On rational $K[\pi, 1]$ spaces and Koszul algebras

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February 3, 1997

1 Introduction

Let X be a connected topological space with finite Betti numbers. Then X is said to be a *rational* $K[\pi, 1]$ *space* if its $\mathbf{Q}$-completion $\mathbf{Q}_\infty(X)$ constructed in [8] is aspheric, that is $\pi_n\mathbf{Q}_\infty(X) = 0$ for all $n > 1$.

Let A be a connected graded algebra of finite type over a field F. Then A is said to be a *Koszul algebra* if $\mathrm{Ext}_A^{p,q}(F, F) = 0$, for every $p \neq q$. (See Definition 2.1 for details.)

Our main result (Theorem 5.1) establishes an equivalence between the topological rational $K[\pi, 1]$ property of X and the algebraic Koszul property of $H^*(X, \mathbf{Q})$, in the case of a *formal* space X. The important class of formal spaces was introduced in [30]. One says that X is formal if $\mathbf{Q}_\infty(X)$ depends only on the cohomology algebra $H^*(X, \mathbf{Q})$, see Section 5 for details. We then derive (in Proposition 5.2) that the Koszulness of $H^*(X, \mathbf{Q})$ implies that $\mathbf{Q}_\infty(X)$ is aspheric, for an arbitrary space X. We also derive (in Corollary 5.3) the following *lower central series (LCS) formula* for a formal rational $K[\pi, 1]$ space X:

$$P(X, -t) = \prod_{n\geq 1}(1 - t^n)^{\phi_n}.$$

Here $\phi_n = \mathrm{rank}(\Gamma_n/\Gamma_{n+1})$, where $\{\Gamma_n\}_{n\geq 1}$ is the lower central series of the fundamental group $\pi_1(X)$, and $P(X, t)$ is the Poincaré polynomial of X. The main technical tool in our proofs is Sullivan's notion of *minimal model* for commutative differential graded algebras, see Section 3.

The rational $K[\pi, 1]$ formal spaces, especially those arising in low- dimensional topology, were studied in [24, 5]. More recently the rational $K[\pi, 1]$ property has come in spotlight in connection with the higher logarithms, see [17]. There is an extensive literature on the Koszul property, see for example [3, 4, 28], and [23] for connections with quantum groups. The rational $K[\pi, 1]$ spaces and Koszul algebras first appeared together implicitly in [24] and explicitly in [6].

The complements of complex hyperplane arrangements provide a rich supply of formal spaces [9] and have received a lot of attention. In particular the rational $K[\pi,1]$ property for these spaces was studied in [13]-[15], [20]-[22]. In this context we are able to prove (in Proposition 5.4) that the above LCS formula implies the rational $K[\pi,1]$ property, for arrangements in $\mathbf{C}^3$.

The connection with the Koszul property is the following. It is known that the Koszulness of A implies *the Koszul duality formula* (see 2.3 (iii)), a useful numerical test involving Hilbert series (see Example 5.7). Note however that this formula does not imply in general the Koszul property, see [27, 28]. Finally it is not hard to see that the LCS formula for a formal space X is actually equivalent to the Koszul duality formula for $A = H^*(X, \mathbf{Q})$, if A is quadratic, i.e. generated in degree 1 with relations in degree 2.

It is therefore interesting to know whether the LCS formula implies the rational $K[\pi,1]$ property in the particular case of arrangement complements, as asked by Falk and Randell in [15]. In this particular case Corollary 5.3 was proved by Falk [13] and Kohno [22]. It was proved in full generality in [24]; here we give a different simple proof, based on the Koszul duality formula. More results related to Koszulness and LCS-type formulae are known for the so-called *fiber-type* arrangements introduced in [14], see [29], and for their recent generalization, the *hypersolvable* arrangements of [19]. (See the remarks in 5.6.)

The paper is organized as follows. In Section 2, we recall the definition and a few basic results about Koszul algebras. In Section 3, we recall the definition of a 1-minimal model and give a concrete realization convenient for our purposes. In Section 4, we prove the main algebraic result (Proposition 4.4). In Section 5, we derive our main results 5.1-5.4 about topological spaces and discuss an example of a hyperplane complement.

2 Koszul algebras: Preliminaries

We collect in this section the definitions and certain basic results about Koszul algebras, see [3, 4, 23]. Let F be a field and let $U = \oplus_{n\geq 0} U_n$ be a positively graded F-algebra. We will always assume that $\dim U_n < \infty$ for all n. U is called connected if $U_0 = F$.

Considering just graded left U- modules we denote by Ext_U the derived functor of the *graded* homomorphism functor. For two such modules M and N the linear space $\mathrm{Ext}_U(M,N)$ is bigraded, i.e. $\mathrm{Ext}_U(M,N) = \oplus_{p,q\geq 0} \mathrm{Ext}_U^{p,q}(M,N)$ where p is the homological (resolution) degree and q is the pure (internal) degree coming from the gradings of U, M and N.

Definition 2.1 A positively graded connected F-algebra U is *Koszul* if

$$\mathrm{Ext}_U^{p,q}(F,F) = 0 \tag{2.1}$$

for every $p \neq q$, where $F = {}_U F$ is the trivial graded left U-module (equal to $U/U_{>0}$).

If V is a finite-dimensional vector space over F let $T = T(V)$ denote the full F-tensor algebra on V. The algebra T is provided with the standard grading $\oplus_n T_n$ where $T_0 = F$ and $T_1 = V$. If U is a connected graded F-algebra then there exists a canonical graded algebra homomorphism $T(U_1) \to U$. The algebra U is called *quadratic* if this homomorphism is surjective and its kernel I is generated, as an ideal of T, by its degree 2 part I_2. We call I the ideal of U.

Definition 2.2 Let U be a quadratic algebra and let I be its ideal. Let $I_2^!$ be the annihilator in $U_1^* \otimes U_1^* = (U_1 \otimes U_1)^*$ of the linear space I_2 and let $I^!$ be the ideal of $T(U_1^*)$ generated by $I_2^!$. Then the quadratic algebra $U^! = T(U_1^*)/I^!$ is called *the quadratic dual of U.*

Observe that $(U^!)^! = U$.

For any graded F-linear space $V = \oplus_n V_n$ with $\dim V_n < \infty$ for all n we denote the Hilbert series of V by $H(V,t)$ $(= \sum_n \dim_F(V_n)t^n)$. The following statements will be used often without reference.

Theorem 2.3 *(i) If U is Koszul then it is quadratic. For any quadratic algebra the condition (2.1) holds for $p < 3$ and $q \neq p$.*

(ii) A quadratic algebra U is Koszul if and only if $U^!$ is Koszul.

(iii) If U is Koszul then $H(U,t) \cdot H(U^!,-t) = 1$.

Quite recently, Positselski [27] and Roos [28] have independently constructed examples showing that the converse of 2.3 (iii) is false.

3 1-minimal models of DGA's

In this section we suppose that F is a field of characteristic 0 and consider algebras over F. Let $M = \oplus_{n \geq 0} M_n$ be a connected DGA. That is M is a connected graded F-algebra which is graded- commutative (i.e. $xy = (-1)^{mn}yx$, for $x \in M_m$ and $y \in M_n$), equipped with a *differential* $d : M \to M$, which is homogenous of degree 1, satisfies $d^2 = 0$ and acts on M as a graded derivation (i.e. $d(x \cdot y) = dx \cdot y + (-1)^n x \cdot dy$, if $x \in M_n$). The DGA homomorphisms are required to be homogenous of degree zero, multiplicative, and to commute with the differentials. Two DGA's are said to be *quasi-isomorphic* if they can be connected by a sequence of DGA maps inducing homology isomorphisms.

Assuming that M is generated by M_1, the canonical (increasing) filtration of M is defined by $M(0) = F$ and $M(n)$ being the subalgebra of M generated by $M(n-1)$ and $d^{-1}(M(n-1)) \cap M_1$. Recall [7] that M is called *minimal* if $M = \wedge M_1$ (i.e. M is freely generated in degree one as a graded commutative algebra) and $\bigcup_n M(n) = M$. For every DGA A with $H^0(A) = F$ there exists a minimal algebra M as above and a DGA homomorphism $f : M \to A$ such that $f^* : H^*(M) \to H^*(A)$ is an isomorphism in dimension 1 and a monomorphism in dimension 2. The DGA M is unique up to isomorphism ([16], Theorem 12.1). M is called a *1-minimal model* of A.

For the rest of the paper we denote by $A = \oplus_{n \geq 0} A_n$ a graded commutative connected algebra. The 1-minimal model of A (provided with $d = 0$) can be realized as dual to an important Lie algebra attached to A. Let $L = \mathbf{L}(A_1^*)$ be the free Lie algebra on A_1^*, graded by bracket length. Notice that L_2 can be identified with $A_1^* \wedge A_1^*$. The multiplication of A generates a comultiplication $A_2^* \to A_1^* \wedge A_1^*$. Define the *holonomy Lie algebra* $G = G(A)$ of A as the factor of L by the Lie ideal generated by the image of the comultiplication. G is a graded Lie algebra and we write $G = \oplus_{n \geq 1} G_n$. We also consider the filtration of G by the Lie ideals $\Gamma_n G$ where $\Gamma_n G$ is the n-th term of the lower central series of G, i.e. $\Gamma_1 G = G$ and $\Gamma_{n+1} G = [\Gamma_n G, G]$. Notice that the grading of G is closely related to the filtration, namely $\Gamma_n G = \oplus_{k \geq n} G_k$. In particular if we denote by $G(n)$ the nilpotent Lie algebra $G/\Gamma_{n+1} G$ then $G(n) \approx \oplus_{i=1}^n G_i$, as F-linear spaces.

For every n we define the DGA freely generated in degree one $M(n) =: \wedge(G(n))^*$. The differential d is dual to the Lie bracket $G(n) \wedge G(n) \to G(n)$ on $(G(n))^*$ and is extended to $M(n)$ by the graded derivation property. The natural projections $G(n+1) \to G(n)$ generate the embeddings $M(n) \subset M(n+1)$ of DGA's and we put $M = \bigcup_{n \geq 1} M(n)$. Notice that $M(1) \approx \wedge A_1$ as a graded algebra, with differential $d = 0$, whence there is a natural DGA homomorphism $M(1) \to A$.

In order to analyze M deeper we put $V_n = G_n^*$, for every $n \geq 1$. Of course $V_1 = A_1$ and $d = 0$ on V_1. More importantly $M(n) = \wedge(\oplus_{i=1}^n V_i)$ and

$$d(V_n) \subset \sum_{i+j=n} V_i \wedge V_j \ , \tag{3.1}$$

for $n > 1$, where the symbol $\wedge$ applied to different spaces means the tensor product. Now extend f to a DGA homomorphism $f : M \to A$ by setting $f = 0$ on $\oplus_{n>1} V_n$.

Proposition 3.1 ([24] , Lemma 1.8 (i)) *Provided with the above homomorphism f, M is a 1-minimal model of A.*

Note that there is a shift of the degrees: what we call here V_n was called V_{n-1} in [24]. This rescaling has features convenient for our purposes. Put

$$C^p(q) = \oplus_{i_1+\cdots+i_p=q} V_{i_1} \wedge \cdots \wedge V_{i_p}\ ,\ C(q) = \oplus_p C^p(q)\ ,$$

and notice that $M = \oplus_q C(q)$ (as a linear space). The following easily seen properties were noticed in the particular case of arrangements in [13] , for a completely different construction of the 1-minimal model M.

Proposition 3.2 *(i) The space $C(q)$ is a subcomplex of M for every q, whence $H^*(M) = \oplus_q H^*(C(q))$.*

(ii) The subspace $B =: \oplus_q H^q(C(q))$ is the subalgebra of $H^*(M)$ generated by $H^1(M) = A_1$. Moreover this algebra is quadratic.

(iii) The homomorphism $f^* : H^*(M) \to A$ is 0 on $\oplus_{p\neq q} H^p(C(q))$.

(iv) If A is quadratic then $B = A$ and f^* is the projection of $H^*(M)$ to B along $\oplus_{p\neq q} H^p(C(q))$.

Proof. (i) The invariance of $C(q)$ with respect to d follows immediately from (3.1) and implies the first statement. (ii) Denote by J the ideal of $M(1) = \wedge V_1 = \oplus_q C^q(q)$ generated by $d(V_2)$. Then $d(C^{q-1}(q)) = J_q$ whence $B = \oplus_q H^q(C(q)) = \wedge V_1/J$. (iii) It follows immediately from the construction of f. (iv) If A is quadratic then $A = \wedge A_1/I$ where I is a homogeneous ideal of $\wedge A_1$ generated by I_2. It follows from the construction of M that $V_1 = A_1$ and $d(V_2) = I_2$. Thus $B = A$. Using (iii) and descending to cohomology we obtain the second statement. $\square$

4 1-minimal models and Koszul algebras

In this section, we prove the main algebraic result. We need to use the universal enveloping algebra U of the Lie algebra G. A more explicit description of U is as follows. Let T be the tensor algebra of the linear space A_1^* and $\tau : A_1^* \otimes A_1^* \to A_1^* \otimes A_1^*$ be the linear map generated by $v \otimes w \mapsto w \otimes v$. Then U is the factor of T by the ideal I generated by $(\mathrm{Im}(1-\tau)) \cap \mathrm{Im}\Delta$, where Δ is the comultiplication $A_2^* \to A_1^* \otimes A_1^*$. This description gives the grading of U (which is induced from the standard grading of T) and shows that U is a quadratic algebra.

Lemma 4.1 *If the algebra A is quadratic then it is the quadratic dual of U, i.e. $A = U^!$.*

Proof. If A is quadratic it is the factor of $\wedge A_1$ by the ideal generated by the kernel of the product $A_1 \wedge A_1 \to A_2$. Equivalently A is the factor of the tensor algebra of A_1 by the ideal generated by the kernel of the product $A_1 \otimes A_1 \to A_2$ and the elements $a + \tau^* a$, $a \in A_1 \otimes A_1$. The result follows since the degree 2 component of this ideal is the annihilator of the degree 2 component of I. $\square$

Remark 4.2 Similarly one can show that in the general case $U^!$ is the factor algebra of $\wedge A_1$ by the ideal generated by the kernel of the multiplication $A_1 \wedge A_1 \to A_2$.

Now we study the Yoneda algebra $\mathrm{Ext} = \oplus_{p,q} \mathrm{Ext}_U^{p,q}(F, F)$.

Proposition 4.3 *The bigraded algebra* $\mathrm{Ext} = \oplus_{p,q} \mathrm{Ext}^{p,q}$ *coincides with the algebra* $H^*(M) = \oplus_p H^p(M)$ *where the second (pure) grading is induced on* $H^p(M)$ *by the grading of* M *by the subcomplexes* $C(q)$.

Proof. We have $\mathrm{Ext}^p = H^p(G, F) = H^p(\mathrm{Hom}(\wedge G, F))$ (see e.g. Chapter XIII of [10]). The space of n-cochains of the complex $\mathcal{K} = \mathrm{Hom}(\wedge G, F)$ consists of the skew symmetric multilinear functions of n variables from G with the differential δ defined by

$$(\delta f)(x_1, \ldots, x_{n+1}) = \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

The functions of pure degree q are those ones that are nonzero only if $\sum_{i=1}^n \deg x_i = q$, where the initial grading on G is used. These functions form a finite dimensional subcomplex $\mathcal{K}(q)$ of $\mathcal{K}$ and $\mathcal{K} = \oplus_q \mathcal{K}(q)$. Since $\mathcal{K}(q)$ is finite dimensional we have

$$\mathcal{K}(q) = (\oplus_{i_1 + i_2 + \cdots + i_p = q} G_{i_1} \wedge G_{i_2} \wedge \cdots \wedge G_{i_p})^* = C(q)$$

as graded linear spaces. Thus one can identify $\mathcal{K}$ and M as linear spaces. The algebra multiplication on M induces then the standard shuffle multiplication of skew symmetric forms from $\mathcal{K}$. The restrictions to the algebra generators of the differentials δ and d clearly coincide, up to sign. Besides both differentials satisfy the graded Leibnitz condition with respect to the algebra multiplication on M and the shuffle multiplication on $\mathcal{K}$. Thus these differentials coincide up to sign, which implies the statement. $\square$

Proposition 4.4 *The graded algebra homomorphism* $f^* : H^*(M) \to A$ *is an isomorphism if and only if* A *is Koszul.*

Proof. Suppose A is Koszul. Then it is quadratic and U is its quadratic dual whence U is also Koszul. Thus

$$H^p(C(q)) = \mathrm{Ext}_U^{p,q}(F,F) = 0 \tag{4.1}$$

for $q \neq p$. Now Proposition 3.2 (iv) implies that f^* is an isomorphism.

Conversely suppose that f^* is an isomorphism. Proposition 3.2 (iii) implies that $A \approx B$ and consequently A is quadratic, by Proposition 3.2 (ii). Thus A is the quadratic dual of U and besides (4.1) holds. This implies that U is Koszul whence A is also Koszul. □

Remark 4.5 Proposition 3.2 also implies that both statements of Proposition 4.4 are equivalent to A being quadratic and $H^*(M)$ being generated in degree 1.

5 Formal topological spaces

In this section we interpret the results of the previous section for topological spaces. A space X (connected and with finite Betti numbers) is called *formal* ([16], p.158) if it has the homotopy type of a simplicial complex K and the DeRham DGA $A_{PL}(K)$ of $\mathbf{Q}$-polynomial forms on the complex K is quasi-isomorphic to $A = H^*(X,\mathbf{Q})$ endowed with zero differential. Note that this coincides with the definition given in the Introduction, as follows e.g. from [7] , 12.2. (An enlighting discussion of this definition can be found in [18], pp.235-236.) Among the well-known examples of formal spaces are the compact Kähler manifolds [12] and the complements of complex hyperplane arrangements [9]. If X is formal then a 1-minimal model of A is also a 1-minimal model of $A_{PL}(K)$ and it is called a 1-minimal model of X. The holonomy Lie algebra $G = G(X)$ of A is then the $\mathbf{Q}$-version of the $\mathbf{R}$-algebra defined by Chen in [11] and it is called the holonomy Lie algebra of X.

Using Proposition 3.1 the holonomy algebra G of a formal space X has the following interpretation in terms of $\pi_1(X)$. Let Γ_n be the n-th term of the lower central series of $\pi = \pi_1(X)$, i.e. $\Gamma_1 = \pi$ and inductively $\Gamma_n = [\Gamma_{n-1}, \pi]$. The graded abelian group $\mathrm{gr}_\Gamma^*(\pi) =: \oplus_{n\geq 1}(\Gamma_n/\Gamma_{n+1})$ has a natural graded $\mathbf{Z}$-Lie algebra structure induced by the group commutator. Then one has a graded Lie algebra isomorphism $G_* \approx \mathrm{gr}_\Gamma^*(\pi_1(X)) \otimes \mathbf{Q}$, see e.g. [16] , Theorem 12.2. In particular we have $\dim G_n = \phi_n$, where $\phi_n = \mathrm{rank}(\Gamma_n/\Gamma_{n+1})$. Applying the Poincaré-Birkhoff-Witt theorem we obtain the following expression for the Hilbert series of the universal enveloping algebra $U = U(X)$

of G:

$$H(U,t)=\prod_{n\geq 1}(1-t^n)^{-\phi_n}\ . \tag{5.1}$$

The space X is a rational $K[\pi,1]$, in the sense of the definition given in the Introduction, if and only if its 1-minimal model M is a *minimal model* of X, i.e. $f^*: H^*(M)\to H^*(X,\mathbf{Q})$ is an isomorphism. See [7] , 12.8 (iii) for a proof. Applying Proposition 4.4 we obtain the following.

Theorem 5.1 *Suppose X is a formal topological space. Then X is a rational $K[\pi,1]$ if and only if $H^*(X,\mathbf{Q})$ is a Koszul algebra.*

The above result may be used to get certain information also for non-formal spaces X.

Proposition 5.2 *Let X be an arbitrary connected space with finite Betti numbers. If $H^*(X,\mathbf{Q})$ is a Koszul algebra then X is a rational $K[\pi,1]$.*

Proof. As mentioned before, the $K[\pi,1]$ property of X may be checked via the minimal model of X. It is known that one may associate to X a DGA $\mathcal{M}$, called the minimal model of X. As a commutative graded algebra $\mathcal{M}$ is freely generated by a graded $\mathbf{Q}$-vector space $W=\oplus_{n\geq 1}W^n$ (see e.g. [7] , 7.7 and 7.8, for the complete definitions and the precise result). For our purposes it will suffice to recall that X is a rational $K[\pi,1]$ if and only if $W^n=0$ for every $n>1$ ([7] , 12.8 (iii)).

Set now $A=H^*(X,\mathbf{Q})$. Our result will follow at once from the fact that W is in general related to A as follows. Firstly A has a so-called *bigraded model* $\mathcal{B}$, see [18] , Proposition 3.4. It is a DGA freely generated by a graded vector space $V=\oplus_{n\geq 1}V^n$, with several other defining properties. Secondly one may associate to X a new (perturbed) differential D on $\mathcal{B}$ and an induced differential D_ζ on V, see [18] , Theorem 4.4 and 4.14. Finally one knows ([18] 4.14) that there is an isomorphism of graded vector spaces $W^*\approx H^*(V,D_\zeta)$ (see [30] , Section 8 for a proof).

If A is Koszul we know from Proposition 4.4 that the 1-minimal model M constructed in Section 3 is actually the minimal model of A. It follows then from [24] , Lemma 1.8 (i) that M is in fact the bigraded model of A. Since the free generators V of M are concentrated by construction in degree one, the same property obviously will hold also for W and therefore X is a rational $K[\pi,1]$. $\square$

The next corollary was obtained in [24] , Lemma 4.5, by using the bigraded models of [18]. We present here a new proof, based on the Koszul duality formula.

Corollary 5.3 *If a formal space X is a rational $K[\pi, 1]$ then the LCS formula holds for X (see the Introduction).*

Proof. By Theorem 5.1 , $A = H^*(X, \mathbf{Q})$ is Koszul. Since U is the quadratic dual of A, Theorem 2.3 (iii) implies that

$$P(X, -t) = H(U, t)^{-1}$$

where $P(X, t)$ is the Poincaré polynomial of X. Then (5.1) completes the proof. □

The most intensively studied class of formal spaces consists of the complements of hyperplane arrangements. Let $\mathcal{A}$ be a finite set of linear hyperplanes in $\mathbf{C}^\ell$ and $X = \mathbf{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. Then $A = H^*(X, \mathbf{Q})$ can be embedded in the DGA of holomorphic differential forms on X (see [9]) whence X is formal. Besides A is determined by combinatorial data, namely by the intersection lattice of $\mathcal{A}$ (see [25]). The Hilbert series $H(A, t) = P(X, t)$ is a polynomial of degree ℓ.

The problems related to the topic of this paper were first studied in the arrangement setting in [2, 20, 21, 22, 14, 13]. Falk [13] studied conditions for X to be a rational $K[\pi, 1]$. He proved that X is a rational $K[\pi, 1]$ for every arrangement of so called fiber-type. This is a topologically defined class that coincides with the combinatorially defined class of supersolvable arrangements (see [25] for definitions). Falk [13] and Kohno [22] proved Corollary 5.3 for complements of hyperplane arrangements.

It is still unknown if X can be a rational $K[\pi, 1]$ for a not supersolvable arrangement or if the LCS formula can hold without X being a rational $K[\pi, 1]$. The following result solves the latter problem negatively in a low dimensional case.

Proposition 5.4 *If $\ell = 3$ and the lower central series formula holds for X then X is a rational $K[\pi, 1]$.*

In order to prove the proposition we need to bring into consideration affine arrangements. Let $\mathcal{A}$ be an arrangement in $\mathbf{C}^\ell$ and $H \in \mathcal{A}$. Pick $\alpha \in (\mathbf{C}^\ell)^*$ with kernel H. Then intersecting every $K \in \mathcal{A} \setminus \{H\}$ with the affine hyperplane $\tilde{H} = \{v \in \mathbf{C}^\ell | \alpha(v) = 1\}$ we obtain a set $\mathcal{A}'$ of affine hyperplanes in $\tilde{H}$. Put $X' = \tilde{H} \setminus \bigcup_{K \in \mathcal{A}'} K$, $A' = H^*(X', \mathbf{Q})$ and $U' = U(X')$. Set also $\phi'_n = \operatorname{rank}(\Gamma_n \pi_1(X')/\Gamma_{n+1}\pi_1(X'))$, $n \geq 1$.

One knows that X' is formal ([25] ,Theorem 5.90). It is also well known that X is homeomorphic to $X' \times (\mathbf{C}^*)$, see [25] , Proposition 5.1. This enables us to drop the dimension in the statement of Proposition 5.4 to 2, in particular

$$H(A', t) = 1 + mt + nt^2. \tag{5.2}$$

This also implies that $A \approx A' \otimes \wedge(e)$, with $\deg(e) = 1$.

Now we can infer the following two equivalences. On one hand X is a rational $K[\pi, 1]$ if and only if X' is a rational $K[\pi, 1]$. Indeed, denoting by $f' : (\wedge V', d') \to (A', 0)$ the 1-minimal model of A', we may infer from the Künneth theorem that $f =: f' \otimes id : (\wedge V' \otimes \wedge(e), d' \otimes id) \to (A, 0)$ will be the 1-minimal model of A. Hence f^* is an isomorphism if and only if f'^* is an isomorphism, as asserted.

On the other hand obviously $\phi_k = \phi'_k$, if $k > 1$, and $\phi_1 = \phi'_1 + 1$. Since $H(A, t) = H(A', t)(1 + t)$, it follows that the LCS formula holds for X if and only if it holds for X'. According to (5.1) and (5.2) the latter is equivalent to

$$H(U', t) = (1 - mt + nt^2)^{-1}.$$

Consequently Proposition 5.4 will follow from the simple lemma below, via Theorem 5.1.

Lemma 5.5 ([26], 3.4, Proposition 2.) *Let U' be a quadratic algebra with*

$$H(U', t) = (1 - mt + nt^2)^{-1}. \tag{5.3}$$

Then U' is Koszul.

Proof of the lemma. Let I be the ideal of U'. The condition (5.3) immediately implies that $m = \dim U'_1$ and $n = \dim I_2$. The natural complex

$$0 \to U' \otimes I_2 \to U' \otimes U'_1 \to U' \to F \to 0$$

is exact except maybe at $U' \otimes I_2$ for every quadratic algebra. The condition (5.3) implies that it is exact whence it is a graded free resolution of F. Thus $\mathrm{Ext}^i_{U'}(F, F) = 0$ for $i \geq 3$, which implies the result. $\square$

Proof of the proposition. The Koszul duality formula 2.3 (iii) together with (5.2) and (5.3) imply that A' and $(U')^!$ have the same Hilbert series. On the other hand one knows that A' is generated in degree one ([25] , Theorem 5.90) and therefore A' is a quotient of $(U')^!$, see Remark 4.2. Consequently $A' \approx (U')^!$ and A' is Koszul. $\square$

Remarks 5.6 The Koszul property for $A = H^*(X, \mathbf{Q})$ was first considered in [29] , where it was proved that A is Koszul for fiber-type arrangements. A substantial generalization of the fiber-type class, namely the hypersolvable class, was recently introduced in [19]. Among other things it was shown there ([19] , Theorem E) that the associated quadratic algebra $U^!$ (see 4.2) is always Koszul for a hypersolvable arrangement. Consequently a generalization of the LCS formula ([19] , Theorem C (ii)) holds for this new class. At the same time ([19] , Theorem D (iv),(v)) the implication of Proposition 5.4 holds for hypersolvable arrangements. Moreover one knows that the only rational $K[\pi, 1]$ hypersolvable complements are the fiber-type ones ([19] , Theorem D (ii),(iv)).

Falk in [13] asked the question if the quadraticity of A is sufficient for X being a rational $K[\pi, 1]$. The following example shows that this is not true.

Example 5.7 Consider the hyperplanes given in $\mathbf{C}^3 = \{(x, y, z)\}$ by the functionals $x, y, x+y, z, x-z, y-z, x+y-2z$. Then the algebra A is quadratic. (In fact it is Kohno's parallel arrangement discussed in [15] and [22]. It is mistakenly written in [22] that X is rational $K[\pi, 1]$ for this arrangement.) One can compute that $H(A', t) = 1+6t+10t^2$. If X is a rational $K[\pi, 1]$ then by Corollary 5.3 and (5.1) the series $(1-6t+10t^2)^{-1}$ equals $H(U', t)$ whence all its coefficients are nonnegative. This implies that the roots of $1-6t+10t^2$ are real as in the proof of [1] , Lemma 3.4, which is a contradiction.

References

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