

# COMPLETELY REDUCIBLE HYPERSURFACES IN A PENCIL

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**ABSTRACT.** We study completely reducible fibers of pencils of hypersurfaces on  $\mathbb{P}^n$  and associated codimension one foliations of  $\mathbb{P}^n$ . Using methods from theory of foliations we obtain certain upper bounds for the number of these fibers as functions only of  $n$ . Equivalently this gives upper bounds for the dimensions of resonance varieties of hyperplane arrangements. We obtain similar bounds for the dimensions of the characteristic varieties of the arrangement complements.

## 1. INTRODUCTION

In this paper we focus on completely reducible fibers of a pencil of hypersurfaces on  $\mathbb{P}^n$  with irreducible generic fiber. Our main result (Theorem 2.1) gives an upper bound for the number  $k$  of these fibers that depends only on  $n$ . For instance for every  $n > 1$  we obtain  $k \leq 5$ . Or for every  $n \geq 4$  a pencil with  $k$  at least 3 is a linear pull-back of a pencil on  $\mathbb{P}^4$ .

Despite the evident classical taste of the result we have not found it in the literature although various restrictions on reducible fibers and even more special completely reducible fibers were studied in [17, 7, 8].

In order to prove the result we use a combination of techniques from the theory of codimension one singular foliations on  $\mathbb{P}^n$  and the theory of hyperplane arrangements. In particular we define a foliation associated with a pencil and consider its Gauss map

$$\mathbb{P}^n \dashrightarrow (\mathbb{P}^n)^\vee.$$

Our most technical result (Theorem 4.1) says that for a pencil with  $k \geq 3$  the Gauss map of the associated foliation is dominant.

It turns out that the union of the linear divisors of all completely reducible fibers of a pencil with  $k$  at least three can be characterized intrinsically, by means of arrangement theory. In fact this characterization in terms of resonance varieties of hyperplane arrangements is contained in the recent paper [5] and this was the starting point of our study. We recall this characterization in a concise form in Theorem 5.1. This and Theorem 2.1 give upper bounds on the dimensions of the resonance varieties of arrangements whence on the dimension of cohomology of the Orlik-Solomon algebras.

Much of the interest in the resonance varieties comes from the fact that they are closely related to the support loci for the cohomology of local systems on arrangement complements - the so called characteristic varieties. These relations have been studied by many authors, see for instance [3],[9]. In a recent preprint [4] Dimca cleared some subtle points in these relations as well as in the relations of characteristic components with pencils. Using this we are able to find upper

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bounds for dimensions of these components too (Theorem 7.2). In order to do this in full generality (at least for the positive dimensional components) we need to generalize Theorem 4.1 to more general pencils having at least two completely reducible fibers and another fiber with irreducible components either of degree one or non-reduced. Since this condition on pencils does not look natural we do not include this generalization (Theorem 7.1) in Theorem 4.1.

We do not know if in all the cases the upper bounds given in Theorems 4.1 and 7.1 are strict. In section 6 we collect some old and new examples of pencils on  $\mathbb{P}^n$  ( $n > 1$ ) with at least three completely reducible fibers.

## 2. MAIN RESULT AND REDUCTION

**2.1. Completely reducible fibers.** Let  $F$  and  $G$  be polynomials of the same degree  $d > 0$  from  $\mathbb{C}[x_0, x_1, \dots, x_n]$  defined up to nonzero multiplicative constants. They define the pencil  $\mathcal{P} = \{aF + bG \mid [a : b] \in \mathbb{P}^1\}$  whose fibers  $aF + bG$  can be identified with hypersurfaces in  $\mathbb{P}^n$ . We are looking for upper bounds for the number of the completely reducible fibers, i.e., products of linear forms. Without loss of generality we can and will always assume that  $F$  and  $G$  are relatively prime (equivalently, the generic fiber of  $\mathcal{P}$  is irreducible) and  $F$  and  $G$  are completely reducible themselves, whence there are at least two completely reducible fibers of  $\mathcal{P}$ . In the trivial case where  $d = 1$ , all fibers of the pencil  $\mathcal{P}$  are hyperplanes whence completely reducible.

The following theorem is the main result of the paper.

**Theorem 2.1.** *If  $\mathcal{P}$  is a pencil of hypersurfaces on  $\mathbb{P}^n$  with irreducible generic fiber and  $k$  is the number of completely reducible fibers of  $\mathcal{P}$  then the following assertions hold*

- (1) *If  $k > 5$  then  $\mathcal{P}$  is a pencil of hyperplanes (equivalently, it is the linear pull-back of a pencil on  $\mathbb{P}^1$ );*
- (2) *If  $k > 3$  then  $\mathcal{P}$  is the linear pull-back of a pencil on  $\mathbb{P}^2$ ;*
- (3) *If  $k > 2$  then  $\mathcal{P}$  is the linear pull-back of a pencil on  $\mathbb{P}^4$ .*

**2.2. Reduction.** The proof of Theorem 2.1 consists of two parts. In this subsection we present an elementary reduction to the non-vanishing of a certain determinant. The rest of the proof requires more techniques and will be given in the following sections.

We are using the notation from the theorem. If  $k \leq 2$  there is nothing to prove. So we assume that  $k \geq 3$ .

Let  $\tilde{Q} = \prod \alpha_i^{m_i}$  denotes the product of the completely reducible fibers of  $\mathcal{P}$  and  $Q = \prod \alpha_i$  denotes the reduced polynomial with the same zero set. Then consider the 1-form

$$\omega = \frac{Q}{\tilde{Q}} \omega_0$$

where  $\omega_0 = FdG - GdF$ .

We will need the following properties of  $\omega$ .

- (i)  $\omega$  does not depend (up to a nonzero multiplicative constant) on the choice of  $F$  and  $G$  from the set of completely reducible fibers of  $\mathcal{P}$ .

A proof is left to the reader.

- (ii)  $\omega$  is a polynomial form.

It suffices to check that  $\alpha_i^{m_i-1}$  (for every  $i$ ) divides the coefficients of  $\omega_0$ . If  $\alpha_i$  divides  $F$  (or  $G$ ) this is clear; otherwise it follows from (i).

For the future use we write  $\omega = \sum_{i=1}^{n-1} a_i dx_i$  for some  $a_i \in \mathbb{C}[x_0, \dots, x_n]$  and denote by  $D$  the determinant of the Jacobi matrix  $\left(\frac{da_i}{dx_j}\right)$ .

(iii) For every linear factor  $\alpha_i$  of  $Q$  we have  $\alpha_i$  divides the coefficients of  $d\alpha_i \wedge \omega$ .

*Proof.* It suffices to check that  $\alpha_i^{m_i}$  divides  $d\alpha_i \wedge \omega_0$ . Again if  $\alpha_i$  divides  $F$  one needs to check that  $\alpha_i^{m_i}$  divides  $d\alpha_i \wedge dF$  which is clear. Otherwise one can apply (i).  $\square$

(iv)  $Q^{n-1}$  divides  $D$ .

*Proof.* It suffices to prove that  $\alpha_i^{n-1}$  divides  $D$  for every  $i$ . Fix  $i$  and change the coordinates so that  $\alpha_i = x_0$ . Applying property (iii) we see that  $x_0$  divides  $a_j$  for  $j = 1, 2, \dots, n$ . In particular, except for the entries of the first row and the first column, all the entries of the matrix  $\left(\frac{\partial a_i}{\partial x_k}\right)$  are divisible by  $x_0$ . Using the cofactor expansion of  $D$  with respect to the first row completes the proof.  $\square$

Now we can prove that the non-vanishing of  $D$  would imply Theorem 2.1.

**Proposition 2.1.** *Suppose  $D$  is not identically 0. Then Theorem 2.1 follows.*

*Proof.* It follows from property (iv) of  $\omega$  that  $\tilde{Q}^{n-1}$  divides

$$\left(\frac{\tilde{Q}}{Q}\right)^{n-1} D.$$

Since  $\deg(\tilde{Q}) = k \deg(F) = kd$  we obtain that

$$\begin{aligned} (n-1)kd &\leq (n+1) \left( 2d - 2 - \deg\left(\frac{\tilde{Q}}{Q}\right) \right) + (n-1) \deg\left(\frac{\tilde{Q}}{Q}\right) \\ &= (n+1)(2d-2) - 2 \deg\left(\frac{\tilde{Q}}{Q}\right). \end{aligned}$$

Therefore

$$k \leq \frac{n+1}{n-1} \left( 2 - \frac{2}{d} \right) - \frac{2}{(n-1)d} \deg\left(\frac{\tilde{Q}}{Q}\right) < 2 \frac{n+1}{n-1}.$$

In particular

$$\begin{aligned} n \geq 5 &\text{ implies } k \leq 2, \\ n \geq 3 &\text{ implies } k \leq 3, \\ n \geq 2 &\text{ implies } k \leq 5. \end{aligned}$$

Theorem 2.1 follows.  $\square$

To prove that the polynomial  $D$  is not 0 we will use foliations on  $\mathbb{P}^n$  and some properties of the hyperplane multi-arrangement  $\tilde{Q} = 0$ .

### 3. THE MULTINET PROPERTY AND ITS GENERALIZATION

First in this section we recall certain results from [5] about the multi-arrangement  $(\mathcal{A}, m)$  defined by  $\tilde{Q} = 0$ . We remark that we recall here only the facts about arrangements that are needed for the proof of Theorem 2.1. For more details see section 5.

The multi-arrangement  $(\mathcal{A}, m)$  consists of the set  $\mathcal{A}$  of hyperplanes  $H_i$  in  $\mathbb{P}^n$  determined by  $\alpha_i = 0$  where  $\alpha_i$  is running through all linear divisors of  $Q$ . Besides to each  $H_i$  the positive integer  $m(H_i) = m_i$  is assigned where  $m_i$  is the exponent of  $\alpha_i$  in  $\tilde{Q}$ .

First we consider the case where  $n = 2$ . In this case, it was proved in [5] that the partition

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$$

of  $\mathcal{A}$  into completely reducible fibers (called classes in [5]) of the pencil  $\mathcal{P}$  can be equivalently characterized by the combinatorics of lines and points. In particular

- (a)  $\sum_{H \in \mathcal{A}_i} m(H)$  is independent of  $i = 1, \dots, k$ ;
- (b) If  $p$  is the point in the base locus of  $\mathcal{P}$  then the sum

$$n(p) = \sum_{H \in \mathcal{A}_i, p \in H} m(H)$$

is independent of  $i = 1, \dots, k$ .

The collection  $(\mathcal{A}, \mathcal{X})$  of lines and points satisfying conditions (a)-(d) is called a  $k$ -multinet. If  $n(p) = 1$  for all  $p \in \mathcal{X}$  (whence  $m(H) = 1$  for all  $H \in \mathcal{A}$ ) then it is a  $k$ -net.

Intersecting an arrangement in  $\mathbb{P}^n$  with a general position plane  $\mathbb{P}^2$  one readily sees that the similar properties hold for arrangements of hyperplanes in  $\mathbb{P}^n$  where the intersections of codimension 2 should be substituted for points. This definition of multinets in  $\mathbb{P}^n$  for  $n > 2$  does not say anything about the intersections of higher codimensions of hyperplanes in a multinet. We can prove and then use a property of these intersections for  $k \geq 3$ . In fact we prove this property under a weaker assumption that we will use in section 7.

**Proposition 3.1.** *Let  $\mathcal{P}$  be a pencil of hypersurfaces on  $\mathbb{P}^n$  with irreducible generic fiber generated by two completely reducible fibers  $F$  and  $G$ . If there exists a third fiber that is a product of linear forms and non-reduced polynomials then*

$$\sum_{\alpha_H | F, p \in H} m(H) = \sum_{\alpha_H | G, p \in H} m(H)$$

for every  $p$  in the base locus of  $\mathcal{P}$ .

*Proof.* Let  $p$  be a point in the base locus of the pencil. Choose affine coordinates  $(x_1, \dots, x_n)$  where  $p$  is the origin and write  $F = F_1 \cdot F_2$  and  $G = G_1 \cdot G_2$  where  $F_2, G_2 \notin \mathfrak{m}$  and all the irreducible components of  $F_1$  and  $G_1$  are in  $\mathfrak{m}$ ,  $\mathfrak{m}$  being the maximal ideal  $(x_1, \dots, x_n)$ . The statement of the lemma is equivalent to the homogeneous polynomials  $F_1$  and  $G_1$  having the same degree.

Our hypothesis implies that there exists a hyperplane or a non-reduced hypersurface in the pencil passing through 0. In the former case, there is a linear form  $\alpha \in \mathbb{C}[x_1, \dots, x_n]$  that divides, say  $K = F - G$ . Thus  $\alpha$  divides  $K_0 = F_2(0)F_1 - G_2(0)G_1$ . If  $\deg F_1 \neq \deg G_1$  then  $\alpha$  divides, say  $F_1$  which is a contradiction.

In the latter case, there is an irreducible polynomial  $f \in \mathfrak{m}$  such that  $f^m$  divides  $K$  for some  $m > 1$  whence  $f$  divides the coefficients of  $\omega = FdG - GdF = KdF - FdK$ . Now put  $R = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$ , i.e.,  $R$  is the radial (or Euler) vector field in  $\mathbb{C}^n$ , and denote by  $i_R$  the interior product of a form with it. Then the Leibniz formula implies that

$$i_R \omega = F_2 G_2 i_R \underbrace{(F_1 dG_1 - G_1 dF_1)}_{\omega_1} + F_1 G_1 i_R (F_2 dG_2 - G_2 dF_2).$$

To prove the lemma it suffices to show that  $i_R \omega_1 = 0$ .

Suppose this is not true. Then we have  $i_R \omega_1 = c F_1 G_1$  for a  $c \in \mathbb{C}^*$  and  $i_R \omega = F_1 G_1 g$  where  $g$  is a polynomial in  $x_1, \dots, x_n$  such that  $g(0) \neq 0$ . Since  $f$  divides the polynomial  $i_R \omega$  it divides  $F_1 G_1$  which is again a contradiction.  $\square$

Sometimes it is convenient to assume that the pencil  $\mathcal{P}$  is not a linear pull-back from a smaller dimensional projective space. We will say in this case that  $\mathcal{P}$  is *essential* in  $\mathbb{P}^n$ . This is equivalent to  $\mathcal{A}$  being *essential*, i.e.,  $\bigcap_{H \in \mathcal{A}} H = \emptyset$ . This can be expressed also by saying that the rank of  $\mathcal{A}$  is  $n$  where rank is the codimension of  $\bigcap_{H \in \mathcal{A}} H$  (see section 5). It is immediate from the multinet property that  $\mathcal{A}$  is essential if and only if the arrangement defined by all the linear divisors of  $F$  and  $G$  is essential.

We will need the following property of essential arrangements that immediately follows from definitions.

**Proposition 3.2.** *Let  $\mathcal{A}$  be the collection of the linear divisors of all the completely reducible fibers of an essential pencil  $\mathcal{P}$  on  $\mathbb{P}^n$ . Then there exist two distinct points  $p_1, p_2 \in \mathbb{P}^n$  such that for  $j = 1, 2$  the subarrangements*

$$\mathcal{B}_j = \bigcup_{p_j \in H, H \in \mathcal{A}} H$$

*have rank  $n - 1$ .*

#### 4. FOLIATIONS AND THE GAUSS MAP

**4.1. Foliations.** In this paper we will adopt an utilitarian definition for codimension one singular foliations on  $\mathbb{P}^n$ , from now on just foliation on  $\mathbb{P}^n$ . A *foliation*  $\mathcal{F}$  on  $\mathbb{P}^n$  will be an equivalence class of homogeneous rational differential 1-forms on  $\mathbb{C}^{n+1}$  under the equivalence relation

$$\omega \sim \omega' \text{ if and only if there exists } h \in \mathbb{C}[x_0, \dots, x_n] \setminus 0 \text{ for which } \omega = h\omega',$$

such that  $i_R \omega = 0$  and  $\omega \wedge d\omega = 0$  for every representative  $\omega$ . Here  $R$  and  $i_R$  are similar to the ones used in the proof of Proposition 3.1 but in  $\mathbb{C}^{n+1}$ . Of course, to ensure the validity of the two conditions for every representative it is sufficient to check it just for one of them.

Among the representatives of  $\mathcal{F}$  there are privileged ones — the homogeneous polynomial 1-forms with singular, i.e. vanishing, set of codimension at least two. Any two such forms that are equivalent differ by a nonzero multiplicative constant. If such a form has coefficients of degree  $d + 1$  then we say that  $\mathcal{F}$  is a degree  $d$  foliation. The shift in the degree is motivated by the geometric interpretation of the degree. It is the number of tangencies between  $\mathcal{F}$  and a generic line in  $\mathbb{P}^n$ .

Outside the singular set the well-known Frobenius Theorem ensures the existence of local submersions with connected level sets whose tangent space at a point is the

kernel of a defining 1-form at this point. These level sets are the local leaves of  $\mathcal{F}$ . The leaves are obtained by patching together level sets of distinct submersions that have nonempty intersection. Although the data is algebraic the leaves, in general, have a transcendental nature.

Now we show how foliations appear from completely reducible fibers of a pencil of hypersurfaces on  $\mathbb{P}^n$ .

**Lemma 4.1.** *Let  $\mathcal{P}$  be the pencil on  $\mathbb{P}^n$  generated by polynomials  $F$  and  $G$  and  $\omega$  the 1-form from 2. Then  $\omega$  defines a foliation on  $\mathbb{P}^n$ .*

*Proof.* One needs to check the two conditions from definition of foliations for  $\omega$  or equivalently, for the closed form  $\eta = \frac{dG}{G} - \frac{dF}{F}$ . The condition  $i_R\eta = 0$  follows immediately since  $F$  and  $G$  are homogeneous polynomials of equal degrees, while the integrability condition is automatically satisfied thanks to the closedness of  $\eta$ .  $\square$

The foliation defined by  $\omega$  will be called the foliation *associated to  $\mathcal{P}$* .

**4.2. Gauss map.** Let  $\omega$  be a homogeneous polynomial differential 1-form on  $\mathbb{C}^{n+1}$  such that  $i_R\omega = 0$  and  $\omega \wedge d\omega = 0$ . Let  $\mathcal{F}$  be the foliation defined by  $\omega$  on  $\mathbb{P}^n$ . The Gauss map of  $\mathcal{F}$  is the rational map

$$\begin{aligned} \mathcal{G}_\omega = G_\mathcal{F} : \mathbb{P}^n &\dashrightarrow (\mathbb{P}^n)^\vee \\ p &\mapsto T_p\mathcal{F}. \end{aligned}$$

that takes every point  $p \in \mathbb{P}^n \setminus \text{sing}(\mathcal{F})$  to the hyperplane tangent to  $\mathcal{F}$  at  $p$ . Under a suitable identification of  $\mathbb{P}^n$  with  $(\mathbb{P}^n)^\vee$  the Gauss map  $\mathcal{G}_\omega$  is nothing more than the rational map defined in homogeneous coordinates by the coefficients of  $\omega$ .

We say that a foliation  $\mathcal{F}$  has degenerate Gauss map when  $\mathcal{G}_\mathcal{F}$  is not dominant, i.e., its image is not dense in  $(\mathbb{P}^n)^\vee$ . On  $\mathbb{P}^2$  a foliation with degenerate Gauss map has to be a pencil of lines. Indeed the restriction of the Gauss map to a leaf of the foliation coincides with the Gauss map of the leaf and the only (germs of) curves on  $\mathbb{P}^2$  with degenerate Gauss map are (germs of) lines. Thus all the leaves of a foliation with degenerate Gauss map are open subsets of lines. In order for this foliation not to have every point singular, these lines should intersect at one point.

On  $\mathbb{P}^3$  the situation is more subtle and a complete classification can be found in [1]. Some results toward the classification of foliations with degenerate Gauss map on  $\mathbb{P}^4$  have been recently obtained by T. Fassarella [6].

In our special case where a foliation is associated to the pencil  $\mathcal{P}$  we can prove that the Gauss map cannot degenerate.

**Theorem 4.1.** *Let  $\mathcal{P}$  be an essential pencil of hypersurfaces on  $\mathbb{P}^n$  with at least three completely reducible fibers. Then the Gauss map of the associated foliation  $\mathcal{F}$  is non-degenerate.*

*Proof.* We again denote by  $\mathcal{A}$  the arrangement defined by the linear divisors of all  $k$  completely reducible fibers of  $\mathcal{P}$  and let  $\mathcal{A} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$  be its partition into fibers. We assume that  $\mathcal{A}_1$  and  $\mathcal{A}_2$  correspond to  $F$  and  $G$  respectively. For each  $H \in \mathcal{A}$  we denote by  $m(H)$  the multiplicity of the respective linear form  $\alpha_H$  in the its fiber.

Now we use induction on  $n$ . Suppose  $n = 2$ . The only way to have a pencil of lines as the foliation associated to  $\mathcal{P}$  is for  $\mathcal{A}$  to be a pencil of lines itself. But a

pencil of lines is not essential in  $\mathbb{P}^2$ . Thus the Gauss map is not degenerate in this case.

Suppose that the result holds for essential pencils in  $\mathbb{P}^n$  and let  $\mathcal{P}$  be an essential pencil in  $\mathbb{P}^{n+1}$ . The foliation associated to  $\mathcal{P}$  can be thus defined by the 1-form

$$\omega_0 = FdG - GdF,$$

where

$$F = \prod_{H \in \mathcal{A}_1} \alpha_H^{m(H)} \quad \text{and} \quad G = \prod_{H \in \mathcal{A}_2} \alpha_H^{m(H)}.$$

Fix two points  $p_1, p_2 \in \mathbb{P}^{n+1}$  with the property from Proposition 3.2 and let  $\pi_1 : \widetilde{\mathbb{P}^{n+1}} \rightarrow \mathbb{P}^{n+1}$  be the blow-up of  $\mathbb{P}^{n+1}$  at  $p_1$ . Using notation from Proposition 3.2 the restriction to exceptional divisor  $E_1 \cong \mathbb{P}^n$  of the strict transforms of the hyperplanes in  $\mathcal{B}_1$  induces a non-degenerate arrangement of hyperplanes in  $E_1$  which we will still denote by  $\mathcal{B}_1$ .

If  $\iota : E_1 \rightarrow \widetilde{\mathbb{P}^{n+1}}$  is the natural inclusion then we claim that the closure of the image of the rational map  $\sigma_1 = \mathcal{G}_{\mathcal{F}} \circ \pi_1 \circ \iota$  (see the diagram) is the hyperplane  $H^{(1)}$  in  $(\mathbb{P}^{n+1})^\vee$  dual to  $p_1$ .

$$\begin{array}{ccc} E_1 \cong \mathbb{P}^n & \xrightarrow{\iota} & \widetilde{\mathbb{P}^{n+1}} \\ & & \downarrow \pi_1 \\ & & \mathbb{P}^{n+1} \end{array} \quad \begin{array}{c} \xrightarrow{\mathcal{G}_{\mathcal{F}} \circ \pi_1} \\ \xrightarrow{\mathcal{G}_{\mathcal{F}}} \end{array} \quad \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\quad} \end{array} \quad (\mathbb{P}^{n+1})^\vee$$

Indeed the arrangement  $\mathcal{B}_1 \subset E_1$  admits a partition

$$\mathcal{B}_1 = \bigcup_{i=1}^k \mathcal{B}_1 \cap \mathcal{A}_i$$

and a function  $m_1 = m|_{\mathcal{B}_1}$  satisfies the multinet properties with the initial multiplicities and the induced partition into classes. This follows from Propositions 3.1 and 3.2. Now the equivalence from section 3 implies that  $\prod_{H \in \mathcal{A}_i \cap \mathcal{B}_1} \alpha_H^{m(H)}$  is a fiber of the pencil  $\mathcal{P}_1$  on  $E_1$  generated by

$$F_1 = \prod_{H \in \mathcal{A}_1 \cap \mathcal{B}_1} \alpha_H^{m(H)} \quad \text{and} \quad G_1 = \prod_{H \in \mathcal{A}_2 \cap \mathcal{B}_1} \alpha_H^{m(H)}.$$

Now consider the associated foliations. On one hand the foliation  $\mathcal{F}_1$  on  $\mathbb{P}^n$  associated to  $\mathcal{P}_1$  can be defined by the 1-form  $F_1 dG_1 - G_1 dF_1$ .

On the other hand the first nonzero jet of  $FdG - GdF$  at  $p_1$  is  $F_1 dG_1 - G_1 dF_1$  viewing now  $\alpha_H$  as linear forms on  $\mathbb{C}^{n+2}$ . More precisely let us assume that  $p_1 = [0 : \dots : 0 : 1]$  and write  $F = F_1 F_2$ ,  $G = G_1 G_2$ . Then the map  $\mathcal{G}_{\mathcal{F}}$  (after division by  $F_2(p_1)G_2(p_1)$ ) can be written in the homogeneous coordinates  $[x_0 : x_1 : \dots : x_n : x_{n+1}]$  of  $\mathbb{P}^{n+1}$  as

$$\begin{aligned} \mathcal{G}_{\mathcal{F}} &= \left[ F_k \frac{\partial G_1}{\partial x_0} - G_1 \frac{\partial F_1}{\partial x_0} + b_0 : \dots : F_1 \frac{\partial G_1}{\partial x_n} - G_1 \frac{\partial F_1}{\partial x_n} + b_n : b_{n+1} \right] \\ &= \left[ F_1 dG_1 - G_1 dF_1 + \sum_{i=0}^{n+1} b_i dx_i \right], \end{aligned}$$

where  $b_i \in \mathfrak{m}^{2d}$  ( $d = \deg F$ ) for  $i = 0, \dots, n, n+1$ ,  $\mathfrak{m}$  being the maximal ideal of  $\mathbb{C}[x_0, \dots, x_n]$  supported at  $0 \in \mathbb{C}^{n+1}$ .

If we consider now  $(x_0 : \dots : x_n) \in \mathbb{P}^n$  as a homogenous system of coordinates on the exceptional divisor  $E_1$  then in these coordinates

$$\sigma_1 = \left[ F_1 \frac{\partial G_1}{\partial x_0} - G_1 \frac{\partial F_1}{\partial x_0} : \dots : F_1 \frac{\partial G_1}{\partial x_n} - G_1 \frac{\partial F_1}{\partial x_n} \right] = [F_1 dG_1 - G_1 dF_1]$$

since the coefficients of  $F_1 dG_1 - G_1 dF_1$  lie in  $\mathfrak{m}^{2d-1}$ . Thus  $\sigma_1$  can be identified with the Gauss map of the foliation  $\mathcal{F}_1$  composed with  $\phi$ , where  $\phi$  is the isomorphism of  $(\mathbb{P}^n)^\vee$  with  $H^{(1)}$  defined by the coordinates.

Using also the point  $p_2$  and applying the inductive hypothesis we have now that the closure of the image of  $\mathcal{G}_{\mathcal{F}}$  contains at least two distinct hyperplanes. Since  $\mathbb{P}^{n+1}$  is irreducible the closure of the image of  $\mathcal{G}_{\mathcal{F}}$  must be  $(\mathbb{P}^{n+1})^\vee$ . This completes the proof.  $\square$

Now Theorem 4.1 and Proposition 2.1 constitute a proof of Theorem 2.1.

**4.3. Invariant hyperplanes.** As another application of the Gauss map we can exhibit an upper bound on the number of hyperplanes invariant with respect to a foliation in  $\mathbb{P}^n$ .

Let  $\mathcal{F}$  be a foliation on  $\mathbb{P}^n$  defined by a polynomial 1-form  $\omega = \sum_{i=0}^n a_i dx_i$ . Recall that the invariance of a hyperplane  $H$  defined by a linear form  $\alpha_H$  with respect to  $\mathcal{F}$  means that  $\alpha_H$  divides all coefficients of the form  $\omega \wedge d\alpha_H$ , i.e., the property (iii) from subsection 2.2 holds for  $\omega$  and  $\alpha_H$ . Then the property (iv) holds also, i.e.,  $\alpha_H^{n-1}$  divides the determinant  $D$  of the matrix  $\left( \frac{\partial a_i}{\partial x_j} \right)$ . Taking as  $\omega$  a polynomial form without a codimension one zero we have  $\deg D = (n+1) \deg \mathcal{F}$ . This gives the following proposition.

**Proposition 4.1.** *If  $\mathcal{F}$  is a foliation on  $\mathbb{P}^n$  with the non-degenerate Gauss map then the number of invariant hyperplanes is at most*

$$\left( \frac{n+1}{n-1} \right) \cdot \deg(\mathcal{F}).$$

**Example 4.1.** Here is an example showing that the bound in Proposition 4.1 is sharp. Let  $\mathcal{F}$  be a foliation on  $\mathbb{P}^n$ ,  $n \geq 2$ , induced by a logarithmic 1-form

$$\sum_{i=0}^n \lambda_i \frac{dx_i}{x_i}$$

where  $\sum \lambda_i = 0$  and no  $\lambda_i$  is equal to zero. Then  $\mathcal{F}$  has degree  $(n-1)$  and its Gauss map is non-degenerate. Also  $\mathcal{F}$  leaves invariant all the  $n+1$  hyperplanes of the arrangement.

For  $n = 2, 3$  there are examples of foliations  $\mathcal{F}$  on  $\mathbb{P}^n$  with  $\deg \mathcal{F} > n-1$  and exactly  $\frac{n+1}{n-1} \deg(\mathcal{F})$  invariant hyperplanes. On  $\mathbb{P}^2$  we are aware of three sporadic examples: Hesse pencil, Hilbert modular foliation [11], and a degree 7 foliation leaving invariant the extended Hessian arrangement of all the reflection hyperplanes of the reflection group of order 1296 - see [12], p. 227. Also there is one infinite

family [13], [5, Example 4.6], consisting of degree  $m$ ,  $m \geq 2$ , foliations leaving invariant the arrangement

$$xyz(x^{m-1} - y^{m-1})(x^{m-1} - z^{m-1})(z^{m-1} - y^{m-1}).$$

On  $\mathbb{P}^3$  we are aware of just one example with  $\deg \mathcal{F} > n - 1$  attaining the bound, see 6.2 below.

## 5. CHARACTERIZATION OF THE UNION OF COMPLETELY REDUCIBLE FIBERS

**5.1. Hyperplane arrangements.** First we need to recall some facts about hyperplane arrangements. Let  $\mathcal{A} = \{H_1, \dots, H_m\}$  be an arrangement of linear hyperplanes in  $\mathbb{C}^n$ . Recall that the rank of  $\mathcal{A}$  is the codimension of  $\bigcap_i H_i$ . In particular if the rank is  $n$  the arrangement is essential. Let  $M = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$  be the complement of  $\mathcal{A}$ . The cohomology ring  $H^*(M)$  is well-known (for instance, see [12]). In particular it does not have torsion and working with complex coefficients does not lose any generality.

Put  $A = H^*(M, \mathbb{C})$  and as before fix for each  $i = 1, 2, \dots, m$  a linear form  $\alpha_i$  with the kernel  $H_i$ . By the celebrated Arnold-Brieskorn theorem (e.g., see [12]) the algebra  $A$  is isomorphic under the deRham map to the subalgebra of the algebra of the closed differential forms generated by  $\{\frac{d\alpha_i}{\alpha_i}, 1 = 1, 2, \dots, m\}$ . Notice that  $A^1$  is a linear space of dimension  $m$ .

Since  $A$  is graded commutative each  $a \in A^1$  induces a cochain complex

$$(A, a): \quad 0 \rightarrow A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \dots \rightarrow A^k \rightarrow A^{k+1} \rightarrow 0$$

where the differential is defined as the multiplication by  $a$ . The degree  $l$  *resonance variety*  $\mathcal{R}^l(\mathcal{A})$  is

$$\mathcal{R}^l(\mathcal{A}) = \{[a] \in \mathbb{P}(A^1) \cong \mathbb{P}^{m-1} \mid H^l(A, a) \neq 0\}.$$

In this paper we will need only the first resonance variety  $\mathcal{R}^1(\mathcal{A})$ . If  $\mathcal{A}' \subset \mathcal{A}$  is a subarrangement then by definition  $\mathcal{R}^1(\mathcal{A}') \subset \mathcal{R}^1(\mathcal{A})$ . The *support* of an irreducible component  $\Sigma$  of  $\mathcal{R}^1(\mathcal{A})$  is the smallest subarrangement  $\mathcal{A}' \subset \mathcal{A}$  such that  $\Sigma \subset \mathcal{R}^1(\mathcal{A}')$ . For us the rank of the support of an irreducible component is important. The irreducible components will be called *global* if the rank of its support equals the rank of  $\mathcal{A}$ . In the rest of the paper we call irreducible components of  $\mathcal{R}^1$  *resonance components*. It is well-known (see [3]) that the resonance components are linear subspaces of  $A^1$ .

Since we study pencils in projective spaces we need to projectivize linear arrangements, i.e., to deal with arrangements of hyperplanes in the projective space  $\mathbb{P}^n$ . We still call it essential if its linear cone is such. More explicitly this means that the intersection of all hyperplanes is empty. According to Proposition 3.2 there are at least two points among intersections of hyperplanes and the subarrangement of the hyperplanes passing through any of those points is isomorphic to a linear arrangement of rank  $n$ . We say that the rank of the essential projective arrangement is  $n$ . More generally the rank of an arbitrary projective arrangement is the rank of its linear cone minus one. For the resonance varieties of a projective arrangement we still use those of the linear cone of it.

**5.2. Pencils and resonance components.** In this subsection we give a characterization of the unions of completely reducible fibers of pencils of hypersurfaces on  $\mathbb{P}^n$  in terms of resonance components. We also interpret the partition of such an arrangement into fibers and give a corollary of our main result for the dimensions of the resonance components of arrangements.

The first part is not really new; the following theorem is just a reinterpretation of results from [5], in particular Corollary 3.12. Although these results have been proved there only for pencils in  $\mathbb{P}^2$  they can be immediately generalized to arbitrary dimensions using intersection with a general position plane.

**Theorem 5.1.** *Let  $\mathcal{A}$  be a projective arrangement in  $\mathbb{P}^n$ . The following are equivalent:*

- (i)  $\mathcal{A}$  supports a resonance component of dimension  $k - 1$  ( $k \geq 3$ );
  - (ii) There is a pencil  $\mathcal{P}$  on  $\mathbb{P}^n$  with  $k$  completely reducible fibers such that  $\mathcal{A}$  is the union of the zero loci of the linear divisors of all  $k$  completely reducible fibers.
- Moreover assume (ii) holds and  $F_1, F_2, \dots, F_k$  are the completely reducible fibers of  $\mathcal{P}$ . Then the 1-forms

$$\omega_i = \frac{dF_i}{F_i} - \frac{dF_k}{F_k}, \quad i = 1, 2, \dots, k - 1,$$

form a basis of the corresponding resonance component of  $\mathcal{A}$ .

Now Theorem 5.1 and our main result Theorem 2.1 give the upper bounds on the dimensions of resonance components in terms of ranks of their supports.

**Corollary 5.1.** *If  $\mathcal{A}$  is an arrangement of hyperplanes in  $\mathbb{P}^n$  and  $\Sigma \subset \mathbb{P}(A^1)$  is an irreducible component of  $\mathcal{R}^1(\mathcal{A})$  then the following assertions hold:*

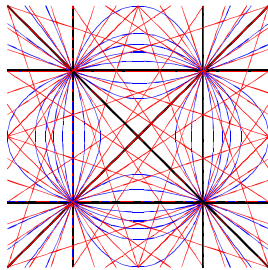
- (1) If  $\dim \Sigma > 3$  then the rank of the support of  $\Sigma$  is one;
- (2) If  $\dim \Sigma > 1$  then the rank of the support of  $\Sigma$  is at most two;
- (3) If  $\dim \Sigma > 0$  then the rank of the support of  $\Sigma$  is at most four.

If the rank of a resonance component is one (for projective arrangement) then the component is called *local* and is very simple. The result (1) of the corollary says that the (projective) dimension of non-local component is at most 3. This in turn implies that the dimension of  $H^1(A, a)$  is at most 3 for  $a$  not from a local component. This result has been proved in [9] for nets and in [5] for multinefs with all multiplicities of lines equal 1. Roughly speaking the results (2,3) of Corollary 5.1 say that the non-triviality of resonance varieties is a low-dimensional phenomenon.

Combining Theorem 5.1 with lemma 4.1 we see that for every resonance component there is a foliation on  $\mathbb{P}^n$ . More directly the irreducible components of  $\mathcal{R}^1(\mathcal{A})$  are precisely the projectivization of maximal linear subspaces  $E \subset A^1$  with dimension at least 2 and isotropic with respect to the product  $A^1 \times A^1 \rightarrow A^2$ , cf. [9, Corollaries 3.5, 3.7]. In particular, if  $E$  is one of these subspaces then all the homogeneous rational 1-forms, whose cohomology classes belong to  $E$ , are proportional over rational functions whence correspond to the same foliation. Moreover the foliation in question admits a rational first integral  $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$  of a rather special kind. If we write  $F = \frac{A}{B}$ , where  $A, B$  are relatively prime homogenous polynomials, then  $sA + tB$  is irreducible for generic  $[s : t] \in \mathbb{P}^1$  and the cardinality of the set

$$\{[s : t] \in \mathbb{P}^1 \mid \text{all irreducible components of } sA + tB \text{ have degree one} \}$$

is  $\dim E + 1$ .


 FIGURE 1.  $A_3$  arrangement (in bold) and Bol's 5-web

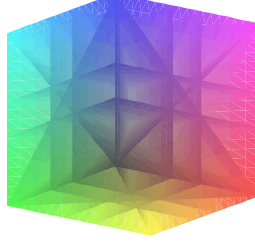
**Remark 5.1** (Webs associated to Arrangements). As we have just explained to each arrangement  $\mathcal{A}$  with  $\mathcal{R}^1(\mathcal{A}) \neq \emptyset$  we can canonically associate a finite collection of foliations on  $\mathbb{P}^n$  induced by the rational maps  $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ , one for each irreducible component of  $\mathcal{R}^1(\mathcal{A})$ . This finite collection forms a global web on  $\mathbb{P}^n$ . The field of web geometry has a venerable history and the present days are witnessing a lot of activity on webs and their abelian relations. From this viewpoint one of the key problems, at least according to Chern, is the classification of planar webs with the maximal number of abelian relations that are not algebraizable, cf. [2, 14]. For a long time the only example that appeared in the literature was Bol's example. It consists of the 5-web formed by four pencils of lines through four generic points on  $\mathbb{P}^2$  and a pencil of conics through these four points. It is a rather intriguing fact that the 5-web, canonically associated to the Coxeter arrangement of type  $A_3$ , is precisely Bol's web. It corresponds to four resonance components supported on pencils of lines and one global component. It would be rather interesting to pursue the determination of the rank of the webs associated to resonant line arrangements in  $\mathbb{P}^2$ .

## 6. EXAMPLES AND OPEN QUESTIONS

**6.1. Pencils on  $\mathbb{P}^2$ .** The inequalities of Theorem 2.1 allow pencils on  $\mathbb{P}^2$  with five completely reducible fibers. However no such pencils (of full rank) are known. This existence problem is not even settled for the case of nets, cf. [18, Problem 2]. The smallest possible example would be a (5,7)-net in  $\mathbb{P}^2$  realizing an orthogonal triple of Latin squares of order 7.

Concerning pencils with four completely reducible fibers just one example is known - the Hesse pencil based on the (4,3)-net. This pencil can be succinctly described as the pencil generated by a smooth cubic and its Hessian.

Concerning pencils on  $\mathbb{P}^2$  with three completely reducible fibers, plenty of them are known, including families with analytic moduli. The existence of such families with analytic moduli can be inferred from the realization result for nets given in [18, Theorem 4.4]. For explicit examples one can consider hyperplane sections of the examples presented in 6.2.

FIGURE 2. Arrangement of 12 hyperplanes on  $\mathbb{P}^3$  associated to  $D_4$ 

**6.2. Pencils on  $\mathbb{P}^3$ .** Let  $d$  be a positive integer and consider the pencil on  $\mathbb{P}^3$  generated by

$$\begin{aligned} F_d &= (x_0^d - x_1^d)(x_2^d - x_3^d) \\ G_d &= (x_0^d - x_2^d)(x_1^d - x_3^d). \end{aligned}$$

Then

$$F_d - G_d = (x_0^d - x_3^d)(x_2^d - x_1^d).$$

The arrangements  $\mathcal{A}_d$  corresponding to these pencils have many interesting properties. Each  $\mathcal{A}_d$  consists of  $6d$  hyperplanes that are the reflecting hyperplanes for the monomial group  $G(d, d, 4)$  generated by complex reflections (e.g., see [12]). For  $d = 1, 2$  the group is the Coxeter group of type  $A_3$  or  $D_4$  respectively. For all  $d \geq 2$  the arrangements  $\mathcal{A}_d$  are essential and carrying each a global irreducible resonance component of dimension one.

Moreover these arrangements are  $(3, d)$ -nets in  $\mathbb{P}^3$  (for  $d \geq 2$ ). This implies that the intersection of  $\mathcal{A}_d$  with a generic plane gives a net in  $\mathbb{P}^2$ . On the other hand, the intersection with a special plane (say, the one defined by  $x_3 = 0$ ) gives the family of multinets mentioned at the end of Section 3 whence representing these multinets as limits of nets.

Let us look closer at the combinatorics of  $\mathcal{A}_d$ , i.e., at the corresponding Latin square (see [18]). For each  $\zeta$  such that  $\zeta^d = 1$  denote by  $H_{i,j}(\zeta)$  the hyperplane defined by  $x_i = \zeta x_j$  ( $1 \leq i < j \leq 4$ ). Then identify the collection of hyperplanes corresponding to the linear divisors of  $F_d$ ,  $G_d$ , and  $F_d - G_d$  via

$$a_\zeta = H_{1,2}(\zeta^{-1}) = H_{1,3}(\zeta) = H_{2,3}(\zeta),$$

and

$$b_\zeta = H_{3,4}(\zeta) = H_{2,4}(\zeta) = H_{1,4}(\zeta).$$

After the identification the Latin square corresponding to  $\mathcal{A}_d$  is the multiplication table of the dihedral group  $D_d$  with  $a_\zeta$  forming the cyclic subgroup of order  $d$  and  $\{b_\zeta\}$  being the complementary set of involutions. Intersecting  $\mathcal{A}_d$  for  $d \geq 3$  with a general plane we obtain a series of 3-nets in  $\mathbb{P}^2$  realizing noncommutative groups (cf. [18]). In particular these nets are not algebraizable. We remark that another non-algebraizable example of a 3-net in  $\mathbb{P}^2$  has been found by J. Stipin in [15]. He has exhibited a  $(3, 5)$ -net that does not realize  $\mathbb{Z}_5$ .

Finally (as we learned from [15]) the general fibers of the above pencil for  $d = 2$  were studied by R. M. Mathews in [10] under the name of *desmic* surfaces. Thus for arbitrary  $d$  these fibers can be considered as generalizations of desmic surfaces.

The foliations  $\mathcal{F}_d$  induced by  $\omega_d = F_d dG_d - G_d dF_d$  have degree  $4d - 2$ . Thus the bound of Proposition 4.1 is attained by  $\mathcal{F}_2$  and no other foliation in the family.

**6.3. Pencils on  $\mathbb{P}^4$ .** We do not know any example of a pencil on  $\mathbb{P}^4$  with three completely reducible fibers that is not a linear pullback of a pencil from a smaller dimension. One can deduce from careful reading of the proof of Proposition 2.1 that the degree of such a pencil must be at least 10.

## 7. PENCILS AND CHARACTERISTIC VARIETIES

Let  $\mathcal{A}$  be an arrangement in  $\mathbb{P}^n$  and  $M$  its complement. If for every  $\rho \in \text{Hom}(\pi_1(M), \mathbb{C}^*)$  we write  $\mathcal{L}_\rho$  for the associated rank one local system then the characteristic varieties  $\mathcal{V}^l(M)$  are defined as follows

$$\mathcal{V}^l(M) = \{\rho \in \text{Hom}(\pi_1(M), \mathbb{C}^*) \mid H^l(M, \mathcal{L}_\rho) \neq 0\}.$$

If  $\mathcal{A}' \subset \mathcal{A}$  is a subarrangement with complement  $M'$  then the inclusion of  $M$  into  $M'$  induces an inclusion of  $\text{Hom}(\pi_1(M'), \mathbb{C}^*)$  into  $\text{Hom}(\pi_1(M), \mathbb{C}^*)$  and also of  $\mathcal{V}^1(M')$  into  $\mathcal{V}^1(M)$ . As in the case of resonance varieties, the support of an irreducible component  $\Sigma$  of  $\mathcal{V}^1(M)$  is the smallest subarrangement  $\mathcal{A}'$  such that  $\Sigma \subset \mathcal{V}^1(\mathcal{A}')$ .

The main result of [3] implies that the projectivization of the tangent cone of  $\mathcal{V}^l(M)$  at the trivial representation is isomorphic to  $\mathcal{R}^l(\mathcal{A})$ . Thus if  $\Sigma \subset \mathcal{V}^1(M)$  is an irreducible component of dimension  $d > 0$  containing  $1 \in \text{Hom}(\pi_1(M), \mathbb{C}^*)$  then the projectivization of its tangent space is an irreducible component of the resonance variety.

There exist translated components of characteristic varieties, i.e., those that do not contain the trivial representation, see [16]. In a recent preprint Dimca [4] found more precise properties of shifted components and clarified the link between the positive dimensional irreducible components of  $\mathcal{V}^1(M)$  and pencils of hypersurfaces. We invite the reader to consult [4] for a more extensive description. Here we will recall just what is strictly necessary for our purposes.

If  $\Sigma$  is a translated component of dimension at least two then after translating it to 1 one obtains an irreducible component of  $\mathcal{V}^1(M)$  through 1.

If  $\Sigma$  is a translated component of dimension one then there exists a pencil of hypersurfaces with generic irreducible fiber and exactly two completely reducible fibers with support contained in the support of  $\Sigma$ . The support of  $\Sigma$  is the union of the hyperplanes that appear as components of fibers of the pencil. Moreover there is at least one extra fiber such that its components are either hyperplanes in the support of  $\Sigma$  or non-reduced hypersurfaces. Combining these results with our methods we can prove the upper bound for the dimension of characteristic components.

The proof of the following theorem repeats almost verbatim the proof of Theorem 4.1 using now Proposition 3.1 in full. So we omit its proof.

**Theorem 7.1.** *Let  $\mathcal{P}$  be an essential pencil of hypersurfaces on  $\mathbb{P}^n$  with irreducible generic fiber, two completely reducible fibers, and a third fiber that is a product of linear forms and non-reduced polynomials. Then the Gauss map of the associated foliation  $\mathcal{F}$  is non-degenerate.*

Now using again Dimca's results and Theorem 7.1 we can prove an analogue of Corollary 5.1 for characteristic varieties.

**Theorem 7.2.** *If  $\mathcal{A}$  is an arrangement of hyperplanes in  $\mathbb{P}^n$  and  $\Sigma \subset \mathcal{V}^1(M)$  is an irreducible component then the following assertions hold:*

- (1) *If  $\dim \Sigma > 4$  then the rank of the support of  $\Sigma$  is one;*
- (2) *If  $\dim \Sigma > 2$  then the rank of the support of  $\Sigma$  is at most two;*
- (3) *If  $\dim \Sigma > 1$  then the rank of the support of  $\Sigma$  is at most four;*
- (4) *If  $\dim \Sigma > 0$  then the rank of the support of  $\Sigma$  is at most six.*

(Notice that the difference with Corollary 5.1 in the dimensions of  $\Sigma$  is due to the projectivization in the corollary.)

*Proof.* The statements (1)-(3) follow immediately from Corollary 5.1 and the relations between resonance and characteristic components. In order to prove (4) it suffices to show that  $n < 7$  if there exists a pencil  $\mathcal{P}$  of hypersurfaces on  $\mathbb{P}^n$  with completely reducible generators  $F$  and  $G$  inducing a full rank arrangement and at least one extra fiber, say  $K = F - G$ , that can be written as

$$K = \tilde{U} \cdot \tilde{V}$$

where  $\tilde{U} \in \mathbb{C}[x_0, \dots, x_n]$  is a product of linear forms and  $\tilde{V} \in \mathbb{C}[x_0, \dots, x_n]$  is a product of non-trivial powers of irreducible polynomials of degree at least two. We denote by  $U, V$  the reduced polynomial with the same zero set of  $\tilde{U}, \tilde{V}$  and point out that

$$(1) \quad 2 \deg \left( \frac{\tilde{V}}{V} \right) \geq \deg(\tilde{V}).$$

If  $\tilde{Q} \in \mathbb{C}[x_0, \dots, x_n]$  denotes the product  $FG$  and  $Q$  denotes the reduced polynomial with the same zero set then

$$\omega = \frac{U}{\tilde{U}} \frac{V}{\tilde{V}} \frac{Q}{\tilde{Q}} \omega_0$$

is a homogeneous polynomial 1-form defining the foliation  $\mathcal{F}$  associated to  $\mathcal{P}$ . In particular

$$\deg(\mathcal{F}) \leq \deg(\tilde{Q}) - 2 - \deg \left( \frac{\tilde{U}\tilde{V}\tilde{Q}}{UVQ} \right).$$

Theorem 7.2 implies that the Gauss map of  $\mathcal{F}$  is non-degenerate. The property (iv) of the forms proved in section 2 can be immediately generalized to  $\omega$  and it implies that  $(\tilde{U}\tilde{Q})^{n-1}$  divides

$$\left( \frac{\tilde{U}\tilde{Q}}{UQ} \right)^{n-1} \det \left( \frac{\partial a_i}{\partial x_j} \right),$$

where  $\omega = \sum a_i dx_i$ . We obtain that

$$\begin{aligned} (n-1) \deg(\tilde{U}\tilde{Q}) &\leq (n+1) \left( \deg(\tilde{Q}) - 2 - \deg \left( \frac{\tilde{U}\tilde{V}\tilde{Q}}{UVQ} \right) \right) + (n-1) \deg \left( \frac{\tilde{U}\tilde{Q}}{UQ} \right) \\ &= (n+1) \left( \deg(\tilde{Q}) - 2 \right) - 2 \deg \left( \frac{\tilde{U}\tilde{Q}}{UQ} \right) - (n+1) \deg \left( \frac{\tilde{V}}{V} \right). \end{aligned}$$

Therefore, if we suppose that  $n \geq 3$ , delete the term  $-2 \deg \left( \frac{\tilde{U}\tilde{Q}}{\tilde{U}\tilde{Q}} \right)$  and use (1) then

$$\begin{aligned} (n-1) \deg(\tilde{Q}) &< (n+1) \left( \deg(\tilde{Q}) - 2 \right) - \frac{n+1}{2} \deg(\tilde{V}) - (n-1) \deg(\tilde{U}) \\ &\leq (n+1) \left( \deg(\tilde{Q}) - 2 \right) - \frac{n+1}{2} \deg(\tilde{U}\tilde{V}) < \frac{3(n+1)}{4} \deg(\tilde{Q}). \end{aligned}$$

In particular  $n < 7$  and the statement follows.  $\square$

We do not know if the bounds given in the theorem are sharp.

We also do not know if there are restrictions on the rank of the support of the zero-dimensional characteristic varieties.

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