r+1}|.$$

Now denote by C the subset of $\mathrm{sing}(A) = \Delta_1 \oplus \cdots \oplus \Delta_r \oplus A_1(r+1)$ consisting of all a such that the following inequalities hold

$$\dim H^{\ell_i}(A(i), a(i)) \geq |\chi_i|,$$
$$\dim H^{\ell_i - 1}(A(i), a(i)) \geq |\chi_i|,$$
$$\dim H^{\ell_{r+1}}(A_1(r+1), a(r+1)) \geq |\chi_{r+1}|.$$

Since C is closed in Zariski topology and contains a set dense in $\mathrm{sing}(A)$, we have $C = \mathrm{sing}(A)$. In other words the above inequalities hold for every $a \in \mathrm{sing}(A)$. Now an application of Proposition 5.13 and the Künneth formula completes the proof. $\square$

As it shown in [21] similar low bounds can be obtained in much more general situation of an E-module with a linear resolution.

7.4. **Linear algebra of a collection of sets.** In order to be able to prove deeper results about the cohomology of (A, a), we need to study a linear algebra set up where a theorem of Vinberg applies. Here $\mathcal{S}$ is a finite set and $\mathcal{X}$ is a non-empty collection of its subsets such that each two elements of $\mathcal{X}$ have at most one common element. We will always assume that both $\mathcal{S}$ and $\mathcal{X}$ are linearly ordered. Let $J = J(\mathcal{X})$ be the incidence matrix of $\mathcal{X}$, i.e., the $|\mathcal{X}| \times |\mathcal{S}|$-matrix with entries $a_{X,i} = 1$ if $i \in X$ and $a_{X,i} = 0$ otherwise. Also let $\mathcal{E}$ be the $|\mathcal{S}| \times |\mathcal{S}|$-matrix whose every entry is 1. Our goal is to analyze the matrix

$$Q = Q(\mathcal{X}) = J^t J - \mathcal{E}.$$

For each matrix R over $\mathbb{Q}$ denote by $N(R)$ the null space of R (consisting of rational column vectors). Notice that $N(\mathcal{E}) = \{u \in \mathbb{Q}^{\mathcal{S}} \,|\, \sum_{i\in\mathcal{S}} u_i = 0\}$. Then for any subspace $N \subset \mathbb{Q}^{\mathcal{S}}$ put $N^0 = N \cap N(\mathcal{E})$. The following observation is immediate.

Lemma 7.13. $N(Q)^0 = N(J)^0$.

Clearly Q is symmetric and its every entry off the diagonal is either -1 or 0. The decomposition of Q in the direct sum of its indecomposable submatrices defines a partition Π of $\mathcal{S}$. This partition can be defined in terms of $\mathcal{X}$ as follows. Let Γ be the graph on $\mathcal{S}$ whose edges are the pairs

$\{i, j\}$ that are not in any $X \in \mathcal{X}$. Then Π is the partition of Γ into its connected components.

We have $Q = \oplus_{P\in\Pi} Q_P$ where each Q_P is an integer matrix satisfying the same conditions as Q and besides being indecomposable. Thus one can apply the Vinberg classification (for example, see [39], pp. 48-49) to Q_P.

Let us recall Vinberg's result. Suppose $R = (a_{ij})$ is a $m \times m$ indecomposable matrix over $\mathbb{Q}$ with two extra properties: (a) $a_{ij} \leq 0$ for $i \neq j$ and (b) $a_{ij} = 0$ implies $a_{ji} = 0$. We call a rational column vector $u^t = (u_1, \dots, u_m)$ positive and write $u > 0$ if all $u_i > 0$ ($u < 0$ if $-u > 0$).

Then one and only one of the following three possibilities holds for R:

(i) R is of *finite type*, i.e., it is positive definite (equivalently for every $u > 0$ we have $Ru > 0$);

(ii) R is of *affine type*, i.e., it is positive semidefinite of rank $m - 1$ and its null space is spanned by a positive vector;

(iii) R is of *indefinite type*, i.e., there exists a positive vector u such that $Ru < 0$.

Applying that classification to Q_P we will call it simply finite, affine or indefinite if it is of the respective type.

The following remark will be useful. Suppose $\mathcal{X}$ does not cover $\mathcal{S}$. Then at least one row of Q consists of -1 whence Q is irreducible and indefinite. If $\mathcal{X}$ covers $\mathcal{S}$ then all the diagonal elements of Q are nonnegative.

It turns out that there are further restrictions on Q_P.

Proposition 7.14. *For matrix $Q(\mathcal{X})$ there are precisely two possibilities:*

(i) For every $P \in \Pi$, Q_P is finite or affine;

(ii) For a unique $P_0 \in \Pi$, Q_{P_0} is indefinite and Q_P is finite for every other $P \in \Pi$.

Proof. It suffices to show that there cannot be distinct $P_1, P_2 \in \Pi$ such that $Q_1 = Q_{P_1}$ is indefinite and $Q_2 = Q_{P_2}$ is either indefinite or affine. Suppose to the contrary that Q_i $(i = 1, 2)$ are such. Then there exist column vectors $u^i > 0$ in $\mathbb{Q}^{P_i}$ such that $Q_1 u^1 < 0$ and $Q_2 u^2 < 0$ or $Q_2 u^2 = 0$. We can also assume that $\sum_{j\in P_1} (u^1)_j = \sum_{j\in P_2} (u^2)_j$. Define $u \in \mathbb{Q}^{\mathcal{S}}$ by $u_j = (u^1)_j$ if $j \in P_1$, $u_j = -(u^2)_j$ if $j \in P_2$ and $u_j = 0$ if $j \in \mathcal{S} \setminus (P_1 \cup P_2)$. Notice that $\sum_{j\in\mathcal{S}} u_j = 0$ whence $\mathcal{E}u = 0$. Then we have using (,) for the standard dot product

$$0 \leq (Ju, Ju) = (Qu, u) + (\mathcal{E}u, u)$$
$$= (Q_1 u^1, u^1) + (Q_2 u^2, u^2) \leq (Q_1 u^1, u^1) < 0$$

which is a contradiction. □

Definition 7.15. *The collection $\mathcal{X}$ is called either affine or indefinite if respectively the possibility (i) or (ii) from Proposition 7.14 realizes for $Q(\mathcal{X})$. In the former case, we denote by $\Pi_1(\mathcal{X})$ the set of elements $P \in \Pi(\mathcal{X})$ such* *that Q_P is affine. In the latter case, $P_0 \in \Pi(\mathcal{X})$ denotes the unique element such that Q_{P_0} is indefinite.*

Corollary 7.16. *Put $N = N(\mathcal{X}) = N(Q(\mathcal{X}))$.*

(i) If $\mathcal{X}$ is affine then $\dim N = |\Pi_1(\mathcal{X})|$ and there is a basis of N consisting of vectors u^P ($P \in \Pi_1(\mathcal{X})$) such that the restriction of u^P to P is positive and its restriction to the other elements of $\Pi(\mathcal{X})$ is 0. In particular, $\dim N^0 = |\Pi_1(\mathcal{X})| - 1$.

(ii) If $\mathcal{X}$ is indefinite then all the vectors from N are 0 on every $P \in \Pi(\mathcal{X}) \setminus \{P_0\}$.

In the latter case of Corollary 7.16, $\Pi(\mathcal{X})$ does not define $\dim N$.

We will also use the following observation.

Proposition 7.17. *If $\mathcal{X}$ is affine then for every $X \in \mathcal{X}$ we have $X \cap P \neq \emptyset$ either for all $P \in \Pi_1(\mathcal{X})$ or for none of them.*

Proof. Suppose to the contrary that there exists $X \in \mathcal{X}$ and $P' \in \Pi_1(\mathcal{X})$ such that $X \cap P' = \emptyset$ but $X \cap P \neq \emptyset$ for $P \in \bar{\Pi} \subset \Pi_1(\mathcal{X})$ with a non-empty $\bar{\Pi}$.

First notice that $|\bar{\Pi}| > 1$. Indeed if $\bar{\Pi} = \{P\}$ then for any $u \in \mathbb{Q}^{\mathcal{S}}$ such that $\sum_{i \in X} u_i = 0$ its restriction to P would be 0. But there exists $v \in N(\mathcal{X})^0$ non-zero on P and P' which contradicts Lemma 7.13. Thus there exist distinct $P_1, P_2 \in \bar{\Pi}$.

Now compare $N = N(\mathcal{X})^0$ and $N' = N(\mathcal{X}')^0$ where $\mathcal{X}' = \mathcal{X} \setminus \{X\}$. On the one hand, since one equation is deleted we have $N \subset N'$. On the other hand, $\Pi(\mathcal{X}')$ can be obtained from $\Pi(\mathcal{X})$ by gluing together the elements intersecting with X. Since P' is untouched, $Q_{P'}$ is unchanged whence $\mathcal{X}'$ is affine. Since P_1 and P_2 get glued $|\Pi_1(\mathcal{X}')| < |\Pi_1(\mathcal{X})|$ whence by Corollary 7.16 $\dim N' < \dim N$. The contradiction completes the proof. □

Remark 7.18. *Let k be an arbitrary field of 0 characteristic. Then the kernel in k^m of any $m \times m$-matrix over $\mathbb{Q}$ is generated by its kernel in $\mathbb{Q}^m$. This allows us to generalize all the notions and results of this subsection to k. The only difference is that when we use the inequalities $u > 0$ or $u < 0$ we apply them only to rational vectors from k^m.*

7.5. **Resonance varieties R^1_m.**

First we define natural subvarieties of $\operatorname{sing}(A)$ reflecting the cohomology dimension.

Definition 7.19. *Let p and m be positive integers such that $p \leq \ell$. Then the resonance variety R^p_m is defined as*

$$R^p_m = \{a \in A_1 | \dim H^p(A, a) \geq m\}.$$

It is clear that R^p_m is really a subvariety of A_1 since it can be described by the vanishing of certain minors of matrices whose entries are linear forms of the coordinates a_i of a. The most interesting of these varieties are $R^p = R^p_1$ and the most studied are R^1_m. The results of Theorem 7.12 can be of course formulated in terms of the resonance varieties R^p_m with large p but we leave this to the reader.

The main goal of this subsection is to study R^1_p (especially for the field k of characteristic 0) using the techniques developed in the previous subsection.

We consider an arbitrary central essential arrangement $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ with the rank of its intersection lattice L denoted by ℓ. Recall that for every $X \in L$ we have the subarrangement $\mathcal{A}_X = \{H \in \mathcal{A} | H \supset X\}$ and denote by $[n]_X$ the set of indexes of elements from $\mathcal{A}_X$.

In order to simplify the notation, we apply the following convention in the rest of the paper. **We identify X with $[n]_X$, i.e., consider X as an element of L and at the same time the set of indexes of H_i that contain X.** In order not to confuse the reader even more, we keep the symbol $\mathcal{A}_X$ for the arrangement of these H_i. We also denote by $L(p)$ the subset of L of all its elements of rank p.

We put $L'(2) = \{X \in L(2) || X| > 2\}$. Notice that $L'(2)$ is the set of all $X \in L$ of rank 2 with irreducible $\mathcal{A}_X$. For every two distinct $i, j \in \bar{n}$ there exists a unique $X_{ij} \in L$ of rank 2 such that $i, j \in X_{ij}$.

In order to study $H^1(A, a)$ for $a = \sum_{i=1}^n a_i e_i \in A_1$ put

$$Z^p(a) = \{x \in A_p | ax = 0\}$$

and notice that $H^1(A, a) = Z^1(a)/ka$.

We start with the following three simple lemmas.

Lemma 7.20. *Let $\ell = 2$. Then $x = \sum_{i=1}^n x_i e_i \in Z^1(a)$ if and only if*

$$a_i \sum_{j=1}^n x_j = (\sum_{j=1}^n a_j) x_i, \text{ for every } i = 1, \dots, n.$$

Therefore

(i) if $a \neq 0$ and $\sum_{i=1}^n a_i = 0$ then $Z^1(a)$ can be described by the equation

$$\sum_{i=1}^n x_i = 0;$$

(ii) if $\sum_{i=1}^n a_i \neq 0$ then $Z^1(a) = ka$;

(iii) if $a = 0$ then $Z^1(a) = A_1$.

If $n = 2$ then the conclusions in cases (i) and (ii) coincide.

The proof of this lemma is a straightforward computation.

The next lemma can be stated for arbitrary p, $1 \leq p \leq \ell$. Recall that for every $x = \sum_{i=1}^n x_i e_i \in A_1$ and every $X \in L$ we put $x(X) = \sum_{i \in X} x_i e_i$.

Lemma 7.21. *We have* $x \in Z^p(a)$ *if and only if* $x(X) \in Z^p(a(X)) \subset A_p(\mathcal{A}_X)$ *for every* $X \in L$ *of rank* $p+1$.

Proof. Recall from Proposition 2.7 that for every p

$$A_p = \oplus_{X \in L(p)} A_p(\mathcal{A}_X) \tag{7.12}$$

and $A(\mathcal{A}_X)$ is a subalgebra of A for every $X \in L$. Thus the lemma follows from the fact that $ax = 0$ if and only if the projections of ax to all the direct summands of (7.12) are 0. □

Combining the two previous lemmas we obtain the following.

Lemma 7.22. *The subspace* $Z^1(a)$ *of* A_1 *can be described by the following system:*

for every $X \in L'(2)$ *such that* $a(X) \neq 0$ *and* $\sum_{i \in X} a_i = 0$,

$$\sum_{i \in X} x_i = 0; \tag{7.13}$$

for every other $X \in L(2)$ *and every pair* $i < j$ *from* X,

$$a_i x_j - a_j x_i = 0. \tag{7.14}$$

Since we are interested only in $a \in \operatorname{sing}(A)$, we restrict our attention to $a \in A_1^0$. Since the algebra A is skew commutative $b \in Z^1(a)$ if and only if $a \in Z^1(b)$. Thus Corollary 5.16 implies that $Z^1(a)^0 = Z^1(a)$ for every $a \in A_1^0 \setminus \{0\}$.

Now we introduce the main objects of the section. For any $a \in A_1^0$ define the subset of $L'(2)$ via

$$\mathcal{X}'(a) = \{X \in L'(2) | \sum_{i \in X} a_i = 0,\ a(X) \neq 0\} \tag{7.15}$$

and the subset $[n](a)$ of $[n]$ via

$$[n](a) = \bigcup_{X \in \mathcal{X}'(a)} X \setminus \mathcal{O}(a)$$

where

$$\mathcal{O}(a) = \{j | a_j = 0 \text{ and } X_{ij} \notin \mathcal{X}'(a) \text{ for some } i \text{ with } a_i \neq 0\}.$$

Notice that if $\ell = 3$, i.e., the projectivization of $\mathcal{A}$ is a line arrangement in the projective plane, then $\mathcal{X}'(a)$ is a set of points and $\mathcal{A}(a)$ is the subarrangement corresponding to $[n](a)$ of certain lines passing through any of the points from $\mathcal{X}'(a)$.

We will need to use the restriction of $\mathcal{X}'(a)$ to $[n](a)$. This new collection $\mathcal{X}(a)$ consists of $X \cap [n](a)$ for all $X \in \mathcal{X}'(a)$. We apply the notation from the beginning of the section to $\mathcal{X}'(a)$ and $\mathcal{X}(a)$. Most importantly, $N(\mathcal{X}(a))$ denotes the subspace of A_1 of all elements that are 0 outside of $[n](a)$ and whose restrictions to $[n](a)$ lie in the null space (over k) of $Q = Q(\mathcal{X}(a))$.

The following theorem is the main technical result of this section. **In it and for the rest of this paper we assume that the characteristic of k is 0.**

Theorem 7.23. *Let $a \in A_1^0 \setminus \{0\}$. If a is not 0 on $[n] \setminus [n](a)$ or $\mathcal{X}(a)$ is indefinite then $a \notin R^1$. If a is 0 on $[n]\setminus[n](a)$ and $\mathcal{X}(a)$ is affine then $Z^1(a) = N(\mathcal{X}(a))^0$. In particular, in the latter case* $\dim Z^1(a) = |\Pi_1(\mathcal{X}(a))| - 1$.

Proof. We break the proof in several parts. We will often omit a from the notation if it is not ambiguous.

(i) **Reduction to the case where a vanishes on the complement to $[n](a)$.**

Suppose that for some $j \in [n] \setminus [n](a)$ we have $a_j \neq 0$. Then $j \notin \mathcal{O}(a)$ whence $j \notin \overline{[n]} = \bigcup_{X\in\mathcal{X}'(a)} X$. This means that for every $i \in [n]$ we have $X_{ij} \notin \mathcal{X}'(a)$ whence (7.14) holds for every $x \in Z^1(a)$. Putting $c = x_j/a_j$ we obtain from (7.14) that $x_i = ca_i$ for every $i \in [n]$ whence $x = ca$. We conclude that $\dim Z^1(a) = 1$ and $a \notin R^1$. From now on we can assume that $a_j = 0$ for every $j \notin [n](a)$.

(ii) **Every element $x \in Z^1(a)$ vanishes on the complement to $[n](a)$.**

Let $x \in Z^1(a)$ and $j \notin [n](a)$. As above the latter means that either $j \notin \overline{[n]}$ or $j \in \mathcal{O}$. In either case there exists $i \in [n]$ with $a_i \neq 0$ and $X_{ij} \notin \mathcal{X}'(a)$. Thus (7.14) holds and implies $x_j = 0$. Thus x can be nonzero only on $[n](a)$. From now on we consider only vectors x that are 0 outside of $[n](a)$.

(iii) **The inclusion $Z^1(a) \subset N(\mathcal{X}(a))^0$.**

By definition of $\mathcal{X}(a)$ and Lemma 7.22, (7.13) holds for every $X \in \mathcal{X}(a)$ (as well as $X \in \mathcal{X}'(a)$) whence $J(\mathcal{X}(a))x = 0$. Since besides $\sum_{i=1}^n x_i = 0$ we have $Q(\mathcal{X}(a))x = 0$, i.e., $x \in N(\mathcal{X}(a))^0$. Thus $Z^1(a) \subset N(\mathcal{X}(a))^0$.

(iv) **The indefinite case.**

Suppose that $\mathcal{X}(a)$ is indefinite and $x \in Z^1(a)$. Since $x \in N(\mathcal{X}(a))$ by (iii), the restriction of x to every element P of $\Pi = \Pi(\mathcal{X}(a))$ lies in the null space of the matrix $Q_P = Q(\mathcal{X}(a))_P$. Since for $P \neq P_0$ this matrix is nondegenerate we have x vanishes on every $P \neq P_0$.

Now we consider the restriction of x to P_0. For every pair $i, j \in P_0$ we have $X_{ij} \notin \mathcal{X}(a)$ by definition of Π. Thus (7.14) holds for those i and j. Furthermore since by our agreement $i, j \notin \mathcal{O}$ we conclude that either both a_i and a_j vanish or neither of them does. Since $a \neq 0$ it cannot vanish on P_0. Now (7.13) implies $x = ca$ on P_0 for some $c \in k$ (cf. (i)). Summing up we see that $x = ca$ whence $\dim Z^1(a) = 1$, i.e., $a \notin R^1$.

(v) **The affine case.**

Suppose $\mathcal{X}$ is affine and $x \in N(\mathcal{X})^0$, i.e., $Q(\mathcal{X})x = 0$ and $\mathcal{E}x = 0$. By Lemma 7.13 we have $J(\mathcal{X})x = 0$, i.e., (7.13) holds for $X \in \mathcal{X}$. Moreover for every $i, j \in [n](a)$ such that $X_{ij} \notin \mathcal{X}$ we have $i, j \in P$ for some $P \in \Pi$. If Q_P is finite then $x_i = x_j = 0$ and (7.14) holds tautologically. If Q_P is affine then the kernel of Q_P is one-dimensional and (7.14) holds again. By Lemma 7.22, $x \in Z^1(a)$ which completes the proof. □

For the sake of clarity let us state the qualitative part of the previous technical result.

Corollary 7.24. *For every $a \in A_1^0$ there exists the subarrangement $\mathcal{A}(a)$ (whose set of indexes contains the support of a) and the collection $\mathcal{X}(a)$ of codimension 2 intersections of hyperplanes from $\mathcal{A}(a)$ that define whether $a \in R^1$ or not. Namely $a \in R^1$ if and only if $\mathcal{X}(a)$ is affine with at least 3 affine components. If this criterion holds then $Z(a) = N(\mathcal{X}(a))^0$, i.e., $Z(a)$ is defined by the system of linear equations*

$$(1)\ x_i = 0,\ \text{if } i \notin [n](a);\ (2)\ \sum_{j \in X} x_j = 0,\ \text{for } X \in \mathcal{X}(a);\ (3)\ \sum_i x_i = 0.$$

Notice that the only information required about a to decide if $a \in R^1$ and, if it does, to compute $Z(a)$ is the set of $X \in L$ of rank 1 and 2 such that $\sum_{i \in X} a_i = 0$.

If we change a we immediately find the following symmetry.

Corollary 7.25. *For every $a \in R^1 \setminus \{0\}$ and $b \in Z^1(a) \setminus \{0\}$ we have $Z^1(b) = Z^1(a)$.*

Proof. Let $b \in Z^1(a)$ and $b \neq 0$. We use the notation introduced before Theorem 7.23. Due to this theorem, $b = 0$ outside of $[n](a)$ and lies in $N(\mathcal{X}(a))^0$ (and in $N(\mathcal{X}'(a))^0$). Also $\mathcal{X}(a)$ is affine. Thus for every $P \in \Pi(\mathcal{X}(a))$ either $b_i \neq 0$ for every $i \in P$ or $b = 0$ on P. The collection of P with the former condition is nonempty, let P' be one of them.

Take $X' \in \mathcal{X}'(a)$ and put $X = X' \cap [n](a)$. Notice that $X \neq \emptyset$ since $a(X') \neq 0$ and $X \in \mathcal{X}(a)$. Since $b \in N(\mathcal{X}'(a))$ we have $\sum_{i \in X'} b_i = 0$. Besides by Proposition 7.17, we have $X \cap P' \neq \emptyset$ whence $b(X) \neq 0$ and $b(X') \neq 0$. This implies that $X' \in \mathcal{X}'(b)$ and $\mathcal{X}'(a) \subset \mathcal{X}'(b)$. By symmetry (since $a \in Z^1(b)$), we have $\mathcal{X}'(a) = \mathcal{X}'(b)$. This and the previous paragraph also imply that $[n](a) \cap \mathcal{O}(b) = \emptyset$ whence $[n](a) \subset [n](b)$. Again by symmetry $[n](a) = [n](b)$ and the result follows. □

Now we are ready to answer the question what the irreducible components of the variety R_1 are.

Corollary 7.26. *The variety R^1 is the union of the finite number of linear spaces $Z(a) = N(\mathcal{X}(a))^0$ that pairwise intersect at 0. Thus these spaces are precisely the irreducible components of R_1.*

Proof. It follows immediately from Theorem 7.23 and Corollary 7.25. □

For convenience of the reader we collect the properties of R^1 that follow from Theorem 7.23 and the above corollaries.

Corollary 7.27. *(i) All the irreducible components of R_1 are linear subspaces of $\Delta = \{\sum_i x_i e_i \in A_1 | \sum_i x_i = 0\}$.*

(ii) Every irreducible component of R^1 can be described in Δ by the sytem of equations $\sum_{i \in X} x_i = 0$ where X is running through some set of elements of L of ranks 1 and 2.

(iii) Pairwise intersection of the components is 0.

(iv) For every m the variety R^1_m is the union of the components of R^1 whose dimensions are greater than m.

Corollary 7.26 gives a description of the irreducible components of R_1 in terms of elements of A_1^0. In order to deal with concrete examples it is convenient for a fixed arrangement $\mathcal{A}$ to have a description of subsets $\mathcal{X}$ of rank two elements of $L = L(\mathcal{A})$ such that $N(\mathcal{X})^0$ is a component of R_1 for $\mathcal{A}$.

Proposition 7.28. *Let $\mathcal{X} \subset L(2)'$ satisfy the following conditions: (i) $\mathcal{X}$ is affine with at least three affine components, (ii) no $X \in \mathcal{X}$ lies in $\bigcup_{P \in \Pi \setminus \Pi_1} P$ where $\Pi = \Pi(\mathcal{X})$. Then $N(\mathcal{X})^0$ is an irreducible component of R_1.*

Proof. Suppose $\mathcal{X}$ satisfies the conditions (i)-(ii). Put $N = N(\mathcal{X})^0$ and fix $a \in N \setminus \{0\}$. If $X \in \mathcal{X}$ then $\sum_{i \in X} a_i = 0$. Besides since $a \neq 0$ and $\mathcal{X}$ satisfies (ii), Proposition 7.17 implies that $a(X) \neq 0$. Thus $X \in \mathcal{X}'(a)$ and $\mathcal{X} \subset \mathcal{X}'(a)$. Moreover since $\mathcal{X}$ is affine, it covers $\mathcal{A}$ and for any $X_{ij} \notin \mathcal{X}$ either both a_i and a_j equal 0 or both do not. This implies that $\mathcal{O}(a) = \emptyset$ whence $[n](a) = [n]$ and $\mathcal{X}(a) = \mathcal{X}'(a)$. Thus $\mathcal{X} \subset \mathcal{X}(a)$.

On the other hand, let $X \in \mathcal{X}(a)$. Suppose $X \notin \mathcal{X}$. Then X lies in a component P of Π and since $a(X) \neq 0$ this component is affine. This implies in turn that $a(X)$ is proportional to a positive rational vector whence $\sum_{i \in X} a_i \neq 0$. This contradiction implies $X \in \mathcal{X}(a)$ and $\mathcal{X} = \mathcal{X}(a)$. Now the result follows from Corollary 7.26. □

There are other irreducible components of R_1 for $\mathcal{A}$ that are not covered by Proposition 7.28, the components coming from subarrangements of $\mathcal{A}$. For instance, all the local components come from proper subarrangements unless $\mathrm{rk} L = 2$. On the other hand, not all components for a subarrangement are components of $\mathcal{A}$. For instance, if $\mathrm{rk} L = 2$ then the unique component of R_1 comes from the whole $\mathcal{A}$.

It is possible to describe the pairs $(\mathcal{B}, \mathcal{X}')$ that define a component for $\mathcal{A}$. Here $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{X}' \subset L'(2)$ is such that its restriction $\mathcal{X}$ to $\mathcal{B}$ satisfies the conditions of Proposition 7.28 for B. We can simplify the description introducing *pure affine* coverings. We call a covering $\mathcal{X}$ of a finite set pure affine if the matrix $Q(\mathcal{X})$ is affine with no finite components. Notice that if we require, in Proposition 7.28, $\mathcal{X}$ to be pure affine then the condition (ii) holds automatically.

We also say that a collection $\mathcal{X} \in L$ covers a subarrangement $\mathcal{B}$ if it covers the set of indexes of $\mathcal{B}$.

Proposition 7.29. *Let $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{X}' \subset L'(2)$ such that its restriction $\mathcal{X}$ to $\mathcal{B}$ is pure affine with at least three components. Then $N(\mathcal{X})^0$ is a component of R_1 if and only if $\mathcal{B}$ is the inclusion maximum subarrangement of $\mathcal{A}$ to which the restriction of $\mathcal{X}'$ is pure affine. Moreover the map ψ : $(\mathcal{B}, \mathcal{X}') \mapsto N(\mathcal{X})^0$ defines an one-to-one correspondence between the pairs with the above property and the irreducible components of R_1.*

Proof. (i) Take a nonzero $a \in R_1$ and consider the irreducible component containing a. By Theorem 7.23, it is equal to $N(\mathcal{X}(a))^0$ where $\mathcal{X}(a)$ is a covering of a subarrangement $\mathcal{A}(a)$. Let $\mathcal{B}(a)$ be the subarrangement of $\mathcal{A}(a)$ that corresponds to the union of the affine components of this covering and keep the symbol $\mathcal{X}(a)$ for the restriction of this covering to $\mathcal{B}(a)$. Then $\mathcal{X}(a)$ is a pure affine covering with at least three components.

If the restriction $\mathcal{X}_1$ of $\mathcal{X}'(a)$ to some $\mathcal{B}_1 \supset \mathcal{B}(a)$ is pure affine then $Z(a) \supset N(\mathcal{X}_1)^0 \supset N(\mathcal{X}(a))^0$ whence these sets coincide. In particular, for every $H_i \in \mathcal{B}_1 \setminus \mathcal{B}(a)$, there exists an element of $N(\mathcal{X}(a))^0$ not vanishing on i. On the other hand, either i is in a finite component or $i \in \mathcal{O}(a)$ whence every element of $N(\mathcal{X}(a))^0$ vanishes on i. This contradiction shows that $\mathcal{B}(a)$ satisfies the condition of the proposition whence ψ is surjective.

(ii) Let $(\mathcal{B}, \mathcal{X}')$ satisfy the condition of the proposition. Fix an $a \in N(\mathcal{X})^0$, $a \neq 0$ and compare $(\mathcal{B}, \mathcal{X}')$ with $(\mathcal{B}(a), \mathcal{X}(a)')$. Take $X \in \mathcal{X}'$. Since $a(X) \neq 0$ and $\sum_{i \in X} a_i = 0$, we have $X \in \mathcal{X}'(a)$. Thus $\mathcal{X}' \subset \mathcal{X}'(a)$.

On the other hand, take $X \in \mathcal{X}'(a)$ and put $\Pi = \Pi(\mathcal{X})$ (a partition of $\mathcal{B}$). Using the same properties of a for this X, there exist at least two distinct $P_1, P_2 \in \Pi$ such that $P_m \cap X \neq \emptyset$ $(m = 1, 2)$. Thus taking an index from each of those intersections, we can find $X_{ij} \in \mathcal{X}'$ and $X_{ij} = X$. This gives $\mathcal{X}' = \mathcal{X}'(a)$.

Now it is easy to check that $\mathcal{B} \subset \mathcal{B}(a)$ and to deduce from the maximality of $\mathcal{B}$ that $\mathcal{B} = \mathcal{B}(a)$. This implies that

$$N(\mathcal{X})^0 = N(\mathcal{X}(a))^0 \tag{7.16}$$

whence it is an irreducible component of R_1.

(iii) The injectivity of ψ follows immediately from (7.16). This completes the proof. □

Another way to define the irreducible components of R^1 is by axiomatizing properties of the partitions $\Pi_1(\mathcal{X}(a))$. These properties which define so called *neighborly* partitions can be found in [25].

7.6. **Examples.** In this subsection we give some examples of irreducible components of R^1 and of notions developed in the previous section.

Local components. Fix an $X \in L'(2)$ and put $\mathcal{X} = \{X\}$, $\mathcal{B} = \mathcal{A}_X$. Then $Q(\mathcal{X})$ is 0-matrix, $\Pi(\mathcal{X})$ is the partition of $\mathcal{B}$ in single elements, and $N(\mathcal{X})^0$ is the subspace of $A_1(\mathcal{A}_X)$ of vectors with the sum of coordinates equal 0. The conditions of Proposition 7.29 are clearly satisfied whence

$N(\mathcal{X})^0$ is an irreducible component of R^1_j for $j \le m-2$, where $m = |\mathcal{A}_X|$. Such components are called *local.*

In order to get some idea about nonlocal components of R^1 we start with affine matrices.

Affine indecomposable matrices. Recall that for every covering $\mathcal{X}$ of a set $[n]$, the respective matrix $Q = Q(\mathcal{X}) = (q_{ij})$ is a symmetric $n \times n$-matrix with $m_i = q_{ii} \in \mathbb{Z}_{\ge 0}$ and $q_{ij} = -1$ or 0 for $i \ne j$. Such matrices Q can be parametrized by a pair (Γ, m) where Γ is a graph on n vertices $v_i, \dots, v_n$ with edges $\{v_i, v_j\}$ for $q_{ij} = -1$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\ge 0}$ is a vector labeling the vertices of Γ. We write $Q = Q(m, \Gamma)$. The decomposition of Q into the direct sum of indecomposable components corresponds to the partition of Γ into connected components.

We start with the following two general observations.

Proposition 7.30. *Let Γ be an arbitrary (finite) connected graph. Then the set $\mathcal{M}(\Gamma)$ of vectors $m \in \mathbb{Z}^n_{\ge 0}$ for which $Q(m, \Gamma)$ is affine, is finite.*

Proof. Suppose that Γ has n vertices. For each $m \in \mathcal{M}(\Gamma)$ denote by Q_r the principal submatrix of $Q = Q(m, \Gamma)$, such that the involved rows have r smallest m_i, and by D_r the determinant of Q_r. Without loss of generality assume that $m_1 \le m_2 \le \cdots \le m_n$. Lemma 4.4 from [39] implies that for $r < n$, matrix Q_r is a direct sum of finite matrices whence $D_r > 0$. Thus for fixed $r < n$ and Q_r, the determinant D_n can be viewed as a polynomial in $m_{r+1}, \dots, m_n$ with a positive coefficient D_r of the highest degree term $m_{r+1} \cdots m_n$. Since $D_n = 0$ the possible value of m_{r+1} is bounded from above whence m_{r+1} can afford only finite set of values. The result follows by induction on r. $\square$

Proposition 7.31. *Let Γ be a connected graph on n vertices with a set E of edges. Let $\mathcal{M}'$ be the set of all positive integer column vectors u ($u^t = (u_1, \dots, u_n)$) such that u_i are mutually relatively prime and each u_i divides $u^{(i)} = \sum_{\{i,j\} \in E} u_j$. Define $m = f(u)$ via $m_i = u^{(i)}/u_i$. Then (i) $m \in \mathcal{M}(\Gamma)$, (ii) u lies in the null space of $Q(m, \Gamma)$, and (iii) $f : \mathcal{M}' \to \mathcal{M}(\Gamma)$ is a bijection.*

Proof. Fix $u \in \mathcal{M}'$ and consider the matrix $Q = Q(f(u), \Gamma)$. Clearly $Qu = 0$ whence Q is affine (see [39], Corollary 4.3) which proves (i) and (ii). Now let Q be an arbitrary integer affine matrix. Then its null space contains a unique positive integer vector with mutually relatively prime coordinates which implies (iii). $\square$

Examples of affine indecomposable matrices include some well-known classes.

1. *Laplace matrices of graphs.* Here G is an arbitrary connected graph on vertices $v_1, \dots, v_n$ and Q is its Laplace matrix, i.e., $Q = Q(d, \Gamma)$ where d_i is the degree of v_i. Clearly d satisfies the formula for m from Proposition 7.31 with $u_i = 1$ for all i.

2. *Affine generalized Cartan matrices.* Here Γ is an extended Dynkin diagram with single bonds and $m_i = 2$ for all i. Notice that for the diagrams of type other than $A_\ell^{(1)}$ the labeling is different from that in the previous example.

3. *Full graphs.* Let Γ be the full graph on n vertices. Then it is not hard to describe all the labelings m so that $Q = Q(m,\Gamma)$ is affine.

Proposition 7.32. *For the full graph Γ, the matrix $Q = Q(m,\Gamma)$ is affine if and only if*

$$\sum_{i=1}^{n} \frac{1}{m_i+1} = 1. \tag{7.17}$$

Proof. According to Proposition 7.31, Q is affine if and only if the following system has a positive integer solution:

$$(m_i+1)x_i = \sum_{j=1}^{n} x_j \ ,1 \le i \le n. \tag{7.18}$$

Clearly (7.17) is equivalent to this. □

Notice that the Laplace matrix of the full graph has $m_i = n-1$ for all i.

4. *Bushes.* Let Γ be the bush on $n+1$ vertices. By that we mean that Γ is a rooted tree whose root v_0 has degree n and all the other vertices $v_1, \dots, v_n$ (changing enumeration slightly) are leaves. An argument similar to that of Proposition 7.32 shows that Q is affine if and only if

$$m_0 = \sum_{i=1}^{n} \frac{1}{m_i}. \tag{7.19}$$

If (7.19) holds then the null space of Q is generated by the vector $u^t = (1, 1/m_1, \dots, 1/m_n)$. As a concrete example, the complete list of vectors m for the extended Dynkin diagram $D_4^{(1)}$ consists of the following 20 vectors (up to permutations of leaves): (4,1,1,1,1) (the Laplace matrix), (3,2,2,1,1), (2,2,2,2,2) (the Cartan matrix), (2,3,3,3,1), (2,6,3,2,1), (2,4,4,2,1), (1,4,4,4,4), (1,3,4,4,6), (1,12,4,3,3), (1,6,6,3,3), (1,12,12,3,2), (1,15,10,3,2), (1,18,9,3,2), (1,24,8,3,2), (1,42,7,3,2), (1,8,8,4,2), (1,12,6,4,2), (1,20,5,4,2), (1,10,5,5,2), (1,6,6,6,2).

Realizable affine matrices. Now recall that the affine matrices we considered in the previous section have been constructed from a 0-1 matrices J. More precisely

$$Q = J^t J - \mathcal{E}. \tag{7.20}$$

Moreover J does not have 0-rows or columns and any two columns both have entries 1 in at most one common row. If (7.20) holds for a matrix J with the above properties then we say that Q is *realizable* by J.

Not all affine matrices are realizable.

Example 7.33. *Let Q be the Laplace matrix of the graph Γ on vertices $\{v_1, \dots, v_5\}$ which is the full graph on $\{v_2, \dots, v_5\}$ extended by the single edge $\{1, 5\}$. We want to observe that Q is not realizable. Moreover consider the indecomposable component S (on rows 1 to 4) of $Q + \mathcal{E}$:*

$$S = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$$

Suppose $S = J^t J$ for a 0-1 matrix J. Then the first column of J has precisely two 1's in some rows and each of the other three columns has 1 in one of these rows. Thus some two of these three columns have 1 in common row which is a contradiction. Therefore S and Q are not realizable.

If an affine matrix is realizable then it may have several different (even after permutations of rows in J) realizations. Clearly the number of realizations is finite. For instance, the Cartan matrix of type $D_4^{(1)}$ has 3 different realizations. For certain ordering on the vertices these realizations are

$$J_i \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where

$$J_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, J_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Realizations with at least three components. More interesting problem for us is to realize decomposable matrices with at least 3 affine components. For a component P, we denote the restrictions of Q and J to it respectively by Q_P and J_P. The set of the components is denoted by Π.

This class turns out to be more restrictive. While studying this class we sometimes apply the dual point of view to a matrix J realizing Q. Each column C_i of it can be considered as a subset of the set R of all rows of J. We always assume that there are no 0-rows in J_P whence the system $\mathcal{C}(P) = \{C_i\}_{i \in P}$ is a covering of R for each $P \in \Pi$. If $P' \in \Pi$ and $P' \neq P$ then C_j for every $j \in P'$ is a *transversal* of $\mathcal{C}(P)$. Recall that it means $C_j \cap C_i \neq \emptyset$ for all $i \in P$ whence $|C_i \cap C_j| = 1$. This immediately implies the inequalities

$$|\bar{P}| \leq |C_j| \leq |P| \tag{7.21}$$

where $j \in P'$ and $\bar{P}$ is a subset of P such that $C_i \cap C_r = \emptyset$ for every $i, r \in \bar{P}$.

The inequalities (7.21) allow us to classify some series of realizable matrices with at least 3 (even two) affine components.

Theorem 7.34. *Suppose $Q = Q(m, \Gamma)$ is a realizable matrix whose graph Γ has at least two connected components that are full graphs with at least two vertices each. Suppose the matrices corresponding to the components are affine. Then there exists an integer $n \geq 2$ such that all the diagonal elements of Q are $n-1$, $p = |\Pi| \leq n+1$, and $|P| = n$ for each $P \in \Pi$. More precisely the realizations of Q are parametrized (up to ordering) by all $(n-1) \times (p-2)$ matrices of n-permutations $\{\sigma(i,j)\}$ $(1 \leq i \leq n-1,\ 1 \leq j \leq p-2)$ with the following properties:*

(i) every $\sigma(i,j)$ is free, i.e., does not have fixed points;

(ii) every two permutations standing in one row or column are disjoint, i.e., the images of any element are distinct under these permutations.

Proof. First notice that for any matrix corresponding to a full graph, all pairs $\{C_i, C_j\}$ in any realization J are disjoint. Then (7.21) implies that $|C_i| = |P|$ for each i and each $P \in \Pi$. Put $n = |P|$. Since each Q_P is at least 2×2-matrix we have $n \geq 2$. Clearly all the diagonal elements of Q are $n-1$.

Now order all the elements of Π as $P_0, P_1, \dots, P_{p-1}$. Ordering the elements inside P_s $(s = 0, 1)$ and the rows of J appropriately we can assume without any loss of generality that (i) $C_i = \{n(i-1)+1, \dots, ni\}$ for $i \in P_0$, i.e., $1 \leq i \leq n$, (ii) $C_j = \{j-n, j, \dots, j+(n-2)n\}$ for $j \in P_1$, i.e., $n+1 \leq j \leq 2n$, and (iii) for every $r > 0$ the first n rows of $J_r = J_{P_r}$ form the identity matrix. Now break all the rows of J into blocks of n subsequent rows from $jn+1$ to $(j+1)n$ $(j = 0, 1, \dots, n-1)$. Notice that the conditions on the columns in J_0 and J_1 say that their blocks are fixed: all the blocks of J_1 equal the identity $(n \times n)$ matrix and the ith block of J_0 has the ith column consisting of entries 1 and others 0. Thus we can focus our attention on the blocks of J_r for $r > 1$.

Since the first block in each J_r equals to the identity matrix, every other block defines an n-permutation. We obtain an $(n-1) \times (p-2)$ matrix of permutations. Clearly the conditions on J are equivalent to the conditions (i) and (ii) of the theorem. All unproved yet assertions follow. □

Realization by arrangements. It is still a different question if there exists an arrangement over a given field for which the matrix Q as in Theorem 7.34 gives a component of R^1. Intersecting an arbitrary arrangement $\mathcal{A}$ with a 3-dimensional subspace in general position to $\mathcal{A}$ and projectivizing we can reduce the question to arrangements of lines in a projective plane.

We can exhibit such arrangements over $\mathbb{C}$ realizing matrices J from Theorem 7.32 for $n = 2$ and $n = 3$. For $n = 2$ it is the Coxeter arrangement of type A_3 given by $\{x, y, z, x-y, x-z, y-z\}$ and for $n=3$ the Hessian arrangement of 12 lines containing 9 flexes of a non-singular cubic (for example, given by $\{x, y, z, x+\zeta^i y+\zeta^j z\}$ where $\zeta = \exp(2\pi\sqrt{-1}/3)$ and i, j run independently through 0,1, and 2). In the next section, we will prove that for $n > 3$ the matrices J do not correspond to arrangements over $\mathbb{C}$.

The arrangement of 9 lines projectively dual to the Hessian arrangement, that is also called the Ceva(3) arrangement (or a representation of the plane of order 3) gives a different example. All the 12 points of this arrangement do not define any component of the resonance variety R^1 since all components of the respective matrix Q are finite. Any subset of 9 points, such that the left out 3 points have all 9 lines passing through them, defines a nonlocal component of R^1 of the type discussed in Theorem 7.34 with $n = p = 3$.

Let us remark about Theorem 7.34 that it completely covers the case of arrangements of lines with only triple multiple points and with R^1 having nonlocal components. Indeed if any row of matrix J has exactly three 1's then Q has at most three components. Thus the number of components is exactly three and each one of them corresponds to a full graph. It follows from the above theorem that there are $3n$ lines (for some n) partitioned in 3 groups of equal size n such that the lines in each group are in general position and each multiple point has a line from each group passing through it. In the next section, we will show that these arrangements (equivalently their duals) do not exist for $n > 5$.

We obtain another series of representable matrices Q using the affine matrices corresponding to bushes. Let Γ be the graph of three connected components, each one being the bush on $r+1$ vertices. Let the restriction of the labeling vector m to each connected component be $(1, r, \ldots, r)$ where 1 labels the root of the bush. Then $Q(m, \Gamma)$ has three affine components and is realizable. As J one can take the incidence matrix of all lines and all points of the complex arrangement of lines of type $\mathcal{A}_{r,1,3}$ given by $\{x, y, z, x^r - y^r, x^r - z^r, y^r - z^r\}$ (see [14], Remark 6.3.).

Cartan matrices. In the rest of this subsection we classify all the realizable matrices with several affine components such that at least three of the components are Cartan matrices. Our argument is straightforward: given an indecomposable matrix we consider all realizations of it and list all possibilities for the columns of other indecomposable components for each realization. Going through this list we can see what other components can be realized jointly with the initial matrix. In order to reduce amount of case by case inspections the following simple observation is useful.

Lemma 7.35. *Let Γ be a graph on vertices $\{v_1, \ldots, v_n\}$, $m = (m_1, \ldots, m_n)$ a labeling of its vertices, and $Q = Q(m, \Gamma)$. Let Π be the partition of the set of vertices of Γ into connected components. Fix $P \in \Pi$ and $v_i \in P$. Suppose J is a realization of Q and J' is obtained from J by deleting the* ith *column and all the rows whose restrictions to $P \setminus \{v_i\}$ are 0. Then J' is a realization* *of* $Q' = Q(m', \Gamma')$ *where* Γ' *is the subgraph of* Γ *on* $\{v_1, \ldots, \hat{v}_i, \ldots, v_n\}$ *and* $m'_j \leq m_j$ *for all* $j \neq i$ *with equality on* $P \setminus \{v_i\}$.

Since we consider Cartan matrices, the labels m have $m_i = 2$ for all i. Notice that in terms of arrangements of lines, this means that we consider only systems of multiple points such that any line passing through them contains precisely 3 of them. Under this condition, the indecomposable affine matrices are Cartan matrices of extended Dynkin diagrams with singular edges, i.e., the types $A_p^{(1)}$ $(p \geq 2)$, $D_p^{(1)}$ $(p \geq 4)$ and $E_p^{(1)}$ $(p = 6, 7, 8)$. Notice that all these graphs except $A_2^{(1)}$ contain $\Gamma_0 = A_3$ as a subgrpah. Thus the most important graph to analyze is Γ_0.

The following facts about Γ_0 can be obtained by simple inspection.

(i) The Cartan matrix of Γ_0 has the unique (up to ordering) realization by

$$J_0 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) There are precisely two maximal sets of columns transversal to the collection of columns of J_0 such that every two of them intersect at no more than one element. Written as matrices they are

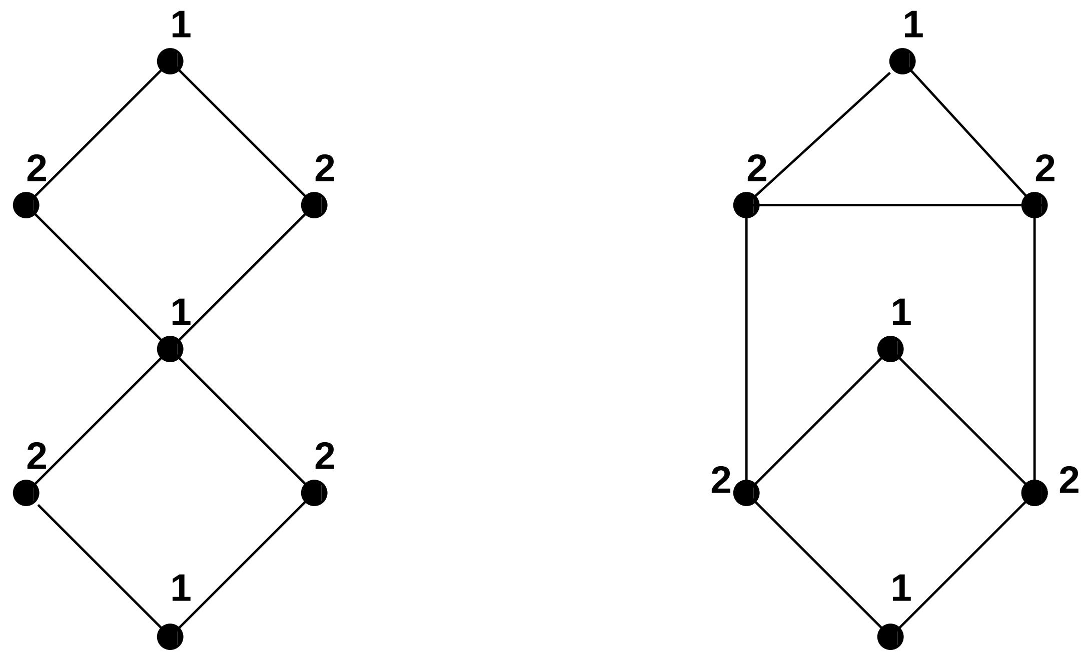
$$T_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The respective symmetric matrices $R_i = T_i^t T_i - \mathcal{E}$ $(i = 1, 2)$ are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

These matrices correspond to the following labeled graphs Γ_1 and Γ_2.



Γ_1 Γ_2

(iii) Now suppose that a graph Γ has (at least three) extended Dynkin diagrams with single edges as its connected components and at least one of them, say Γ', contains A_3 as a subgraph (i.e., this component is not $A_2^{(1)}$). Suppose the respective Cartan matrix $Q = Q(m, \Gamma)$ $(m = (2, \dots, 2))$ is realizable by J. Denote by J' the submatrix of J that consists of the columns of J corresponding to vertices of Γ' and by J'' the matrix of the other columns. Then Lemma 7.35 implies that J'' can be obtained from either of J_1 and J_2 by adding three new rows that form the identity block with the first three rows of J_i and have only 0 in the other positions, and then deleting some columns (maybe none). The addition of three new rows does not change Γ_i, just makes the labeling $(2, 2, \dots, 2)$. Thus one should be able to delete some vertices from either Γ_1 of Γ_2 and obtain a graph having at least two extended Dynkin diagrams as its connected components. It is impossible (by inspection) and we get a contradiction. Thus Q cannot be realized.

(iv) The only remaining case is where each component of Q is the Cartan matrix of $A_2^{(1)}$. This case is covered by Theorem 7.34. Moreover since the permutation group on $\{1, 2, 3\}$ has only two elements without a fixed point there is the unique realization of Q (up to ordering). The maximum number of components Q can have is 4 and if it has 4 affine components it is realized by the matrix J_C

$$J_C = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If it has three affine components then it is realized by either 9 or 10 or 11 first columns of J_C.

Recall that matrix J_C was realized by the collection of all the multiple points of the Hessian arrangement. Moreover this realization is projectively unique over $\mathbb{C}$ (cf. [68], Example 4.2). Thus we obtain the following result:

Theorem 7.36. *Let $\mathcal{A}$ be a projective arrangement of lines with each line containing exactly 3 multiple points. Let the set of all multiple points define a component of R^1. Then $\mathcal{A}$ consists of 9 to 12 lines. If it has 12 lines it is the Hessian arrangement.*

Similarly it is possible to study the case where $m_i \leq 2$ for every i. The only extra possibility here is given by three components each being the affine matrix corresponding to the bush with 2 leaves.

7.7. **Problems.**

While quite a bit is proved about R^1_m, a very little is known about R^p_m with $p > 1$. A particular interesting problem is to describe the irreducible components of R^p_m by explicit equations (recall, it is known that they are linear subspaces of A_1). We conjecture that these equations can be chosen of the type

$$\sum_{H_i \supset X} x_i = 0$$

for certain $X \in L$ and $x = \sum_i x_i e_i \in A_1$. The determinantal formula from [58] suggests that this may be true but it is proved only for $p = 1$ (see Corollary 7.24) and for very few cases besides.

Another approach to the case $p > 1$ would be to study the whole ideal $Z(a)$ of A annihilating $a \in A_1$. For instance, it would be interesting to describe the case where $Z(a)$ is generated in degree 1.

8. Cohomology of local systems on the complement

Historically, relations of $H^*(A,a)$ with the cohomology of local systems on the hyperplane complement M gave the first motivation for studying the former. In this section, we do not give an in any way complete survey of the topic in the title. For that we direct the reader to [51, 28, 47] and references there. We discuss only some results that can be applied to $H^*(A,a)$ and in particular to proving stronger versions of certain results of the previous section. The results of 8.3 are from [46].

8.1. **Cohomology of generic local systems and $H^*(A,a)$.** In this section all arrangements are over $\mathbb{C}$ and projectivized. In other words, $\mathcal{A} = \{H_1,\dots,H_n\}$ is a set of projective hyperplanes in $\mathbb{P}^{\ell-1}$. As before $M = \mathbb{P}^{\ell-1} \setminus \bigcap_i H_i$. If A is the OS algebra (over $\mathbb{C}$) of the corresponding central arrangement in $\mathbb{C}^\ell$ then the algebra $H^*(M,\mathbb{C})$ can be identified with the graded subalgebra $\bar{A}$ of A generated by $\bar{A}_1 = \Delta = \{x = \sum_i x_ie_i | \sum_i x_i = 0\}$.

If $a \in \bar{A}_1$ then it defines a local system $\mathcal{L}(a)$ on M associated with the one-dimensional representation of $\pi_1(M)$ sending its generator corresponding to H_i to $\exp(2\pi i a_i)$. In other words, the 1-form $\omega_a = \sum_i a_i d\alpha_i/\alpha_i$ on M defines the integrable connection $\nabla_a = d + \omega_a$ on the trivial vector bundle of rank 1 (here d is the DeRham differential). The system $\mathcal{L}(a)$ is formed by the flat sections with respect to this connection. The cohomology $H^*(M,\mathcal{L}(a))$ can be defined as the cohomology of the complex

$$\Omega = (0 \to \Omega^* \to \Omega^1 \to \cdots \to \Omega^{\ell-1} \to 0)$$

which is the global deRham complex on M provided with the twisted differential $d + \omega_a\wedge$. Corollary 4.5 gives the embedding of the complexes $(\bar{A},a) \subset \Omega$ that sends $e_i - e_j$ to $d\alpha_i/\alpha_i - d\alpha_j/\alpha_j$.

Let us record simple but important properties of cohomology that follow immediately from definitions.

Lemma 8.1. *For every $a \in \bar{A}_1$*

(i) $H^*(\bar{A},a) = H^*(\bar{A},\lambda a)$ for every $\lambda \in \mathbb{C}^*$;

(ii) $H^*(M,\mathcal{L}(a)) = H^*(M,\mathcal{L}(a+z))$ for every $z = (z_i) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n z_i = 0$.

The following result relates $H^*(M,\mathcal{L}(a))$ to $H^*(\bar{A},a)$ when a is sufficiently generic. The proof of this result is well known and can be found either in the original papers [23, 57] or in the book [51].

Theorem 8.2. *Suppose that $a \in \bar{A}_1$ and for all $X \in L$ such that $\mathcal{A}_X$ is indecomposable, the sum $\sum_{i\in X} a_i$ is not a positive integer. Then the embedding $(\bar{A},a) \subset \Omega$ is a quasiisomorphism. In particular,*

$$\dim H^p(M,\mathcal{L}(a)) = \dim H^p(\bar{A},a)$$

for every p.

The idea of the proof is to blow up $\mathbb{P}^{\ell-1}$ along the subspaces X and then apply the quasiisomorphism theorem from [17].

Now from Theorem 8.2 and Lemma 8.1 we obtain the following.

Proposition 8.3. *For every $a \in \bar{A}_1$ and every p*

$$\dim H^p(M, \mathcal{L}(a)) \geq \dim H^p(\bar{A}, a).$$

Proof. Let $a \in \bar{A}_1$. Fix a real $\epsilon > 0$ such that for every $X \in L$ and $0 < t < \epsilon$ the number $(1-t)\sum_{i \in X} a_i$ is not a positive integer. Then by Theorem 8.2 and Lemma 8.1(i) we have for every p

$$H^p(M, \mathcal{L}((1-t)a) = H^p(\bar{A}, (1-t)a) = H^p(\bar{A}, a).$$

Using the upper semicontinuity of the dimension of cohomlogy for $t \to 0$ we obtain the result. □

Corollary 8.4. *For every $a \in \bar{A}_1$ and p*

$$\dim H^p(M, \mathcal{L}(a)) \geq \sup_{z \in \mathbb{Z}^n} \dim H^p(\bar{A}, a+z).$$

8.2. **Characteristic varieties.** The characteristic varieties can be defined for $H^*(M, \mathcal{L}(a))$ similarly to the definition of the resonance varieties for $H^*(A, a)$. The set of all rank 1 local systems on M can be identified with the torus $H^1(M, \mathbb{C}^*) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. For given integers $m, p \geq 1$ the *characteristic variety* Σ_m^p of M is the algebraic subvariety of the torus consisting of those local systems $\mathcal{L}$ for which $\dim H^p(M, \mathcal{L}) \geq m$. The varieties Σ_m^p are also called *cohomology support loci.*

The most results known about Σ_m^p are for the case $p = 1$. From now on we consider only this case and put $\Sigma^1 = \Sigma_1^1$. By a twisted version of a theorem by Lefschetz (see [61]), $\pi_1(M)$ does not change if one substitutes for $\mathcal{A}$ its intersection with a generic plane. By this reason we will always assume that $\mathcal{A}$ is an arrangement of lines in the (complex) projective plane $\mathbb{P}^2$.

The varieties Σ_m^1 have an equivalent definition in terms of the Alexander invariant of M. Let Λ be the group ring of $H^1(M)$. Since $H^1(M)$ is the free Abelian group on n fixed generators, Λ can be identified with the ring of Laurent polynomials of n indeterminates. If M' is the maximal Abelian cover of M then $B = H_1(M')$ is a module over Λ. This module is often called the *Alexander invariant* of M (or of $\mathcal{A}$). Any integer $m \geq 1$ and any presentation

$$\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_0 \longrightarrow B \longrightarrow 0$$

define the ideal F_m of Λ generated by all the minors of f of codimension $m-1$. Then Σ_m^1 is the subvariety of $(\mathbb{C}^*)^n$ defined by the ideal $F_m \otimes \mathbb{C}$ of $\Lambda \otimes \mathbb{C}$.

The connections between R^1 and Σ^1 are known for the components of the latter passing through the identity point $\mathbf{1}$ of the torus ${\mathbb{C}^*}^n$. We state the following facts without proofs.

(1). For any irreducible component $\mathcal{W}$ of Σ^1_m of positive dimension there exist s $(s > m)$ points $p_1 \dots , p_s$ in $\mathbb{P}^1$ and a map $f : M \to C = \mathbb{P}^1 \setminus \{p_1, \dots , p_s\}$ such that $\mathcal{W} = f^*(H^1(C, \mathbb{C}^*))$ ([6, 46]).

(2). Let W be an irreducible component of R^1_m. Then the image of W under the map $a \mapsto \mathcal{L}(a)$ is a universal covering of an irreducible component (passing through $\mathbf{1}$) of Σ^1_{s-1} for some $s > m$ ([46]). Any positive dimensional component of Σ^1 containing $\mathbf{1}$ is covered in this sense by a component of R^1.

(3). The tangent cone of Σ^1_m at the point $\mathbf{1}$ coincides with R^1_m ([14]).

(4). For all but finitely many cosets modulo $\mathbb{Z}^n$ of an element $a \in R^1$ we have

$$\dim H^1(M, \mathcal{L}(a)) = \sup_{z\in\mathbb{Z}^n} \dim H^1(A, a + z)$$

([46]).

8.3. **Pencils of curves.** Using the relations of $H^1(A, a)$ with $H^1(M, \mathcal{L}(a))$ we can make certain results of subsection 7.6 stronger. As in that subsection, we can restrict our considerations to arrangements of lines in a projective plane $\mathbb{P}^2$.

First let us show that the matrix $Q = Q(\mathcal{X})$ corresponding to a subset $\mathcal{X}$ of points of $\mathbb{P}^2$ has a natural geometric interpretation. Let P' be the blowup of $\mathbb{P}^2$ at the points from $\mathcal{X}$. Let F be a free Abelian group generators of which correspond to the proper preimages in P' of the lines from $\mathcal{A}$.

Proposition 8.5. *Intersection form on P' induces a symmetric bilinear form on F which has $-Q$ as its matrix.*

Proof. The entries of the bilinear form induced on F by the intersection form of P' are the following. If two lines in P intersect at a point from $\mathcal{X}$ then the entry corresponding to these lines is 0; if the lines intersect at a point outside of $\mathcal{X}$ then the entry is 1; the diagonal entry is $1 - m$ where m is the number of points from $\mathcal{X}$ on this line (all this since a blowup at a point decreases the intersection index by 1). Comparing with the definition of Q in subsection 7.5 we conclude the proof. □

In order to study relations between components of R^1 and Σ^1 it suffices to focus on *essential* components. We call a component W of R^1 essential if in the notation of Proposition 7.29, $\mathcal{B} = \mathcal{A}$. An essential component is defined by a covering of $\mathcal{A}$ by a collection $\mathcal{X}$ of subsets and we will record this sometimes using $W(\mathcal{X})$ for W. For instance, a local component defined by $X \in L'(2)$ is essential if all the lines of the arrangement are passing through X.

A component $\mathcal{W}$ of Σ^1 is essential if it is covered (in the sense of property 2 in the previous subsection) by an essential component $W(\mathcal{X})$ of R^1. We will write sometimes $\mathcal{W}(\mathcal{X})$ for $\mathcal{W}$. Equivalently a component $\mathcal{W}$ of Σ^1 is essential if there is no subarrangement $\mathcal{A}'$ of $\mathcal{A}$ such that for some component $\mathcal{W}'$ of the characteristic variety of $\mathcal{A}'$ one has $\mathcal{W} = i^*(\mathcal{W}')$ where the map of the character groups $i^* : H^1(M(\mathcal{A}'), \mathbb{C}^0) \to H^1(M(\mathcal{A}), \mathbb{C}^0)$ is induced by inclusion. Notice that all the essential components of Σ^1 have positive dimensions and contain $\mathbf{1}$.

Let $\mathcal{W}$ be an essential component of Σ^1. Our immediate goal is to construct a pencil of curves in P' corresponding to $\mathcal{W}$.

Let $f : M \to \mathbf{P}^1 \setminus \{p_1, \dots, p_m\}$ be the map corresponding to the component $\mathcal{W}$ by the property (1) in the previous subsection. The map f defines a rational map $\tilde{f} : \mathbb{P}^2 \to \mathbb{P}^1$ regular outside a set of codimension 2 (cf. [49]), i.e., a finite set of points. Since f is regular outside of $\bigcup_{H\in\mathcal{A}} H$ we have

$$\bigcup \tilde{f}^{-1}(p_i) \subset \bigcup_{H\in\mathcal{A}} H. \tag{8.1}$$

The curves that are the closures of fibers of $\tilde{f}$ do not have common irreducible components. Indeed, if we assume that C is such, then by (8.1) C must be a line and $\mathcal{W}$ is the image of a component in the arrangement obtained from $\mathcal{A}$ by deleting C. This contradicts the assumption that $\mathcal{W}$ is essential.

Put $f' = \tilde{f} \circ \sigma$ where $\sigma : P' \to \mathbb{P}^2$ is the blowup. Denote the pencil of the closures of the fibers of f' by $\pi(\mathcal{W})$ and call it the *pencil corresponding to* $\mathcal{W}$. The fibers $(f')^{-1}(p_i)$ we call *special.* The previous paragraph implies that $\pi(\mathcal{W})$ has no base points. Also it follows from Bertini's theorem that a generic fiber of $\pi(\mathcal{W})$ is nonsingular.

Theorem 8.6. *Let $\mathcal{W}(\mathcal{X})$ be an essential component of Σ^1. Then the set of special fibers of the pencil $\pi = \pi(\mathcal{W}(\mathcal{X}))$ is in one-to-one correspondence with the partition $\Pi(\mathcal{X})$, i.e., with the blocks of matrix $Q(\mathcal{X})$.*

Proof. Let D_i be the free abelian group generated by the lines from $\mathcal{A}$ whose proper transforms belong to $(f')^{-1}(p_i)$. Clearly subgroups D_i and D_j of F are orthogonal with respect to the intersection form. On the other hand, neither of the exceptional curves in P' belongs to a special fiber of π and by Zariski's connectedness theorem the special fibers of π are connected (cf. [49], 3.24). Thus the intersection form is indecomposable on D_i.

Next recall ([35], exp. X) that for each fiber of a fibration of a surface there is unique linear combination z_0 of components of this fiber with positive integer coefficients such that

(a) for any combination z of the components of this fiber $(z, z) \leq 0$ with equality if and only if z is an integer multiple of z_0;

(b) $(z, z_0) = 0$ for any z supported on the fiber.

This implies that the intersection form on each D_i is affine and the proof is complete. $\square$

Existence of a pencil whose special fibers form a partition of arrangement imposes strong restrictions on the arrangement. For example, one can show the following.

Proposition 8.7. *Suppose $\mathcal{A}$ is an arrangement of $m \cdot n$ $(m \geq 3, n \geq 2)$ lines for which*

(a) the matrix Q corresponding to all the multiple points of intersection is the sum of m equal blocks and the lines corresponding to each block are in general position;

(b) any multiple point contains a line from each group;

(c) each line contains n multiple points.

Then $m \leq 5$.

Proof. It follows from (c) that each group is affine. Hence there is a pencil π whose special fibers correspond to the blocks of Q. Let us calculate the Euler characteristic of P', i.e., the blowup of the projective plane with the center at the multiple points of the arrangement. It follows from (c) and (a) there are n^2 multiple points and hence the Euler characteristic of P' is $\mathcal{E}_1 = 3+n^2$. On the other hand, P' is fibered over $\mathbb{P}^1$ with m special fibers consisting of n lines in general position and with generic fiber a nonsingular plane curve of degree n. Each of the above special fibers has the Euler characteristic equal to $2n - \frac{n(n-1)}{2}$. A generic fiber has the Euler characteristic $3n - n^2$. This gives $\mathcal{E}_1 \geq \mathcal{E}_2$ where $\mathcal{E}_2 = (2-m) \cdot (3n - n^2) + m \cdot (2n - \frac{n(n-1)}{2})$ (inequality is due to possible presence of other singular fibers and the semicontinuity of the Euler characteristic). But $\mathcal{E}_2 > \mathcal{E}_1$ for $m \geq 6$ which yields the result. $\square$

Notice that if in the above theorem $m = n+1$ then the proof shows that $\mathcal{E}_1 = \mathcal{E}_2$ if and only if $n = 2, 3$. In these cases we have a pencil with no singular fibers other than those composed of lines of the arrangement.

A similar argument gives a restriction on the size of arrangements having a given number of multiple points on each line and a large $\dim R_1$.

Proposition 8.8. *Let $\mathcal{A}(r, m)$ be a class of arrangements with each line having at most m multiple intersection points and $\dim R_1 = r$. Let Π be the partition of an arrangement from the class corresponding to a component of R^1 of dimension r. Then there is a function $F(r, m) \leq 2m + 1$ such that the number of lines in each of $r + 1$ elements of Π does not exceed $F(r, m)$. In particular the number of the lines in the arrangement is at most $(r + 1)F(r, m) < (r + 1)(2m + 1)$.*

Proof. The argument is similar to the one used in the previous proposition. Let d be the maximum of the numbers of lines in each special fiber of the pencil. Then the Euler characteristic of P' is greater than $\mathcal{E}_2 = (2-r)(3d - d^2) + r(2d - d(d-1)/2)$ (use semicontinuity of the Euler characteristic). On the other hand, (using the blowup) it is greater than or equal to $\mathcal{E}_1 = 3 + d \cdot m \cdot r$. One has $\mathcal{E}_2 > \mathcal{E}_1$ and hence a contradiction unless $\mathcal{E}_2 - \mathcal{E}_1$ as a polynomial in d has a positive root $F(r, m)$ (which is a bound for d).

Direct calculation shows that when $r \to \infty$ the positive root of $\mathcal{E}_2 - \mathcal{E}_1$ is an increasing function of r and its limit is $2m + 1$. $\square$

8.4. **Problems.** Relations between the resonance and respective characteristic varieties are still mysterious. The point is that there may exist components of the latter not passing through **1**. Such components of dimension 0 have been known for a while. Components of dimension one were found by Suciu in [63] which solved several problems from [47]. It is unknown if these components can be detected by the algebra A or, more generally, by the combinatorics of the arrangement. They cannot be detected in general by the resonance varieties so one should bring other combinatorial invariants in the picture. Any progress in this direction would be very interesting.

References

[1] Anick, D., Non-commutative graded algebras and their Hilbert series, J. Algebra **78** (1982), 120-140.
[2] Aomoto, K, Un théorème du type de Matsushima-Murakami concernant l'intégrale des fonctions multiformes, J. Math. Pures Appl. **52** (1973), 1-11.
[3] Aomoto, K., Les équations aux différences linéaires des fonctions multiformes, J. Fac. Sci. Univ. Tokyo, sec. IA **22** (1975), 271-297.
[4] Aomoto, K., Poincaré series of the holonomy Lie algebra attached to a configuration of lines, preprint, Nagoya Univ., 1984.
[5] Aramova, A., Avramov, L.A., and Herzog, J., Resolutions of monomial ideals and cohomology over exterior algebras, Trans. AMS **352** (2000), 579-594.
[6] Arapura, D., Geometry of cohomology support loci for local systems I, J. Algebraic Geometry **6** (1997), 563-597.
[7] Backelin, J. and Fröberg, R., Koszul algebras, Veronese subrings and rings with linear resolutions, Rev. Roumaine Math. Pures Appl. **30** (1985), 85-97.
[8] Beilinson, A., Ginzburg, V., and Soergel, W., Koszul duality patterns in representation theory, J. AMS **9** (1996), 473-527.
[9] Bousfield, A. and Guggenheim, V., On PL DeRham theory and rational homotopy type, Mem. AMS **8**, No. 179 , Providence, RI, 1976.
[10] Brieskorn, E., Sur les groupes de tresses, Séminaire Bourbaki 1971/72, LNM **317**, Springer-Verlag, 1973, 21-44.
[11] Björner, A. and Ziegler, G., Broken circuit complexes: factorizations and generalizations, J. Comb. Theory (B) **51** (1991), 96-126.
[12] Cartan, H. and Eilenberg, S., Homological algebra, Princeton Univ. Press, Princeton, 1956.
[13] Chen, K.T., Iterated integrals of differential forms and loop space cohomology, Ann. Math. **97** (1973), 217-246.
[14] Cohen, D. and Suciu, A., Characteristic varieties of arrangements, preprint, math.AG/9801048.
[15] Cohen, D. and Orlik, P., Arrangements and local systems, preprint 2000.
[16] Crapo, H., A higher invariant for matroid, J. of Combinatorial Theory **2** (1967), 406-417.
[17] Deligne, P., Equations différentielles à points singuliers réguliers, LNM **163**, Springer-Verlag, 1970.
[18] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., Real Homotopy Theory of Kähler Manifolds, Invent. Math. **29** (1975), 245-274.

[19] Eagon, J., Reiner, V., Resolutions of Stanley-Reisner rings and Alexander duality, J. Pure Appl. Algebra **130** (1998), 265-275.
[20] Denham, G. and Yuzvinsky, S., Annihilators of ideals of exterior algebras, preprint 2000.
[21] Eisenbud, D., Popescu, S., and Yuzvinsky, S., Hyperplane arrangement cohomology and monomials in the exterior algebra, preprint 1999.
[22] Eisenbud, D. and Schreyer, F.-O., Free resolutions over exterior algebras, preprint 2000.
[23] Esnault, H., Schechtman, V., and Viehweg, E., Cohomology of local systems of the complement of hyperplanes, Invent. math. **109** (1992), 557-561; Erratum, ibid. **112** (1993) 447.
[24] Falk, M., The minimal model of the complement of an arrangement of hyperplanes, Trans. AMS **309** (1988), 543-556.
[25] Falk, M., Arrangements and cohomology, Annals Combinatorics **1** (1997), 135-157.
[26] Falk, M., Line closed matroids and quadratic algebras, preprint 2000.
[27] Falk, M. and Randell, R., The lower central series of a fiber-type arrangement, Invent. Math. **82** (1985), 77-88.
[28] Falk, M. and Randell, R., On the homotopy theory of arrangements, Complex Analytic Singularities, Adv. Studies in Pure Math., No. **8**, North-Holland, Amsterdam, 1986, 101-124.
[29] Falk, M. and Randell, R., On the homotopy theory of arrangements, II, preprint 1999.
[30] Falk, M. and Terao, H., β**nbc** bases for cohomology of local systems on hyperplane complements, Trans. AMS **349** (1997), 189-202.
[31] Folkman, J., The homology groups of a lattice, J. Math. and Mech. **15** (1966), 631-636.
[32] Fröberg, R., Determination of a class of Poincarè series, Math. Scand. **37** (1975), 29-39.
[33] Grayson, D. and Stillman, M., *Macaulay 2*, Version 0.8.54; available at `http://www.math.uiuc.edu/Macaulay2.`
[34] Griffiths, P. and Morgan, J., Rational homotopy theory and differential forms, Birkhäuser, Boston, 1981.
[35] Groupes de Monodromie in Geometry Algebrique (SGA 7 II), LNM **340**, Springer-Verlag, 1973.
[36] Hain, R., The existence of higher logarithms, Compositio Math. **100** (1996), 247-276.
[37] Halperin, S. and Stasheff, J., Obstructions to Homotopy Equivalences, Adv. in Math. **32** (1979), 233-279.
[38] Hilbert, D. and Cohn-Vossen, S., Geometry and the imagination, Chelsea, New York, 1952.
[39] Kac, V., Infinite dimensional Lie algebras, Cambridge University Press, 1990.
[40] Kohno, T., On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of surfaces, Nagoya Math. J. **92** (1983), 21-37.
[41] Kohno, T., Série de Poincaré-Koszul associée aux groupes de tresses pures, Invent. Math. **82** (1985), 57-75.
[42] Kohno, T., Rational $K[\pi, 1]$ arrangements satisfy the lower central series formula, preprint, Nagoya University, 1985.
[43] Kohno, T.: Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. **62**, Ser. A (1986), 144-147.
[44] Libgober, A., Characteristic varieties of algebraic curves, preprint, math.AG/9801070.
[45] Libgober, A., First order deformations for rank one local systems with a non-vanishing cohomology, preprint 1999.
[46] Libgober, A. and Yuzvinsky, S., Cohomology of the Orlik-Solomon algebras and local systems, Compositio Math. **121** (2000), 337-361.

[47] Libgober, A. and Yuzvinsky, S., Cohomology of local systems, to appear in Advances Studies in Pure Math., Singularities and Arrangements, Sapporo-Tokyo.

[48] Markl, M. and Papadima, S., Homotopy Lie algebras and fundamental groups via deformation theory, Ann. Inst. Fourier, Grenoble **42**, 4 (1992), 905-935.

[49] Mumford D., Algebraic Geometry I, Complex Projective varieties, Springer-Verlag, 1976.

[50] Orlik, P. and Terao, H., Arrangements of hyperplanes, Springer-Verlag, Berlin, 1992.

[51] Orlik, P. and Terao, H., Arrangements and hypergeometric integrals, preprint 2000.

[52] Orlik, P., Solomon, L., Combinatorics and topology of complements of hyperplanes, Invent. math. **56** (1980), 167-189.

[53] Papadima, S. and Yuzvinsky, S., On rational $K[\pi, 1]$ spaces and Koszul algebras, J. Pure Appl. Algebra 144 (1999), 157-167.

[54] Positselski, L., Relation between the Hilbert series of dual quadratic algebras does not imply Koszulity, Func. Anal. Appl. **29** (1995), 213-217.

[55] Provan, J., Decompositions, shellings, and diameters of simplicial complexes and convex polyhedra, Thesis, Cornell Univ., 1977.

[56] Roos, J.-E., On the characterization of Koszul algebras. Four counter-examples, CR Acad. Sci. Paris, Serie I, **321** (1995), 15-20.

[57] Schechtman, V., Terao, H., and Varchenko A., Local systems over complements of hyperplanes and the Kac-Kazhdan conditions for singular vectors, J. Pure Appl. Algebra, **100** (1995), 93-102.

[58] Schechtman, V., and Varchenko, A., Hypergeometric solutions of Knizhnik-Zamolodchikov equations, Lett. Math. Phys. **20** (1990), 279-283.

[59] Schechtman, V., Varchenko, A., Arrangements of hyperplanes and Lie algebra homology, Invent. Math. **106** (1991), 139-194.

[60] Shelton, B. and Yuzvinsky, S., Koszul algebras from graphs and hyperplane arrangements, J. London Math. Soc. (2) **56** (1997), 477-490.

[61] Simpson, C., Some families of local systems over smooth projective varieties, Ann. Math. **138** (1993), 337-425.

[62] Stanley, R., Cohen-Macaulay rings and constructible polytops, Bull. AMS **87** (1975), 133-135.

[63] Suciu, A., Translated tori in the characteristic varieties of complex hyperplane arrangements, preprint AG/9912227.

[64] Sullivan, D., Infinitesimal computations in topology, Publ. Math. IHES **47** (1977), 269-331.

[65] Varchenko, A., The Euler beta-function, the Vandermonde determinant, Legendre's equation, and critical values of linear functions on a configuration of hyperplanes, I, Math. USSR Izvestia **35** (1990), 543-571; II, ibid **36** (1991), 155-167.

[66] Wachs, M. and Walker, J., On geometric semilattices, Order **2** (1986), 367-385.

[67] Yuzvinsky, S., Cohomology of the Brieskorn-Orlik-Solomon algebras, Comm. in Algebra **23** (1995), 5339-5354.

[68] Ziegler, G., Matroid representations and free arrangements, Trans. AMS **320** (1990), 525-541.

[69] Ziegler, G., Matroid shellability, β-systems, and affine arrangements, J. Alg. Combinatorics **1** (1992), 283-300.

E-mail address: `yuz@math.uoregon.edu`

UNIVERSITY OF OREGON, EUGENE, OR 97403