Topological complexity of generic hyperplane complements

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ABSTRACT. We prove that the topological complexity of (a motion planning algorithm on) the complement of generic complex essential hyperplane arrangement of n hyperplanes in an r-dimensional linear space is $\min\{n+1, 2r\}$.

1. Introduction

In this paper we continue the theme started in [**3**] - studying the topological (motion planning) complexity $\mathbf{TC}(M)$ of the complement M of a complex hyperplane arrangement. The number $\mathbf{TC}(X)$ was defined for any path-connected topological space X by M.Farber in [**1, 2**]. This number is of fundamental importance for the motion planning problem: $\mathbf{TC}(X)$ determines character of instabilities for all motion planning algorithms in X.

The main result of this paper can be stated as follows:

THEOREM 1.1. *Let M be the complement of a complex central essential arrangement of n hyperplanes in the linear space V of dimension $r > 0$. Then* $\mathbf{TC}(M) = \min\{n+1, 2r\}$.

2. The motion planning problem

In this section we recall the definitions and results from [**1, 2**] that we will use later in this paper.

Let X be a connected topological space X that is homotopy equivalent to a CW complex. Let PX be the space of all continuous paths $\gamma : [0,1] \to X$, equipped with the compact-open topology, and let $\pi : PX \to X \times X$ be the map assigning the end points to a path: $\pi(\gamma) = (\gamma(0), \gamma(1))$. The map π is a fibration whose fiber is the based loop space ΩX. The topological complexity of X, denoted by $\mathbf{TC}(X)$, is the smallest number k such that $X \times X$ can be covered by open sets $U_1, \dots, U_k$, so that for every $i = 1, \dots, k$ there exists a continuous section $s_i : U_i \to PX, \pi \circ s_i = 1$.

According to [**2**], *a motion planner* in X is defined by finitely many subsets $F_1, \dots, F_k \subset X \times X$ and continuous maps $s_i : F_i \to PX$, where $i = 1, \dots, k$, such that:

(a) the sets $F_1, \dots, F_k$ are pairwise disjoint (i.e.,$F_i \cap F_j = \emptyset$, $i \neq j$), and cover $X \times X$;

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(b) $\pi \circ s_i = 1_{F_i}$ for any $i = 1, \ldots, k$;
(c) each F_i is an ENR.

The subsets F_i are *local domains* of the motion planner; the maps s_i are *local rules*.

In [**2**] it is shown that: *the minimal integer k, such that a smooth manifold X admits a motion planner with k local rules, equals* $\mathbf{TC}(X)$.

The other properties of $\mathbf{TC}(X)$ we will need are:

(i) $\mathbf{TC}(X)$ depends only on the homotopy type of X.

(ii) $\mathbf{TC}(X) \leq 2\dim(X) + 1$.

(iii) $\mathbf{TC}(X \times S^1) \leq \mathbf{TC}(X) + 1$.

Next result provides a lower bound for $\mathbf{TC}(X)$ in terms of the cohomology ring $H^*(X)$ with coefficients in a field. The tensor product $H^*(X) \otimes H^*(X)$ is also a graded ring with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1| \cdot |u_2|} u_1 u_2 \otimes v_1 v_2$$

where $|v_1|$ and $|u_2|$ are the degrees of the cohomology classes v_1 and u_2. The cohomology multiplication $H^*(X) \otimes H^*(X) \to H^*(X)$ is a ring homomorphism. Let $Z \subset H^*(X) \otimes H^*(X)$ be the kernel of this homomorphism. The ideal Z is called *the ideal of zero-divisors* of $H^*(X)$. The *zero-divisors-cup-length* is the length of the longest nontrivial product in the ideal of zero-divisors.

(iv) The topological complexity $\mathbf{TC}(X)$ is greater than the zero-divisors-cup-length of $H^*(X)$.

The topological complexity $\mathbf{TC}(X)$, as well as the Lusternik-Schnirelmann category $\mathrm{cat}(X)$, are particular cases of the notion of *Schwarz genus* (also known as *sectional category*) of a fibration; it was introduced and thoroughly studied by A.Schwarz in [**5**].

3. Hattori theorem

In this section we recall the necessary definitions from arrangement theory and the famous result of Hattori. The details can be found in [**4**].

Let V be a complex linear space of a positive dimension r. An arrangement $\mathcal{A}$ in V is a set $\{H_1, \ldots, H_n\}$ of n hyperplanes for some n. The arrangement is *essential* if $\bigcap_{i=1}^n H_i = 0$. In particular for an essential arrangement $n \geq r$. Fix for each i a functional $\alpha_i \in V^*$ such that $\ker\alpha_i = H_i$. The arrangement is *generic* if for any subset $I \subset \overline{n} = \{1, \ldots, n\}$ with $|I| = r$ the respective set of functionals is linearly independent. In particular each generic arrangement is essential.

For a generic arrangement $\mathcal{A}$ the homotopy type of the space $M = M(\mathcal{A}) = V \setminus \bigcup_{i=1}^n H_i$ is easy to describe. First, in order to give a precise reference we need to reduce $\mathcal{A}$ to an arrangement of affine hyperplanes. For that choose an element of $\mathcal{A}$, say H_n, put $\overline{H_n} = \{v \in V | \alpha_n(v) = 1\}$, and put $\overline{\mathcal{A}} = \{H_i \cap \overline{H_n} | i = 1, \ldots, n-1\}$. The arrangement $\overline{\mathcal{A}}$ consists of affine hyperplanes in the affine space $\overline{H_n}$ of dimension $r-1$. Moreover since $\mathcal{A}$ is generic the affine arrangement $\overline{\mathcal{A}}$ is *in general position*, i.e., the intersection of any p hyperplanes from it has codimension p for $p \leq r-1$ and is empty for $p > r-1$. In particular $|\overline{\mathcal{A}}| = n-1$. Since $M = M(\mathcal{A})$ is the total space of a trivial fiber bundle over $\overline{M} = M(\overline{\mathcal{A}})$ with the fiber $\mathbf{C}^*$ we have the homotopy equivalence $M \approx \overline{M} \times S^1$ (cf. [**4**], Proposition 5.1).

Now we state Hattori's theorem [**4**], Theorem 5.21. Denote by T^m the (compact) torus of dimension m and for every $I \subset \overline{m}$ put

$$T_I^m = \{(z_1, \dots, z_m) \in T^m | z_j = 1, \text{ for } j \notin I\}.$$

THEOREM 3.1. *Let $n > r > 1$. For any general position arrangement of $n-1$ affine hyperplanes in $(r-1)$ - dimensional space its complement has the homotopy type of $\overline{M_0}$ where $\overline{M_0}$ is the skeleton of dimension $r-1$ of the canonical CW-complex of T^{n-1}, i.e.,*

$$\overline{M_0} = \bigcup_{|I|=r-1} T_I^{n-1}.$$

COROLLARY 3.2. *For any generic arrangement of n linear hyperplanes in r dimensional space its complement M has the homotopy type of M_0 where*

$$M_0 = S^1 \times \bigcup_{|I|=r-1} T_I^{n-1}.$$

PROOF. For $n > r$ it follows immediately from Hattori's theorem. For $n = r$ (in particular for $r = 1$) the arrangement consists of all coordiante hyperplanes whence $M \approx (\mathbf{C}^*)^r \approx T^r = M_0$. □

The property (i) of $\mathbf{TC}(X)$ allows us to focus in the rest of the paper on calculating $\mathbf{TC}(M_0)$. We will always denote by n the number of hyperplanes in the generic central arrangement $\mathcal{A}$ we will consider and by r the dimension of the ambivalent space V.

4. Low bound

In this section we use the definition of M_0 to describe $H^*(M_0; \mathbf{C})$ and to exhibit a low bound on $\mathbf{TC}(M_0)$ using the property (iv).

Denote by $E(n) = \oplus_{i=0}^n E(n)_i$ the exterior algebra over $\mathbf{C}$ with n generators of degree one. Also for every k, $0 \le k \le n$, put $E(n)^k = E(n)/\oplus_{i>k} E(n)_i$ (a truncated exterior algebra).

From the description of M_0 in Corollary 3.2 we have

$$H^*(M_0, \mathbf{C}) = E(1) \otimes E(n-1)^{r-1}$$

where the tensor product is taken in the category of graded algebras. In particular we have the following lemma.

Denote by e_0 a generator of $H^*(S^1) = E(1)$ and by $e_1, \dots, e_{n-1}$ the generators of $H^*(\overline{M_0}) = E(n-1)^{r-1}$. Also for every $I = \{i_1 < i_2 < \dots < i_k\} \subset \overline{n-1}$ put $e_I = e_{i_1} \cdots e_{i_k}$.

LEMMA 4.1. *The set $\{e_0 e_I | I \subset \overline{n-1},\ |I| = r-1\}$ is a basis of the linear space $H^r(M_0, \mathbf{C})$.*

Now we define the elements in the ideal of zero divisors of $H^*(M_0) \otimes H^*(M_0)$ corresponding to the generators. Namely put $\overline{e}_i = 1 \otimes e_i - e_i \otimes 1$ for every $i = 0, 1, \dots, n-1$.

PROPOSITION 4.2. *Let $k = \min\{n-1, 2r-2\}$ and $J \subset \overline{n-1}$ with $|J| = k$. Then $\pi = \overline{e}_0 \prod_{i \in J} \overline{e}_i \neq 0$.*

PROOF. The linear space $H^*(M_0)\otimes H^*(M_0)$ is double graded by the subspaces $H^s\otimes H^t$, $0\le s,t\le r$. It suffices to prove that $(r,k+1-r)$-component $\pi_{r,k+1-r}$ of π does not vanish. Clearly this component is

$$\pi_{r,k+1-r}=\sum_{I\subset J,|I|=r-1}\pm e_0e_I\otimes e_{J\setminus I}.$$

Since $|J\setminus I|=k+1-r\le r-1$ and $H^*(\overline{M_0})=E(n-1)^{r-1}\subset H^*(M_0)$ all monomials $e_{J\setminus I}$ belong to a basis of $H^{k+1-r}(M_0)$. The monomials e_0e_I belong to a basis of $H^{r-1}(M_0)$ by Lemma 4.1. Hence all the summands of $\pi_{r,k+1-r}$ belong to a basis of $H^*(M_0)\otimes H^*(M_0)$ whence $\pi_{r,k-r}\ne 0$. This completes the proof. $\square$

Now the property (iv) of $\mathbf{TC}(X)$ immediately implies the following.

COROLLARY 4.3.

$$\mathbf{TC}(M)=\mathbf{TC}(M_0)\ge\min\{n+1,2r\}.$$

5. Motion planning

In this section we prove that the upper bound for $\mathbf{TC}(M_0)$ coincides with the low bound from the previous section.

First since $M_0\approx\overline{M_0}\times S^1$ we have by property (iii)

$$\mathbf{TC}(M_0)\le\mathbf{TC}(\overline{M_0})+1.$$

Now suppose $n+1\ge 2r$. Since $\dim\overline{M_0}=r-1$ we have using property (ii) that $\mathbf{TC}(\overline{M_0})\le 2r-1$ whence

$$\mathbf{TC}(M_0)\le 2r=\min\{n+1,2r\}.$$

Thus we have to consider only the case $n+1<2r$. To find the upper bound in this case we constract an explicit motion planning for $\overline{M_0}$ with n rules.

THEOREM 5.1. *For arbitrary $r\le n$ there exists a motion planning for $\overline{M_0}$ with n rules.*

PROOF. First for every $J\subset\overline{n-1}$ we define the close subset F'_J of $T^{n-1}\times T^{n-1}$ via

$$F'_J=\{(u,u')|u_j=u'_j \text{ if and only if } j\in J\}$$

and put $F_J=F'_J\cap(\overline{M_0}\times\overline{M_0})$. Then we put $F_i=\bigcup_{|J|=i}F_J$ for every $i=0,1,\dots,n-1$. The sets F_i are pairwise disjoint and cover $\overline{M_0}\times\overline{M_0}$ whence we can take them as the local domains of the motion planning we are constructing. Since the sets F_J are also pairwise disjoint it suffices now to construct local rules on them, i.e., (continuous) sections $s_J:F_J\to P\overline{M_0}$.

For that define an auxiliary function $\tau:S^1\to[0,1]$ by treating S^1 (in the rest of the proof) as the set of all complex numbers of norm 1 and putting

$$\tau(z)=\begin{cases}\frac12(1-\frac{|z-1|}{\sqrt2}) & \text{if } |z-1|\le\sqrt2,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that $\tau(1)=\frac12$. Also for two points $z\ne z'\in S^1$, $z=\exp[\sqrt{-1}\phi]$, $z'=\exp[\sqrt{-1}\phi']$, where $0\le\phi,\phi'<2\pi$, define the path $\zeta_{z,z'}$ on S^1 via $\zeta_{z,z'}(t)=\exp[\sqrt{-1}(t\phi+(1-t)\phi')]$ (i.e., the moving with a constant speed from z to z' along the natural orientation of $\mathbf{C}$).

Now for $(u,u')=((u_1,\dots,u_{n-1}),(u'_1,\dots,u'_{n-1}))\in T^{n-1}\times T^{n-1}$ we define $s_J(t)=(s_{J,j}(t))_{j\in\overline{n-1}}$ via $s_{J,j}(t)=u_j=u'_j$ for every $t\in[0,1]$ if $j\in J$. If $j\notin J$ we put

$$s_{J,j}(t)=\begin{cases}u_j & \text{if } 0\le t<\tau(u_j),\\ \zeta_{u_j,u'_j}\left(\frac{t-\tau(u_j)}{1-\tau(u_j)-\tau(u'_j)}\right) & \text{if } \tau(u_j)\le t\le 1-\tau(u'_j),\\ u'_j & \text{if } 1-\tau(u'_j)<t\le 1.\end{cases}$$

It is clear from the definition that s_J is continuous and $s_J(0)=u$, $s_J(1)=u'$. Also since τ is continuous and $\zeta_{z,z'}$ depends continosly on (z,z') on $S^1\times S^1$ with the diagonal deleted we see that s_J is conitiously depending on (u,u') on F_J. It is left to check only that $s_J(t)\in\overline{M_0}$ for every $t\in[0,1]$. In other words we need to check that for every t we have $s_{J,j}(t)=1$ for at least $n-r$ values of j.

Suppose that $u\in T_I^{n-1}$ and $u'\in T_{I'}^{n-1}$, $|I|=|I'|=r-1$. Consider the complements $\overline{I}=\overline{n-1}\setminus I$ and $\overline{I'}=\overline{n-1}\setminus I'$. Put $I_0=\overline{I}\cap\overline{I'}$ and fix a bijection $\phi:\overline{I}\setminus I_0\to\overline{I'}\setminus I_0$ putting $j'=\phi(j)$ for every $j\in\overline{I}\setminus I_0$. Then if $j\in I_0$ we have $j\in J$ whence $s_{J,j}(t)=u_j=u'_j=1$ for every t. Suppose $j\in\overline{I}\setminus I_0$. Then $\tau(u_j)=\tau(1)=\frac{1}{2}$ whence $s_{J,j}=u_j=1$ for $t\le\frac{1}{2}$. On the other hand, $\tau(u'_{j'})=\tau(1)=\frac{1}{2}$ whence $s_{J,j'}(t)=1$ for $t\ge\frac{1}{2}$. Collecting this data we see that indeed for arbitrary t there are $n-r$ values of j such that $s_{J,j}(t)=1$ which completes the construction of the motion planning whence also the proof. □

Corollary 5.2. $\mathbf{TC}(\overline{M_0})\le\min\{n,2r-1\}$ *whence* $\mathbf{TC}(M)=\mathbf{TC}(M_0)\le\min\{n+1,2r\}$ *and Theorem 1.1 follows.*

In all cases where the topological complexity has been computed for hyperplane arrangement complements it coincides with the low bound given by the zero-divisors-cup-length (see property iv in section 2). This justifies the following conjecture.

Conjecture 5.3. *For every complex central hyperplane arrangement with the complement M the topological complexity $\mathbf{TC}(M)$ is greater by 1 than the zero-divisors-cup-length of $H^*(M,\mathbf{C})$.*

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