DERIVATIONS OF AN EFFECTIVE DIVISOR ON THE COMPLEX PROJECTIVE LINE

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Abstract. In this paper we consider an effective divisor on the complex projective line and associate with it the module D consisting of all the derivations θ such that $\theta(I_i) \subset I_i^{m_i}$ for every i, where I_i is the ideal of p_i. The module D is graded and free of rank 2; the degrees of its homogeneous basis, called the exponents, form an important invariant of the divisor. We prove that under certain conditions on (m_i) the exponents do not depend on $\{p_i\}$. Our main result asserts that if these conditions do not hold for (m_i) then there exists a general position of n points for which the exponents do not change. We give an explicit formula for them. We also exhibit some examples of degeneration of the exponents, in particular those where the degeneration is defined by vanishing of certain Schur functions. As application and motivation, we show that our results imply Terao's conjecture (about the combinatorial nature of the freeness of hyperplane arrangements) for certain new classes of arrangements of lines in the complex projective plane.

1. Introduction

Arguably, the most intriguing conjecture in the theory of hyperplane arrangements is the Terao conjecture about the combinatorial character of the freeness of arrangements. The first interesting and open case is formed by 3-arrangements, i.e., arrangements of projective lines in the complex projective plane $\mathbb{CP}^2$. Recent progress has been made by M. Yoshinaga (see Section 7) who found a new relation between the freeness of an arrangement $\mathcal{A}$ and its restriction $\mathcal{A}_H$ to an arbitrary hyperplane $H \in \mathcal{A}$. More precisely the hyperplanes of $\mathcal{A}_H$ have natural multiplicities, which allows one to consider the multi-arrangement $\tilde{\mathcal{A}}_H$. Then if $\mathcal{A}$ has rank $\ell+1$ and is free with the exponents $\{e_0 = 1, e_1, \dots, e_\ell\}$ then $\tilde{\mathcal{A}}_H$ must be free with the exponents $\{e_1, \dots, e_\ell\}$. For arrangements in $\mathbb{CP}^2$ this necessary condition is also sufficient (even if it is checked for only one line). This result brings to light multi-arrangements of points (i.e., effective divisors) on $\mathbb{CP}^1$.

In the rest of the paper we will consider multi-arrangements $\tilde{\mathcal{A}} = \{p_1, p_2, \dots, p_n\}$ in $\mathbb{CP}^1$ with $n \geq 2$ and the positive integer multiplicity m_i of p_i $(i = 1, 2, \dots, n)$. We usually order the points subject to $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_n$ and call $m = (m_1, m_2, \dots, m_n)$ the multiplicity vector of $\tilde{\mathcal{A}}$. Also we put $\tilde{n} = \sum_{i=1}^{n} m_i$. If $m = (1, 1, \dots, 1)$ then $\tilde{\mathcal{A}} = \mathcal{A}$ and is called a simple arrangement.

Choosing an appropriate coordinate system on $V = \mathbb{C}^2$ we can view the symmetric algebra $S = \mathcal{S}(V^*)$ as the polynomial algebra $\mathbb{C}[x, y]$ and fix linear forms

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$\alpha_i = x - z_i y \in S$ ($z_i \in \mathbb{C}$) such that $ker(\alpha_i) = p_i$ for all i. Then the defining polynomial of $\tilde{\mathcal{A}}$ is $Q = \prod_{i=1}^{n} (x - z_i y)^{m_i}$.

The graded S-module of derivations

$$D(\tilde{\mathcal{A}}) = \{\theta \in Der_{\mathbb{C}} S \mid \theta(\alpha_i) \in \alpha_i^{m_i} S\}$$

is known to be always free (as a reflexive module over a ring of homological dimension 2) of rank 2 whence it is freely generated by two derivations whose degrees e_1 and e_2 are uniquely determined by $\tilde{\mathcal{A}}$. We always assume that $e_1 \le e_2$ and call the pair $exp(\tilde{\mathcal{A}}) = (e_1, e_2)$ the exponents of $\tilde{\mathcal{A}}$. As it was noticed by G. Ziegler [7] (who first considered $D(\tilde{\mathcal{A}})$) the exponents are not in general determined by m, unlike in the case of simple arrangement where $e_1 = 1$ and $e_2 = n - 1$, (see [2]).

The main result of this paper (Theorem 3.1) is that under certain conditions on the multiplicity vector m there exists a general position set (more precisely, a nonempty set, open in the Zariski topology) of n points on $\mathbb{CP}^1$ such that

$$exp(\tilde{\mathcal{A}}) = \left(\left\lfloor \frac{1}{2} \sum_{i=1}^{n} m_i \right\rfloor, \left\lceil \frac{1}{2} \sum_{i=1}^{n} m_i \right\rceil \right)$$

for every n-tuple of points in this position. The proof of the theorem is broken in several steps and occupies Sections 3-5. The main part of the proof is in Section 5 where we prove that a certain determinant is not identically zero as a polynomial in z_i's.

Also we study all the cases where m does not satisfy these conditions. We prove that in these cases the exponents are determined by m and give explicit formulas for them (Section 2). In Section 6, we consider examples of degeneration of the exponents, in particular a case where the degeneration is defined by the vanishing of Schur functions for rectangular diagrams. Finally in Section 7 we recall the Terao conjecture and Yoshinaga theorem and exhibit several new classes of 3-arrangements for which our results imply the conjecture.

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2. Multiplicity vectors that determine exponents

First, for convenience of the reader we recall Zieler's generalization ([7], p. 351) of Saito's criterion ([2], Theorem 4.19) restricted to our case. Let $\theta_1, \theta_2 \in D(\tilde{\mathcal{A}})$ and in some coordinate system on $\mathbb{CP}^1$ we have $\theta_i = f_{i1}\partial_x + f_{i2}\partial_y$ for some homogeneous $f_{ij} \in S$ of the same degree where $i, j = 1, 2$ and ∂_x and ∂_y are derivative with respect to x and y. Then (θ_1, θ_2) is a basis of $D(\tilde{\mathcal{A}})$ if and only if the determinant $f_{11}f_{22} - f_{12}f_{21}$ is equal to $\tilde{Q}$ multiplied by a nonzero constant.

Now we mention two simple properties of the exponents$(e_1, e_2), e_1 \le e_2$, which we use frequently in the rest of the paper.

1. $e_1 + e_2 = \tilde{n}$.

This follows from the version of Saito's criterion of the freeness for multi-arrangements.

2. Suppose $\mathcal{A}_1 \subset \mathcal{A}_2$ are two multi-arrangements and $\tilde{n}(\mathcal{A}_2) = \tilde{n}(\mathcal{A}_1) + 1$. If $exp(\mathcal{A}_1) = (e_1, e_2)$ and $exp(\mathcal{A}_2) = (e'_1, e'_2)$ then either $e'_1 = e_1$ (whence $e'_2 = e_2 + 1$) or $e'_2 = e_2$ (whence $e'_1 = e_1 + 1$).

Indeed since $D(\mathcal{A}_2) \subset D(\mathcal{A}_1)$ we have $e_i' \geq e_i$, $i = 1, 2$. Now the property follows from property 1.

In the rest of the section we will consider several cases of multiplicity vectors m where the exponents (e_1, e_2) are uniquely determined by m and do not depend on the position of the points. In most of these cases we will be able to exhibit a derivation of minimal degree from $D(\tilde{\mathcal{A}})$ as a function of m.

In the rest of the paper we always assume that a coordinate system is fixed on $\mathbb{CP}^1$ so that the defining polynomial of $\tilde{\mathcal{A}}$ is

$$\tilde{Q} = x^{m_1} y^{m_2} \prod_{i=3}^{n} (x - z_i y)^{m_i} \tag{2.1}$$

for some $z_i \in \mathbb{C}\backslash\{0\}$. Then any derivation from $D(\tilde{\mathcal{A}})$ has the form $f x^{m_1} \partial_x - g y^{m_2} \partial_y$ for some $f, g \in S$ (∂_x and ∂_y are derivatives with respect to x and y).

We can exhibit two derivations in $D(\tilde{\mathcal{A}})$ for future use. Put $\theta_1 = \frac{\tilde{Q}}{x^{m_1}} \partial_y$. It is easy to see that θ_1 is a homogeneous derivation having minimal degree among all the elements of $D(\tilde{\mathcal{A}})$ with $f = 0$. Also put $\theta_2 = \frac{\tilde{Q}}{Q} \theta_E$, where $\theta_E = x\partial_x + y\partial_y$ is the Euler derivation. Again it is easy to check that θ_2 is a homogeneous derivation having minimal degree among all elements of $D(\tilde{\mathcal{A}})$ proportional to θ_E.

Case 2.1. Let $m_1 \geq \sum_{i=2}^{n} m_i$. Then θ_1 is a derivation of minimal degree in $D(\tilde{\mathcal{A}})$. In particular $(e_1, e_2) = (\sum_{i=2}^{n} m_i, m_1)$.

Proof. Since any homogeneous derivation θ from $D(\tilde{\mathcal{A}})$ with $f \neq 0$ satisfies $deg(\theta) \geq m_1 \geq \sum_{i=2}^{n} m_i = deg(\theta_1)$, θ_1 has the smallest degree in $D(\tilde{\mathcal{A}})$ and the statement follows. □

Case 2.2. If $\tilde{n} \leq 2n - 2$ then θ_2 is a derivation of minimal degree in $D(\tilde{\mathcal{A}})$. In particular

$$(e_1, e_2) = (\tilde{n} - n + 1, n - 1).$$

Proof. Notice $deg(\theta_2) = \tilde{n} - n + 1 \leq 2n - 2 - n + 1 = n - 1$. Suppose that there is $\theta \in D(\tilde{\mathcal{A}})$ such that $deg(\theta) < deg(\theta_2) \leq n - 1$. Since $D(\tilde{\mathcal{A}}) \subset D(\mathcal{A})$ and the latter module is generated by θ_E together with a derivation of degree $n - 1$ we have that $\theta = h\theta_E$ for a polynomial h. Applying θ to α_i we see that h is divisible by $\frac{\tilde{Q}}{Q}$ which contradicts the condition on degree of θ.

Hence the degree of θ_2 is minimal and the result follows. □

Remark 2.1. (i) Cases 2.1 and 2.2 meet at the case $m = (n - 1, 1, \ldots, 1)$. For this m the derivations found in the cases give a minimal generating system of degrees $(n - 1, n - 1)$.

Indeed assume $\tilde{n} \leq 2n - 2$ and $m_1 \geq \sum_{i=2}^{n} m_i$. Since every $m_i \geq 1$ we have $\sum_{i=2}^{n} m_i$ and m_1 both are not less than $n - 1$ whence $\tilde{n} = 2n - 2$, $m_1 = n - 1$ and $m_i = 1$ for every $i > 1$.

(ii) The previous remark allows us to handle the case where $\tilde{n} = 2n - 1$. Since $n \geq 2$ there exists an $m_i > 1$. Decreasing this m_i by 1 we obtain a new multiarrangement with exponents $(n - 1, n - 1)$. By property 2, the exponents of the initial arrangement are $(n - 1, n)$. In particular they are determined by m.

Together with Case 2.2, these comments imply that if the average of all m_i is less than 2 then the exponents are determined by m. If the average equals 2, this is false. We however can give one example of m for which it is true.

Example 2.2. If $m_i = 2$ for all $1 \leq i \leq n$ then the exponents are (n, n). Indeed we can exhibit two derivations of degree n satisfying the Saito criterion.

Let an arrangement $\mathcal{A}$ be defined by $Q = xy \prod_{i=3}^{n} (x - z_i y)$ for some $n-2$ numbers $z_i \in \mathbb{C}$ and $\tilde{Q} = Q^2$. If $n = 2$ we can take $\theta_1 = x^2 \partial_x$ and $\theta_2 = y^2 \partial_y$ and the result obviously follows.

Suppose $n > 2$ and for every i, $3 \leq i \leq n$, put $Q_i = \frac{Q}{x - z_i y}$, $h_1 = \sum_{i=3}^{n} Q_i$, and $h_2 = \sum_{i=3}^{n} z_i Q_i$. Then put

$$\xi_1 = h_1 \theta_E + \frac{y}{x} Q \partial_y$$

and

$$\xi_2 = h_2 \theta_E - \frac{x}{y} Q \partial_x.$$

One can check straightforwardly that $\xi_i \in D(\tilde{\mathcal{A}})$ $(i = 1, 2)$ and the matrix of the coordinates of these derivations has determinant equal to $(n-2)Q^2$.

Let us remark that just a slightly different basis for this case was previously constructed in [3], Proposition 5.4.

If $n = 2$ or 3 then of course the exponents are determined by m since there is only one position of 2 or 3 points on $\mathbb{CP}^1$ up to a projective isomorphism. For $n = 4$ even for the average of m_i equal 2 the situation is different. It follows from the cases above that the vectors $(5, 1, 1, 1)$, $(4, 2, 1, 1)$, and $(2, 2, 2, 2)$ determine exponents uniquely (they are $(3, 5)$ for the first vector and $(4, 4)$ for two others). On the other hand, for the vectors $(3, 3, 1, 1)$ and $(3, 2, 2, 1)$ the exponents depend on the position of 4 points on $\mathbb{CP}^1$ (see section 6).

3. The main theorem

In this section, we are concerned with an arbitrary multiplicity vector m not satisfying the strict inequalities studied in the previous section. Our main result asserts that m uniquely determines the exponents (e_1, e_2) for n points in general position on $\mathbb{CP}^1$. (In this paper we understand by a general position set of n points in $\mathbb{CP}^1$, a nonempty set in the configuration space $(\mathbb{CP}^1)^n$ open in the Zariski topology.) Roughly speaking these general position exponents are as close to being equal as possible. More precisely we are going to prove the following.

Theorem 3.1. *Suppose m has the following properties: $m_1 \leq \sum_{i=2}^{n} m_i$ and $\tilde{n} \geq 2n - 2$. Then there exists a general position of n points such that*

$$exp(\tilde{\mathcal{A}}) = \left(\left\lfloor \frac{1}{2} \tilde{n} \right\rfloor, \left\lceil \frac{1}{2} \tilde{n} \right\rceil \right)$$

for every multi-arrangement $\tilde{\mathcal{A}}$ of points in this position having m as the multiplicity vector.

The proof will be broken into several parts according to the following plan. First we focus on the case when $\tilde{n}$ is even. Then for $\tilde{n}$ odd the result can be deduced similarly to Remark 2.1 (i) (cf. section 5). For $\tilde{n}$ even the condition guaranteeing the existence of a derivation in $D(\tilde{\mathcal{A}})$ of degree less than $\tilde{n}/2$ can be naturally expressed as the vanishing of the determinant of a square matrix M of size $\sum_{i=3}^{n} m_i$ whose entries are monomials in z_i, $i = 3, 4, \ldots, n$. Thus the determinant of M is a polynomial, d, in these indeterminates and to prove the theorem it suffices to show that d is not identically 0. For this, in turn, it suffices to show that the leading monomial of d (in some linear ordering of the monomials) has a nonzero coefficient.

Now the proof branches out. Since the set of rows of M is by construction partitioned into blocks corresponding to the points $p_3, p_4, \ldots, p_n$, it is natural to use the Laplace formula for d. In most cases the coefficient of the leading term is the product of minors of M, one from each block of rows, and it is not hard to show that none of these minors vanishes. However, there are cases where the leading term is not a single product but a sum of several products of minors. This case is harder and requires a deeper analysis of the minors.

4. The matrix M

In this section, we assume that $\tilde{n}$ is even and put $e = \frac{1}{2}\tilde{n} - 1$. We assume also that $m_i \geq 2$ for every i and $m_1 < \sum_{i=2}^{n} m_i$. Equivalently, $e - m_1 \geq 0$ whence also $e - m_2 \geq 0$. Any derivation $\theta \in D(\tilde{\mathcal{A}})$ of degree e is of the form $\theta = x^{m_1} f(x,y)\partial_x - y^{m_2} g(x,y)\partial_y$ where $f(x,y), g(x,y) \in S$, $deg(f(x,y)) = e - m_1$ and $deg(g(x,y)) = e - m_2$. Write

$$f(x,y) = \sum_{j=0}^{e-m_1} f_j x^{e-m_1-j} y^j$$

and

$$g(x,y) = \sum_{j=0}^{e-m_2} g_j x^{e-m_2-j} y^j$$

where $f_j, g_j \in \mathbb{C}$. Besides since $\theta \in D(\tilde{\mathcal{A}})$, for all $i \geq 3$ we have $\theta(x - z_i y) \in (x - z_i y)^{m_i} S$. Sacrificing homogeneity, we put $y = 1$ and obtain the equivalent condition

$$\partial_x^k(\theta(x - z_i y)|_{y=1})|_{x=z_i} = 0 \tag{$*$}$$

for all $0 \leq k \leq m_i - 1$ where ∂_x^k is the k^{th} derivative with respect to x. Treating the coefficients f_i and g_j as unknowns we consider (*) as a system of equations for these unknowns. Explicitly the equation corresponding to the point p_i $(3 \leq i \leq n)$ and some k $(0 \leq k \leq m_i - 1)$ is

$$\sum_{j=0}^{e-m_1} f_j \frac{(e-j)!}{(e-j-k)!} z_i^{e-j-k} + z_i \sum_{j=0}^{e-m_2} g_j \frac{(e-m_2-j)!}{(e-m_2-j-k)!} z_i^{e-m_2-j-k} = 0$$

where we agree that $z_i^\ell = 0$ for $\ell < 0$.

The matrix M is the matrix of coefficients of this system of equations which are polynomials in z_i, $i = 3, 4, \ldots, n$. A simple computation shows that the matrix is square - the number of rows as well as columns is $\sum_3^n m_i$. Moreover the set of rows of M is partitioned into $n-2$ blocks $L_3, L_4, \dots, L_n$ where the block L_i does not contain z_j with $j \neq i$. The set of columns of M is partitioned into two blocks: the f-block and the g-block consisting of the coefficients at f_i and g_j respectively. The sizes of the blocks are $e-m_1+1$ and $e-m_2+1$ respectively. Notice that the f-block of the block L_i is a Wronskian matrix of the functions $x^e, x^{e-1}, \dots, x^{m_1}$ for their derivatives up to $m_i - 1$ evaluated at z_i, i.e., an entry of the block has the form $\partial_x^k(x^{e-\ell})|_{x=z_i}$, $k = 0, 1, \dots, m_i - 1, \ell = 0, 1, \dots, e-m_1$. The g-block of the block L_i is a Wronskian matrix of the functions $x^{e-m_2}, x^{e-m_2-1}, \dots, 1$ for the same range of derivatives as for the f-block, multiplied by x and then evaluated at z_i. So its typical entry has the form $(x\partial_x^k(x^{e-m_2-\ell}))|_{x=z_i}$, $k = 0, 1, \dots, m_i - 1, \ell = 0, 1, \dots, e - m_2$.

By construction of M the vanishing of its determinant at some $(n-2)$ -tuple $(z_3, \dots, z_n)$ of pairwise different nonzero complex numbers is equivalent to the existence of a derivation of degree e in $D(\tilde{\mathcal{A}})$ where $\tilde{\mathcal{A}}$ is the multi-arrangement defined by the formula (2.1). We can view z_i as indeterminates and $\det(M)$ as a polynomial in these $n-2$ indeterminates. In the space $\mathbb{C}^{n-2}$ of all $n-2$-tuples of points the complement to the zero locus of d is an open set in the Zariski topology. Thus Theorem 3.1 under the conditions on m assumed in this section would follow from the following result.

Theorem 4.1. *The polynomial d is not identically 0.*

The rest of the section is devoted to the proof of Theorem 4.1 which we break into several parts.

Lemma 4.2. *The Wronskian of the power functions $x^{\lambda_1}, \dots, x^{\lambda_k}$ where (λ_i) is a strictly decreasing k-tuple of nonnegative integers is a monomial in x of degree $\sum_{j=1}^{k} \lambda_j - \sum_{r=1}^{k-1} r$ with the coefficient*

$$(-1)^{\lfloor \frac{k}{2} \rfloor} \prod_{1 \le i < j \le k} (\lambda_i - \lambda_j).$$

Proof. The first part of the statement follows immediately from the fact that every summand in the general formula for the determinant is the product of exactly one entry from each row and each column.

The second part of the statement can be proved by setting $x = 1$ and transforming the obtained matrix to a Vandermonde matrix by applying row operations and induction on number of rows. The sign $(-1)^{\lfloor \frac{k}{2} \rfloor}$ comes from the unusual ordering of the rows of the Vandermonde determinant. □

In order to apply Lemma 4.2 we represent d via (repeated) Laplace's development corresponding to the fixed partition $(L_3, \dots, L_n)$ of the set of rows of M. For that we need to consider ordered partitions $\beta = (B_3, \dots, B_n)$ of columns such that $|B_i| = |L_i| = m_i$ for every i. Each β defines a permutation $P(\beta)$ of all the columns of M. This gives a bijection of the set of these partitions with the set of permutations of the columns of M such that the orderings of columns inside subsequent intervals, B_i, of length m_i are induced from M. Then we have

$$d = \sum_{\beta} \operatorname{sign}(P(\beta)) \prod_{i=3}^{n} d(L_i, B_i)$$

where the summation is over all the ordered partitions β as above and $d(L_i, B_i)$ is the $m_i \times m_i$-minor of M with rows from L_i and columns from B_i.

Remark 4.3. The usual formula for signs of summands in Laplace's development differs from ours. Our formula can be easily checked by considering the signs of products of diagonal elements of each minor $d(L_i, B_i)$(see [1]).

The degrees of entries in each column are determined by the degree r of the top element in the column; for brevity we say that *the column has degree* r.

In order to prove that the polynomial d does not vanish identically it suffices to find a monomial of it with a nonzero coefficient. We will use for this purpose the leading term of d in the lexicographic order generated by the reverse ordering of the indeterminates $z_n < z_{n-1} < \cdots < z_3$. The proof of Theorem 4.1 is immediate when the leading term is the product of (uniquely determined) minors, one from each block L_i. The simplest case when this happens is the case of 'no overlapping'.

Proposition 4.4. Suppose that the sets of degrees of (the top rows of) columns in the f-block and those in the g-block are disjoint (equivalently $m_1 + m_2 > \frac{\tilde{n}}{2}$). Then Theorem 4.1 holds.

Proof. Indeed in this case there are no two columns of same degrees. There is the unique partition β of the columns such that the corresponding product of minors is the leading term of d; it is the partition corresponding to the identity permutation of the columns. The respective minors do not vanish by Lemma 4.2. □

In the rest of the proof we focus on the case where there is some overlapping. This means that several columns on the right flank of the f-block have the same degrees as several columns on the left flank of the g-block. The number of columns of this kind in the f-block is the same as this number for the g-block and equals $s = \frac{1}{2}\tilde{n} + 1 - (m_1 + m_2)$. We denote by O the set of these columns and denote its columns by $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ meaning that the degrees of a_i and b_i are the same $(i = 1, 2, \ldots, s)$ and $deg(a_i) = deg(a_{i+1}) + 1$ $(i = 1, 2, \ldots, s-1)$. Notice that $deg(a_1) = deg(b_1) = m_1 + s - 1 = e - m_2 + 1$ and the number of columns with higher degrees is $e - deg(a_1) = m_2 - 1$. The only minimal linearly dependent, over S, sets of columns in any block L_j are $\{a_{i+1}, b_i\}$ $(i = 1, 2, \ldots, s-1)$.

Even when overlapping occurs there still may be a unique choice of minors from the L_i's whose product equals the leading term of d. However, the simple partition that worked in the non-overlapping case may not work now because some minors may vanish.

To analyze the situation deeper we need more notation. Consider a partition $\beta = (B_3, \ldots, B_n)$ of columns such that the corresponding product of minors equals the leading monomial of d and the respective permutation $P(\beta)$. Then $P(\beta)$ is the identity permutation on the columns outside of O.

To express this in terms of partitions denote by B_{i_1} and B_{i_2} the first and the last blocks intersecting with O. Then the blocks $B_3, \ldots, B_{i_1 - 1}$ and $B_{i_2+1}, \ldots, B_n$ are determined uniquely as well as r_1 columns in B_{i_1} and r_2 columns in B_{i_2} not in O (for some $0 \le r_j < m_{i_j}$, $j = 1, 2$). For convenience, put $m'_i = m_i$ if $i_1 < i < i_2$ and $m'_{i_j} = m_{i_j} - r_j$ for $j = 1, 2$. In particular $\sum_{i=i_1}^{i_2} m'_i = 2s$.

Now we can describe a (perhaps not unique) partition of O recursively.

Proposition 4.5. Let $i_1 \leq i \leq i_2$. Suppose j is the minimal index such that the column a_j is not chosen for all blocks $B_{i'}$ with $i' < i$ and k is similar index for b_k.

If $j \leq k$ then the block B_i has the following columns from O:

(1) $a_j, a_{j+1}, \dots, a_{k-1}, a_k, b_k, b_{k+1}, \dots, b_{m'_i+j-2}$	$if\ m'_i \geq k-j+2$
(2) $a_j, a_{j+1}, \dots, a_{m'_i+j-1}$	$if\ m'_i \leq k-j$
(3) $a_j, a_{j+1}, \dots, a_{k-1}, b_k$	$if\ m'_i = k-j+1$
(4) $a_j, a_{j+1}, \dots, a_{k-1}, a_k$	$if\ m'_i = k-j+1$ and either $i = i_2$ or $m'_{i+1} \leq 2$.

If $j > k$ then $m'_i = 2$, $j = k+1$ and the columns in $B_i \cap O$ are

$$b_k, b_j.$$

Proof. We use induction on i. It is obvious for $i = i_1$ and follows from the inductive hypothesis for $i > i_1$ that the indexes of columns a_t and b_t available for B_i form the intervals $j \leq t \leq s$ and $k \leq t \leq s$ respectively. Choosing the columns of maximal available degrees and avoiding linear dependent pairs we immediately obtain the first three cases for $j \leq k$.

The fourth case is more subtle. If $m'_i = k-j+1$ then the columns from a_j to a_k have the maximal degrees and are independent whence seem to be appropriate for B_i. Assume however that $i \neq i_2$ and $m'_{i+1} \geq 3$. Then with the above choice of columns for B_i the block B_{i+1} has columns $b_k, b_{k+1}, b_{k+2}, \dots$. However the choice of columns for B_i from the third case gives the same degree minor in L_i and allows the higher degree choice $a_k, a_{k+1}, b_{k+1}, \dots$ for a minor in L_{i+1}. Thus, the choice (4) in this case does not work. In the case where $m'_{i+1} = 1$ or 2 this obstruction disappears and both choices (3) and (4) for B_i give nonzero minors of the same degree.

Finally consider the case $j > k$. This condition means that the column a_k was used in the previous blocks and b_k was not. Applying the inductive hypothesis we can see that it is possible if and only if the choice (4) occurs in L_{i-1} whence $m'_i = 2 = k-j+1$. The choice of columns for B_i is now clear. □

Remark 4.6. Proposition 4.5 can be used to show that even if $O \neq \emptyset$ there may be just one product of minors equal to the leading term of d. Moreover, it is possible in general to compute the index r ($i_1 \leq r \leq i_2$) of the block, where the non-uniqueness starts, directly from the vector m. Since we won't use this in the paper we omit this computation.

We will need however some properties of r which we collect in the following corollary that follows immediately from Proposition 4.5.

Corollary 4.7. Suppose that for a multiplicity vector m there are at least two different partitions of O whose corresponding products of minors equal the leading monomial of d. Then

(1) The union of $\bigcup_{r'<r} B_{r'}$ and first $m_r - 1$ columns of B_r consists of all the columns of degree larger than $deg(a_t)$ for some $t \leq s$ satisfying the equality

$$2(t-1) + m_2 = \sum_{i=3}^{r} m_i.$$

(2) Either $t = s$ or $m_i = 2$ for all $i > r$.

(3) The set of columns $O' = \{a_t, a_{t+1}, \dots, a_s, b_t, b_{t+1}, \dots, b_s\}$ is partitioned by the blocks from B_r to B_{i_2}. This partition has the following properties:

(a) the block B_r has only one column from O', either a_t or b_t (which implies that B_{i_2} has either a_s or b_s and no other columns from O);

(b) if the second column of a block is a_j then the first column of the next block is b_j and vice versa;

(c) if the first column of a block is a_j then its second column is either a_{j+1} or b_{j+1};

(d) if the first column of a block is b_j then its second column is b_{j+1}.

The ordered partitions of O' having the properties of the previous Corollary are in one-to-one correspondence with some permutations of O' that we call admissible.

Example 4.8. Using the notation of the previous Corollary, if $k = 1$ and $s = 2$ then there exists the following 3 admissible partitions of $O = O'$

$$a_1|b_1b_2|a_2,\ b_1|a_1a_2|b_2,\ b_1|a_1b_2|a_2.$$

It is easy to explicitly compute minors involved in Corollary 4.7. Under the conditions of Proposition 4.5 consider any product of minors giving maximal degree monomial of d. Recall that the minor $\mu_i = d(L_i, B_i)$ is a monomial in z_i with some coefficient $c_i \in \mathbb{Z},\ i = 3, \dots, n$.

Lemma 4.9. *The coefficients, c_i, are as follows.*

If $i_2 - r > 1$ then

(i) $c_i = -1$ for $r < i \leq i_2$ if both columns of μ_i are from the f-block or both columns are from g-block and $c_i = -2$ if the columns are from different blocks.

If $i_2 = r + 1$ then

(ii) $c_{i_2} = (-1)^{\lfloor \frac{m_{i_2}}{2} \rfloor}\pi(m_{i_2} - 1)!$ if the first column of μ_{i_2} is b_s and $c_{i_2} = (-1)^{\lfloor \frac{m_{i_2}}{2} \rfloor}\pi m_{i_2}!$ if the first column is a_s where $\pi = \prod_{j=1}^{m_{i_2}-2} j!$ for $m_{i_2} > 2$ and $\pi = 1$ otherwise.

For arbitrary $i_2 - r \geq 1$

(iii) $c_r = (-1)^{\lfloor \frac{m_r}{2} \rfloor}\pi'(m_r - 1)!$ if the last column of μ_r is a_k and $c_r = (-1)^{\lfloor \frac{m_r}{2} \rfloor}\pi' m_r!$ if the last column is b_k where $\pi' = \prod_{j=1}^{m_r-2} j!$ for $m_r > 2$ and $\pi' = 1$ otherwise.

This lemma follows immediately from Corollary 4.7 and Lemma 4.2.

Now we are ready to prove that even when the leading monomial of the determinant d is not just a product of minors but the sum of several of those, its coefficient does not vanish. We prove this by computing that coefficient.

For each admissible permutation P of $O' = (a_k, \dots, a_s, b_k, \dots, b_s)$ (i.e., a permutation that defines a partition satisfying Corollary 4.7) consider the product $\prod(P) = \prod c_i$ where $r \leq i \leq i_2$ and c_i is defined for the block B_i in Lemma 4.9. Notice that the product $\pi'(m_r - 1)!$ can be factored out of $\prod(P)$ for each P along with the product $\prod_{i=3}^{r-1} c_i \prod_{i=i_2+1}^{n} c_i$. More precisely for an arbitrary admissible permutation P put

$$\prod(P) = ((-1)^{\lfloor \frac{m_r}{2} \rfloor + \lfloor \frac{m_{i_2}}{2} \rfloor}\pi'(m_r - 1)\prod_{i=3}^{r-1} c_i \prod_{i=i_2+1}^{n} c_i)C(P)$$

for some $C(P) \in \mathbb{C}$.

Proposition 4.10. If $i_2 = r + 1$ then

$$\sum_P \operatorname{sign}(P)C(P) = \pi(m_{i_2} - 1)!(1 - m_r m_{i_2}); \tag{4.1}$$

if $i_2 > r + 1$ then

$$\sum_P \operatorname{sign}(P)C(P) = (-1)^{\lceil \frac{u+2}{2} \rceil}((m_r - 1)u + 1) \tag{4.2}$$

where the sum is taken over all the partitions P of O' satisfying Corollary 4.7 and $u = i_2 - r + 1$.

Proof. (i) Suppose $i_2 = r + 1$. Lemma 4.9 (ii) and (iii) implies that there are only two admissible permutations. The formula (4.1) follows also from that lemma.

(ii) Suppose that $i_2 > r+1$, i.e., $u > 2$. Corollary 4.7 implies that $m_{i_2} = 2$ whence the left hand side of (4.2) depends only on m_r and u. We denote it $\sigma(m_r, u)$ and put $\sigma(u) = \sigma(2, u)$.

Let us consider first the most important case where $m_r = 2$. The key observation here is the following recursive formula:

$$\sigma(u) = \sigma(u - 2) + (-1)^u 2\sigma(u - 1)$$

for every $u \geq 4$. Consider first all permutations P with a_k as the first element. Then the beginning of P is $a_k|b_k b_{k+1}|a_{k+1}$. That shows that the the $C(P) = C(P')$ where P' is the restriction of P to the rest of O'. Observe that the size of the set is $u - 2$ and $\operatorname{sign}(P') = \operatorname{sign}(P)$. This explains the first summand of the formula.

Then consider all permutations P with b_k as the first element. This defines the beginning of P as $b_k|a_k$. Thus $C(P) = (-2)C(P')$ where P' is again the restriction of P to the rest of O', this time of size $u - 1$. Besides $\operatorname{sign}(P') = (-1)^{u-1}\operatorname{sign}(P)$ as it is easy to see by counting transpositions. That explains the second summand.

Augment the recursive formula by the initial conditions $\sigma(2) = 3$ and $\sigma(3) = -4$ (for $u = 3$ there are 3 admissible orderings giving the summands equal to 2,2,-8, cf. Example 4.8). Using the recursive formula and the initial conditions, formula (4.2) for $m_r = 2$ can be proved by induction on u.

If $m_r > 2$ then the recursive formula is a little different;

$$\sigma(m_r, u) = \sigma(u - 2) + (-1)^u m_r \sigma(u - 1)$$

although the proof is the same. Substituting (4.2) for $m_r = 2$ in this formula we complete the proof. □

Proposition 4.10 completes the proof of Theorem 4.1.

5. Proof of Theorem 3.1

In this section we finish the proof of Theorem 3.1.

Recall that we consider multi-arrangements $\tilde{\mathcal{A}}$ with the following conditions on the multiplicity vector $(m_1, m_2, \ldots, m_n)$ such that $m_1 \geq m_2 \geq \cdots \geq m_n$:

(i)$m_1 \leq \sum_{i=2}^{n} m_i$;

(ii)$\tilde{n} = \sum_{i=1}^{n} m_i \geq 2n - 2$.

If $\tilde{n}$ is even then condition (i) allows us to write a square matrix M whose determinant $d = \det M$ is a polynomial in the coordinates of the points of $\mathcal{A}$. The equation $d = 0$ describes the locus in $(\mathbb{CP}^1)^{n-2}$ of all arrangements $\mathcal{A}$ such that $e_1(\tilde{\mathcal{A}}) < \frac{n}{2}$. For the case $m_n > 1$ (i.e., $m_i > 1$ for every i), we proved in the previous section that d is not identically 0 which implies the statement of the theorem for this case.

Now we consider the general case where multiplicity vector is arbitrary (satisfying (i) and (ii)). We apply induction on the number, α, of multiplicities equal 1 using as the base results of the previous section for $\alpha = 0$. Suppose $\alpha > 0$ (in particular $m_n = 1$) and consider $m' = (m_1, \dots, m_{n-1})$. If $\tilde{n}$ is odd then the result follows immediately from the inductive hypothesis for m', the monotonicity of e_1, and the fact that the pre-image of a general position set under a coordinate projection is in general position. Thus we can assume that $\tilde{n}$ is even.

Assume that the result does not hold for m. Since $\tilde{n}$ is even we can again bring the matrix M into consideration. The assumption implies that d is the 0 polynomial, i.e., every arrangement with the multiplicity vector m has

$$e_1 \leq \frac{\tilde{n}}{2} - 1. \tag{5.1}$$

In particular if we fix a multi-arrangement $\tilde{\mathcal{A}}'$ of $n-1$ points in general position with the multiplicity vector m' then adjoining any extra point q (with multiplicity 1) we obtain a multi-arrangement satisfying (5.1). Applying the inductive hypothesis to m' we obtain that for $\tilde{\mathcal{A}}'$ we have $\exp(\tilde{\mathcal{A}}') = (\frac{\tilde{n}}{2} - 1, \frac{\tilde{n}}{2})$. If θ is a (unique up to a nonzero multiplicative constant) derivation from $D(\tilde{\mathcal{A}}')$ of the degree $\frac{\tilde{n}}{2} - 1$ then for any point, q, θ will preserve q. Thus, θ is in $D(\mathcal{B})$ where $\mathcal{B}$ is an arbitrary (simple) arrangement. Choose $\mathcal{B}$ of a cardinality greater than $\frac{\tilde{n}}{2} + 1$. Then $\exp(\mathcal{B}) = (1, e)$ where the exponent 1 corresponds to the Euler derivation θ_E and $e > \frac{\tilde{n}}{2}$. Hence, θ is proportional to θ_E, i.e., $\theta = P\theta_E$ for a homogeneous polynomial P. Since besides $\theta \in D(\tilde{\mathcal{A}})$ we have $deg(P) \geq \tilde{n} - n$ (see Section 2) which implies

$$\tilde{n} - n + 1 \leq \frac{\tilde{n}}{2} - 1. \tag{5.2}$$

The inequality (5.2) is equivalent to $\tilde{n} \leq 2n - 4$ which contradicts the condition (ii) of the theorem. This contradiction completes the proof. $\square$

6. Degeneration of the exponents

In this section, we consider several cases of degeneration of exponents, i.e., examples of (classes of) multi-arrangements satisfying the conditions of Theorem 3.1 and having $e_1 < \frac{\tilde{n}}{2}$. We assume for convenience that $\tilde{n}$ is even and as before represent the defining polynomial of a multi-arrangement as $\tilde{Q} = x^{m_1} y^{m_2} \prod_{i=3}^{n} (x - z_i y)^{m_i}$ ($m_i \leq m_j$ for $j < i$). It is necessary for degeneration that the $(n-2)$-tuple $(z_3, \dots, z_n)$ annihilates the polynomial d.

Remark 6.1. The polynomial d has a factor whose zero locus does not contain acceptable $n-2$-tuples. Factoring z_i from each row of a block L_i of the matrix M, then subtracting similar rows of different blocks and factoring differences $(z_i - z_j)$ we obtain

$$d = \prod_{k=3}^{n} z_k^{m_k} \prod_{3 \leq j < i \leq n} (z_j - z_i)^{m_i} d_1$$

for some polynomial d_1.

We conjecture that d_1 is not divisible by any z_i or $z_i - z_j$. There could be however $(n-2)$-tuples in the zero locus of d_1 that do not define an arrangement (with n points). For instance, if $m = (6,4,3,2,1)$ then $d_1(1,1,1) = 0$.

Examples.

1. $m = (3,3,1,1)$. This example was considered first by G.Ziegler [7]. An arbitrary multi-arrangement with the multiplicity vector m is projectively equivalent to $x^3y^3(x-y)(x-zy)$ ($z \in \mathbb{C} \setminus \{0,1\}$) and the matrix M is

$$M = \begin{pmatrix} 1 & -1 \\ z^3 & -z \end{pmatrix}.$$

The polynomial $d_1 = z+1$ vanishes at $z = -1$. Thus the only degenerate case is $x^3y^3(x^2-y^2)$ that have exponents (3,5). Notice that the degeneration here happens exactly in the case where the quadruple (p_1,p_2,p_3,p_4) of points is harmonic (i.e., its cross-ratio equals -1).

2. An interesting generalization of the previous example is given by the vector $m = (k,k,1,\ldots,1)$ where $k \geq 3$ and $n = k+1$. A respective multi-arrangement can be written as $x^ky^k\prod_{i=1}^{k-1}(x - z_iy)$. As a straightforward computation shows this arrangement has $e_1 = k$ (the minimal possible) if and only if the set $\{z_1, z_2, \ldots, z_{k-1}\}$ equals to the set of all roots of degree $(k-1)$ of a nonzero (complex) number. Notice that $\tilde{n} = 3k-1$ whence m satisfies the conditions of Theorem 3.1. Thus, this series of examples shows that the value of e_1 can decrease with respect to the general position by any positive integer.

3. Let us generalize the previous example further. Put $m = (m_1, m_2, 1, \ldots, 1)$ with an arbitrary n satisfying $m_1 - m_2 < n-2 < m_1 + m_2$. (We still assume that $\tilde{n} = m_1 + m_2 + n - 2$ is even which is not very important). It is easy to check that the conditions on m_1, m_2 and n imply the conditions (i) and (ii) of Theorem 3.1 whence for an arrangement in general position $e_1 = \frac{\tilde{n}}{2}$. The matrix M for this example differs from a Vandermonde matrix by the only one gap in the degrees, down from $m_1 - 1$ to $\frac{m_1-m_2+n-4}{2}$ (after factoring out z_i from the ith row for every i). In other words $d_1 = \pm s_\lambda$ where s_λ is the Schur function for the rectangular partition diagram with the base $\frac{m_1+m_2-n}{2}$ and the height $\frac{-m_1+m_2+n-2}{2}$. Thus we obtain the description of the set of arrangements with degenerate e_1 as the zero locus of the special 'rectangular' Schur function.

4. Let us recall Example 1 in this section. It has $\tilde{n} = 8$ which is the smallest value of $\tilde{n}$ that can allow different e_1. Besides the example 1, there is only one more vector m with $\tilde{n} = 8$ such that e_1 is not determined by m. It is $m = (3,2,2,1)$. Indeed m satisfies the conditions (i) and (ii) of Theorem 3.1 and for an arrangement $x^3y^2(x-z_1y)^2(x-z_2y)$ the condition for $e_1 = 3$ is $d_1 = -2z_1 + z_2 = 0$. This defines one (up to projective isomorphism) arrangement with $z_1 = 1$ and $z_2 = 2$. Notice that again the quadruple $(0,-2,\infty,-1)$ of points in $\mathbb{CP}^1$ is harmonic (cf. the order of points from Example 1).

One more class of examples with a similar flavor is given by Proposition 7.4.

7. The Terao conjecture

As we mentioned in the Introduction the main motivation for this work is the Terao conjecture and Yoshinaga theorem for 3-arrangements.

Conjecture 7.1. (**Terao,** [2]) If two hyperplane arrangements over the same field have isomorphic intersection lattices and one of them is free then the other is free as well.

This conjecture was posed (as a question) more than 20 years ago ([4]) and is still open. The first non-trivial (and already hard) case is formed by arrangements in $\mathbb{C}^3$ or equivalently in $\mathbb{CP}^2$. The main evidence in favor of the conjecture is a theorem by Terao ([2], Theorem 4.61) that for a free arrangement its exponents are precisely the roots of the characteristic polynomial of its intersection lattice whence are determined by the intersection lattice. We also mention a result from ([6]) that in the space of all arrangements with a fixed intersection lattice the free ones form a set open in the Zariski topology.

Some recent progress has been made by Yoshinaga in [5]. We paraphrase here the relevant part of his Theorem 2.2.

Theorem 7.2. (**Yoshinaga,** [5]) *If an arrangement $\mathcal{A}$ of lines in $\mathbb{CP}^2$ is free with exponents $1, e_1, e_2$ then its restriction to every $\ell \in \mathcal{A}$ has (as a multi-arrangement) exponents e_1, e_2. If the characteristic polynomial of $\mathcal{A}$ has roots $1, d_1, d_2$ and the restriction of $\mathcal{A}$ to some line has exponents d_1, d_2 then $\mathcal{A}$ is free.*

Yoshinaga's theorem implies in particular that if an arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ gives a counterexample to the Terao conjecture then its restrictions to all lines allow simultaneous degeneration of their minimal exponents. Since the multiplicity vectors of the restrictions are determined by the intersection lattice of $\mathcal{A}$, any time the multiplicity vector on one of the lines determines the exponents we obtain a class of arrangements (or lattices) satisfying the conjecture.

Let us give some examples to this effect. To avoid any confusion let us emphasize that the multiplicity $\bar{m}(p)$ of a point p for an arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ is the number of all lines of the arrangement passing through p. If $p \in \ell \in \mathcal{A}$ then the multiplicity of p in the restricted to ℓ multi-arrangement is $m(p) = \bar{m}(p) - 1$.

Terao's conjecture holds for the following classes of arrangements $\mathcal{A}$ in $\mathbb{CP}^2$.

1. There exists a line in $\mathcal{A}$ whose intersections with other lines of $\mathcal{A}$ are concentrated at no more than 3 points.
2. There exists a line in $\mathcal{A}$ such that all points on it have multiplicities not larger than 3.
3. There exists a line in $\mathcal{A}$ such that the multiplicity vector of the restricted multi-arrangement satisfies $m_1 \geq \sum_{i=2}^{n} m_i$.
4. There exists a line in $\mathcal{A}$ such that the average of the multiplicities of the restricted multi-arrangement is less than 2.

Remark 7.3. If $\mathcal{A}$ is a line arrangement in $\mathbb{CP}^2$ such that $|\mathcal{A}| \leq 8$ then either condition 1 or 4 holds. Hence, for $|\mathcal{A}| \leq 8$ Terao's conjecture holds.

In fact we can prove that a larger class of arrangements satisfy Terao's conjecture.

Proposition 7.4. Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$. If there exists a intersection point p such that $\bar{m}(p) > \frac{1}{2}(|\mathcal{A}| - 3)$ then Terao's conjecture holds for $\mathcal{A}$ (i.e. for all arrangements with the intersection lattice isomorphic to that of $\mathcal{A}$).

Proof. Suppose p satisfies the condition and denote by $\ell_1, \ell_2, ..., \ell_k$ all the lines of $\mathcal{A}$ passing through p where $k = \bar{m}(p)$. There are two alternatives.

(i) There are two lines $\ell,\ \ell' \in \mathcal{A}$ such that $p' = \ell \cap \ell' \notin \ell_i$ for every i. Then the points $\ell \cap \ell_1, ..., \ell \cap \ell_k, p'$ are pairwise distinct. this implies that the number k' of intersection points on ℓ satisfies $k' \geq k + 1 > \frac{1}{2}(|\mathcal{A}| - 1)$ whence the average of the multiplicities of the restriction of $\mathcal{A}$ to ℓ is less than 2. The result follows from (4) above.

(ii) Every point of intersection lies on a line passing through p. This implies that the intersection lattice of $\mathcal{A}$ is supersolvable (see [2], pp. 30,31). Since every arrangement with a supersolvable lattice is free ([2], Theorem 4.58) the result follows. □

In particular, Remark 7.3 generalizes to the following.

Corollary 7.5. Terao's conjecture holds for line arrangements in $\mathbb{CP}^2$ of cardinality less than 11.

Proof. Indeed every arrangement with at most 10 lines either satisfies the condition 2 above or the condition of the previous proposition. □

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