Cohomology of the Orlik-Solomon algebras and local systems

ANATOLY LIBGOBER

Department of Mathematics, University of Illinois, Chicago, Ill 60607

SERGEY YUZVINSKY

Department of Mathematics, University of Oregon, Eugene, OR 97403

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1 Introduction

One of the central questions in the theory of hyperplane arrangements is the problem of expressing topological invariants of the complements to arrangements in terms of combinatorics. Probably the first non trivial result in this direction is due to Arnold, Brieskorn and Orlik-Solomon who calculated the cohomology algebra of the complement (referred below to as the Orlik-Solomon algebra) in terms of the intersection lattice of an arrangement . The question to which extent the fundamental group of the complement can be described in combinatorial terms turned out to be rather difficult (cf. [8], [15]). In this paper we shall show how certain *invariants* of the fundamental group, namely its characteristic varieties containing the trivial character of the fundamental group, can be calculated from the intersection lattice. As for the most questions about fundamental groups, due to Lefschetz type theorems, the study of these invariants sufficient to carry out for arrangements of lines. The arrangements for which these invariants are non trivial have an interesting combinatorial structure namely they admit certain partition of the collection of lines into disjoint groups. We show that such decompositions can be detected either combinatorially, using methods going back to Vinberg and Kac and which were designed for the classification of Kac-Moody algebras, or algebro-geometrically, using pencils of planes curves whose singular members form the arrangement. Moreover we were able in some cases to give classification of arrangements for which these invariants are non trivial and it suggests that a complete classification may be a feasible task.

As we mentioned, the invariants of the fundamental group in question are the characteristic varieties. For an arrangement with the complement M such a variety belongs to the torus of characters of the fundamental group $Hom(\pi_1(M), \mathbf{C}^*)$. More precisely the characteristic variety of M is formed by those characters of $\pi_1(M)$ for which the corresponding local system has non-vanishing cohomology in dimension one. It follows from a theorem of Arapura [1] that these subvarieties are unions of translated subtori and we are interested in subtori which contain the trivial character (i.e. for which no translation is needed).

The starting point in our study is the connection between the following two kinds of cohomology that has been known for some time. The first kind is the cohomology $H^*(A, a)$ of the Orlik-Solomon algebra A of an arrangement provided with the differential of multiplication by an element $a \in A$ of degree 1. The second kind is the cohomology $H^*(M, \mathcal{L}(a))$ of the complement M of the arrangement with the coefficients in the local system $\mathcal{L}(a)$ defined by the 1-differential form corresponding to a. According to [4, 12] these cohomologies coincide if a is sufficiently generic, more precisely if the residue of the differential form at certain intersections of hyperplanes is not a positive integer. The problem of vanishing of the cohomology of either kind also was studied in several papers (e.g. see [8, 18]).

M.Falk in [5] looked at $H^1(A, a)$ from a different point of view. He defined the variety R_1 of all the elements $a \in A_1$ with $H^1(A, a) \neq 0$ and considered its isomorphism class as an (full) invariant of A (its quadratic closure). In [9], the characteristic varieties of algebraic curves were used for the case of arrangements of lines in the projective plane. For this case, a new sufficient condition was obtained for the isomorphism of the two kinds of cohomology above. This method was further developed in [3]. In particular, the conjecture of Falk that R_1 is the union of linear subspaces of A_1 was proved there.

As for the relation between the two types of the cohomology we propose the following conjecture (Conjecture 4.4).

For every p and almost every $a \in A_1^*$ among those with $H^p(M, \mathcal{L}(a)) \neq 0$ one has

$$\dim H^p(M, \mathcal{L}(a)) = \sup_{N \in \mathbf{Z}^n} \dim H^p(A^*, a + N)$$

where the identification $A_1 = \mathbb{C}^n$ is given by the natural basis of A_1 (in one-to-one correspondence with the set of hyperplanes). Choice of precise meaning of "almost all" can be made stronger (resp. weaker), e.g. "for all a for which the corresponding local system is outside of a finite (resp. Zariski closed) set in the space of local systems with non-vanishing pth cohomology". The fact that the left hand side is not smaller than the right hand side is a simple corollary of [4] (see Corollary 4.3). We prove the stronger version of the conjecture for $p=1$ and $a \in R_1$ in Theorem 5.3.

To address this conjecture it is natural to try to find out more about the components of R_1. In [5], a basic piece of data for describing a component was so called neighborly partition of the set of hyperplane. Then the component was given by a system of linear and quadratic equations. We suggest to start from a linear system. Let us associate with every $a \in A_1$ such that $\sum_{i=1}^n a_i = 0$ the set $\mathcal{X} = \mathcal{X}(a)$ of all the elements X of rank 2 of the intersection lattice of the arrangement such that $\sum_{H_i \supset X} a_i = 0$. Treating elements X as sets of hyperplanes intersecting at X we can describe $\mathcal{X}$ by its incidence matrix J. Our main tool is the symmetric matrix $Q = J^t J - E$ where E has all the entries 1. Since all the off-diagonal entries of Q are either 0 or -1 we can apply the Vinberg classification and its generalization (see Theorem 2.2) to it. Since besides any 1-cocycle of A is in the null-space of Q we obtain some new information about $H^1(A,a)$. The main piece of it is that $\dim H^1(A,a)$ is defined by $\mathcal{X}(a)$ (see 3.5). This implies that the set of irreducible components of R_1 is in a one-to-one correspondence with the set of null-spaces of the matrices Q ($a \in A_1$) and distinct components intersect only at 0. In fact, a component V coincides with the set of cocycles for any complex (A,a) with $a \in V \setminus \{0\}$. Moreover, if $a \in R_1$ then the respective matrix Q should have at least three irreducible components of affine type (see Section 2 for definition). This condition turns out to be quite restrictive and allows us to classify certain classes of lattices (matroids) with $R_1 \neq 0$ (see Section 6).

On the other hand, intersecting a projective hyperplane arrangement by a plane in general position, i.e. using a Lefschetz type argument referred to above, we obtain an arrangement of lines in $\mathbf{P}^2$. Blowing up $\mathbf{P}^2$ at the set $\mathcal{X}'$ of points corresponding to $\mathcal{X}$ gives a variety P' that carries quite a lot of information about the initial arrangement. For instance, we recover Q as the minus intersection form on P'. To each component of R_1 having a positive dimension we can associate a pencil of curves, i.e. an algebraic map of the complement of $\mathcal{X}'$ onto the complement of a certain finite subset of $\mathbf{P}^1$. The structure of the matrix Q is closely related to the structure of this pencil. The xistence of the pencils imposes restrictions on the euler characteristic of P' which in turn yields strong restrictions on the arrangement. We obtain the classification and nonexistence results mentioned above using a combination of combinatorial and topological points of view.

The layout of the paper is as follows. In section 2, we study properties of matrices Q using the Vinberg classification and generalizing it. In section 3, we use these properties to obtain information about $H^1(A,a)$. In section 4, we discuss relations between $H^p(A,a)$ and $H^p(M,\mathcal{L}(a))$ and state the conjecture. In section 5, we generalize results from [1] and prove a version of the conjecture for $p=1$. Sections 6 considers from combinatorial and section

7 from algebro-geometric points of view examples and classification of some types of Q both realizable and non-realizable by matroids and arrangements. In particular we obtain complete classification of arrangements with positive dimensional R_1 and with each line containing exactly three points, the arrangements with matrix Q satisfying certain properties and bound on the number of lines in an arrangement with fixed r, k where $r = \dim R_1$ and k is the maximal number of vertices on each line

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2 Matrices of collections of subsets

In this section, I is a finite set and $\mathcal{X}$ is a non-empty collection of its subsets such that each two elements of $\mathcal{X}$ have at most one common element. We will always assume that both I and $\mathcal{X}$ are linearly ordered. Let $J = J(\mathcal{X})$ be the incidence matrix of $\mathcal{X}$, i.e. the $|\mathcal{X}| \times |I|$-matrix with entries $a_{X,i} = 1$ if $i \in X$ and $a_{X,i} = 0$ otherwise. Also let E be the $|I| \times |I|$-matrix whose every entry is 1. Our goal in this section is to analyze the matrix

$$Q = Q(\mathcal{X}) = J^t J - E.$$

For each real matrix R denote by $V(R)$ the null space of R (consisting of real column vectors). Notice that $V(E) = \{u \in \mathbb{R}^I | \sum_{i \in I} u_i = 0\}$. Then for any subspace $V \subset \mathbb{R}^I$ put $V^* = V \cap V(E)$. The following observation is immediate.

Lemma 2.1 $V(Q)^* = V(J)^*$.

Clearly Q is symmetric and its every entry off the diagonal is either -1 or 0. The decomposition of Q in the direct sum of its indecomposable principle submatrices defines a partition Π of I. This partition can be defined in terms of $\mathcal{X}$ as follows. Let Γ be the graph on I whose edges are the pairs i, j that are not in any $X \in \mathcal{X}$. Then Π is the partition of Γ into its connected components.

We have $Q = \oplus_{K \in \Pi} Q_K$ where each Q_K is a integer matrix satisfying the same conditions as Q and besides being indecomposable. Thus one can apply the Vinberg classification (e.g., see [7], pp. 48-49) to Q_K.

Let us recall Vinberg's result. Suppose $R = (a_{ij})$ is a $m \times m$ real indecomposable matrix with two extra properties: (a) $a_{ij} \leq 0$ for $i \neq j$ and (b) $a_{ij} = 0$ implies $a_{ji} = 0$. We call a real column vector $u^t = (u_1, \ldots, u_m)$ positive and write $u > 0$ if all $u_i > 0$ ($u < 0$ if $-u > 0$).

Then one and only one of the following three possibilities holds for R:

(i) R is of finite type, i.e., it is positive definite (equivalently for every $u > 0$ we have $Ru > 0$);

(ii) R is of affine type, i.e., it is positive semidefinite of rank $m-1$ and its null space is spanned by a positive vector;

(iii) R is of indefinite type, i.e., there exists a positive vector u such that $Ru < 0$.

Applying that classification to Q_K we will call it simply finite, affine or indefinite if it is of the respective type.

The following remark will be useful. Suppose $\mathcal{X}$ does not cover I. Then at least one row of Q consists of -1 whence Q is irreducible and indefinite. If $\mathcal{X}$ covers I then all diagonal elements of Q are non-negative.

Now we can prove the main result of the section.

Proposition 2.2 *For matrix $Q(\mathcal{X})$ there are precisely two possibilities:*

(i) For every $K \in \Pi$, Q_K is finite or affine;

(ii) For a unique $K_0 \in \Pi$, Q_{K_0} is indefinite and Q_K is finite for every other $K \in \Pi$.

Proof. It suffices to show that there cannot be distinct $K_1, K_2 \in \Pi$ such that $Q_1 = Q_{K_1}$ is indefinite and $Q_2 = Q_{K_2}$ is either indefinite or affine. Suppose to the contrary that Q_i $(i = 1, 2)$ are such. Then there exist column vectors $u^i > 0$ in $\mathbb{R}^{K_i}$ such that $Q_1 u^1 < 0$ and $Q_2 u^2 < 0$ or $Q_2 u^2 = 0$. We can also assume that $\sum_{j\in K_1} u^1_j = \sum_{j\in K_2} u^2_j$. Define $u \in \mathbb{R}^I$ by $u_j = u^1_j$ if $j \in K_1$, $u_j = -u^2_j$ if $j \in K_2$ and $u_j = 0$ if $j \in I \setminus (K_1 \cup K_2)$. Notice that $\sum_{j\in I} u_j = 0$ whence $Eu = 0$. Then we have using (,) for the standard dot product

$$0 \le (Ju, Ju) = (Qu, u) + (Eu, u)$$
$$= (Q_1 u^1, u^1) + (Q_2 u^2, u^2) \le (Q_1 u^1, u^1) < 0$$

which is a contradiction. $\square$

Definition 2.3 *The collection $\mathcal{X}$ is called either affine or indefinite if respectively possibility (i) or (ii) from Proposition 2.2 realizes for $Q(\mathcal{X})$. In the former case, we denote by $\Pi_1(\mathcal{X})$ the set of elements $K \in \Pi(\mathcal{X})$ such that Q_K is affine. In the latter case, $K_0 \in \Pi(\mathcal{X})$ is always the unique element such that Q_{K_0} is indefinite.*

Corollary 2.4 *Put* $V = V(\mathcal{X}) = V(Q(\mathcal{X}))$.

(i) If $\mathcal{X}$ *is affine then* $\dim V = |\Pi_1(\mathcal{X})|$ *and there is a basis of* V *consisting of vectors* u^K $(K \in \Pi_1(\mathcal{X}))$ *such that the restriction of* u^K *to* K *is positive and its restriction to the other elements of* $\Pi(\mathcal{X})$ *is 0. In particular,* $\dim V^* = |\Pi_1(\mathcal{X})| - 1$.

(ii) If $\mathcal{X}$ *is indefinite then all the vectors from* V *are 0 on every* $K \in \Pi(\mathcal{X}) \setminus \{K_0\}$.

In the latter case of Corollary 2.4, $\Pi(\mathcal{X})$ does not define $\dim V$.
We will also use the following observation.

Proposition 2.5 *If* $\mathcal{X}$ *is affine then for every* $X \in \mathcal{X}$ *we have* $X \cap K \neq \emptyset$ *either for all* $K \in \Pi_1(\mathcal{X})$ *or for none of them.*

Proof. Suppose to the contrary that there exists $X \in \mathcal{X}$ and $K' \in \Pi_1(\mathcal{X})$ such that $X \cap K' = \emptyset$ but $X \cap K \neq \emptyset$ for $K \in \bar{\Pi} \subset \Pi_1(\mathcal{X})$ with a non-empty $\bar{\Pi}$.

First notice that $|\bar{\Pi}| > 1$. Indeed if $\bar{\Pi} = \{K\}$ then for any $u \in \mathbb{R}^I$ such that $\sum_{i \in X} u_i = 0$ its restriction to K would be 0. But there exists $v \in V(\mathcal{X})^*$ non-zero on K and K' which contradicts Lemma 2.1. Thus there exist distinct $K_1, K_2 \in \bar{\Pi}$.

Now compare $V = V(\mathcal{X})^*$ and $V' = V(\mathcal{X}')^*$ where $\mathcal{X}' = \mathcal{X} \setminus \{X\}$. On one hand, since one equation is deleted we have $V \subset V'$. On the other hand, $\Pi(\mathcal{X}')$ can be obtained from $\Pi(\mathcal{X})$ by gluing together the elements intersecting with X. Since K' is untouched, $Q_{K'}$ is unchanged whence $\mathcal{X}'$ is affine. Since K_1 and K_2 get glued $|\Pi_1(\mathcal{X}')| < |\Pi_1(\mathcal{X})|$ whence by Corollary 2.4 $\dim V' < \dim V$. The contradiction completes the proof. □

3 $H^1(A, a)$

In this section, $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a linear arrangement of hyperplanes in a space of dimension ℓ over an arbitrary field and L is its intersection lattice. By $L(k)$ we denote the subset of L of all elements of rank (i.e., codimension) k. Every $X \in L$ defines the subarrangement $\mathcal{A}_X = \{H \in \mathcal{A} | H \supset X\}$. We will treat elements $X \in L$ also as subsets of $\bar{n} = \{1, \ldots, n\}$ putting $i \in X$ if $H_i \in \mathcal{A}_X$. In this sense, we put $L'(2) = \{X \in L(2) || X| > 2\}$. Notice that $L'(2)$ is the set of all $X \in L(2)$ with indecomposable $\mathcal{A}_X$. For every two distinct $i, j \in \bar{n}$ there exists a unique $X_{ij} \in L(2)$ such that $i, j \in X_{ij}$.

Now let us fix an infinite field F of a characteristic different from 2 and recall the definition of the Orlik-Solomon algebra $A = A(\mathcal{A})$ (depending on

L and F only). Let E be the graded exterior algebra over F with generators $e_1, \ldots, e_n$. Define the linear map $\partial : E_p \to E_{p-1}$ by

$$\partial(e_{i_1} \cdots e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} e_{i_1} \cdots \hat{e}_{i_j} \cdots e_{i_p}.$$

Then A is the factor algebra of E by the homogeneous ideal generated by $\partial(e_{i_1} \cdots e_{i_p})$ for all dependent sets $\{H_{i_1}, \ldots, H_{i_p}\}$. The grading on E induces a grading $A = \oplus_p A_p$ on A. We keep the notation $e_1, \ldots, e_n$ for the images of e_i in A. Notice that these elements form a basis of A_1 that allows us to identify A_1 with F^n. As in the previous section, we put $A_1^* = \{a \in A_1 | \sum_{i=1}^n a_i = 0\}$ and $U^* = U \cap A_1^*$ for every subset U of A_1. For every $a \in A_1$ the multiplication by a defines the differential of degree 1 on A. The respective cohomology is denoted by $H^*(A, a)$.

The ultimate goal of this section is to study $H^1(A, a)$ as a function of a and the set $R_1 = \{a \in A_1 | H^1(A, a) \neq 0\}$ (cf. [5]). For that we put

$$Z(a) = \{x \in A_1 | ax = 0\}$$

and notice that $H^1(A, a) = Z(a)/Fa$.

The following three lemmas are spread through the literature (e.g. see [3, 5, 9, 18]) and straightforward.

Lemma 3.1 *Let* $\text{rank} L = 2$ *and* $a = \sum_{i=1}^n a_i e_i \in A_1$. *Then* $x = \sum_{i=1}^n x_i e_i \in Z(a)$ *if and only if*

$$a_i \sum_{j=1}^{n} x_j = (\sum_{j=1}^{n} a_j) x_i, \text{ for every } i = 1, \ldots, n.$$

Therefore

(i) if $a \neq 0$ *and* $\sum_{i=1}^n a_i = 0$ *then* $Z(a)$ *can be described by the equation*

$$\sum_{i=1}^{n} x_i = 0;$$

(ii) if $\sum_{i=1}^n a_i \neq 0$ *then* $Z(a) = Fa$*;*

(iii) if $a = 0$ *then* $Z(a) = A_1$.

If $n = 2$ *then the conclusion in case (i) coincides with that in case (ii).*

For every $x = \sum_{i=1}^n x_i e_i \in A_1$ and every $X \in L$ we put $x(X) = \sum_{i \in X} x_i e_i$.

Lemma 3.2 *Let* $\mathcal{A}$ *be arbitrary and* $a \in A_1$. *Then* $x \in Z(a)$ *if and only if* $x(X) \in Z(a(X)) \subset A_1(\mathcal{A}_X)$ *for every* $X \in L(2)$.

Combining the two previous lemmas we obtain the following.

Lemma 3.3 *Using the notation of Lemma 3.2, $Z(a)$ can be described by the following system:*

for every $X \in L'(2)$ such that $a(X) \neq 0$ and $\sum_{i \in X} a_i = 0$,

$$\sum_{i \in X} x_i = 0; \tag{3.1}$$

for every other $X \in L(2)$ and every pair $i < j$ from X,

$$a_i x_j - a_j x_i = 0. \tag{3.2}$$

It is known that if $a \in A_1 \setminus A_1^*$ then $H^*(A, a) = 0$ ([18]). Therefore we restrict our considerations to $a \in A_1^*$. Since the algebra A is skew commutative $b \in Z(a)$ if and only if $a \in Z(b)$. Thus $Z(a)^* = Z(a)$ for every $a \in A_1^* \setminus \{0\}$.

Now we introduce the main objects of the section. For any $a \in A_1^*$ define the subset of $L'(2)$ via

$$\mathcal{X}(a) = \{X \in L'(2) | \sum_{i \in X} a_i = 0, \ a(X) \neq 0\}$$

and the subset of $\bar{n}$ via

$$I(a) = \bigcup_{X \in \mathcal{X}(a)} X \subset \bar{n}.$$

Notice that if $\mathcal{A}$ is a line arrangement then $\mathcal{X}(a)$ is a set of points and $I(a)$ is the subarrangement of all the lines passing through any of the points from $\mathcal{X}$.

The following theorem is the main result of this section. In it we apply the notation from Section 2 to the collection (in fact, covering) $\mathcal{X}(a)$ on $I(a)$. In particular $V(\mathcal{X}(a))$ denotes the subspace of A_1 of all elements that are 0 outside of $I(a)$ and whose restrictions to $I(a)$ lie in the null space of $Q = Q(\mathcal{X}(a))$.

Theorem 3.4 *Let $a \in A_1^* \setminus \{0\}$. If $\mathcal{X}(a) = \emptyset$ or $\mathcal{X}(a)$ on $I(a)$ is indefinite then $a \notin R_1$. If this collection is affine then $Z(a) = V(\mathcal{X}(a))^*$, in particular* $\dim Z(a) = |\Pi_1(\mathcal{X}(a))| - 1$.

Proof. Let us keep in mind that for every $j \in \bar{n} \setminus I(a)$ and $i \in \bar{n}$ we have $X_{ij} \notin \mathcal{X}(a)$ whence (3.2) holds for every $x \in Z(a)$. Thus if there exists $j \in \bar{n} \setminus I(a)$ such that $a_j \neq 0$ then $\dim Z(a) = 1$ and $a \notin R_1$. In particular this covers the case where $\mathcal{X}(a) = \emptyset$.

From now on we can assume that $\mathcal{X}(a) \neq \emptyset$ and $a_j = 0$ for every $j \notin I(a)$. For every $j \notin I(a)$, i with $a_i \neq 0$ and $x \in Z(a)$, the equalities (3.2) imply $x_j = 0$. Thus x can be non- zero only on $I(a)$ and we can identify x with its restriction to $I(a)$. By definition of $\mathcal{X}(a)$ and Lemma 3.3, (3.1) holds for every $X \in \mathcal{X}(a)$ whence $J(\mathcal{X}(a))x = 0$. Since besides $\sum_{i=1}^{n} x_i = 0$ we have $Q(\mathcal{X}(a))x = 0$, i.e., $x \in V(\mathcal{X}(a))^*$. Thus $Z(a) \subset V(\mathcal{X}(a))^*$.

Now suppose that $\mathcal{X}$ is indefinite and $x \in Z(a)$. Since $x \in V(\mathcal{X}(a))$ we conclude that x is 0 on each $K \in \Pi = \Pi(\mathcal{X}(a))$ such that $K \neq K_0$. Let us study the coordinates of x on K_0. If $i, j \in K_0$ are such that $X_{ij} \notin \mathcal{X}(a)$ then (3.2) holds. Since K_0 is an element of Π and $a \neq 0$, (3.2) holds for every $i, j \in K_0$ and $x = ca$ on K_0 for some $c \in F$. Summing up we see that $x = ca$ whence $\dim Z(a) = 1$, i.e., $a \notin R_1$.

Now suppose $\mathcal{X} = \mathcal{X}(a)$ is affine and $x \in V(\mathcal{X})^*$, i.e., $Q(\mathcal{X})x = 0$ and $Ex = 0$. By Lemma 2.1 we have $J(\mathcal{X})x = 0$, i.e., (3.1) holds for $X \in \mathcal{X}$. Moreover for every $i, j \in I(a)$ such that $X_{ij} \notin \mathcal{X}$ we have $i, j \in K$ for some $K \in \Pi$. If Q_K is finite then $x_i = x_j = 0$ and (3.2) holds tautologically. If Q_K is affine then the kernel of Q_K is one-dimensional and (3.2) holds again. By Lemma 3.3 $x \in Z(a)$ that completes the proof. □

Corollary 3.5 *For every $a \in A_1^*$ the dimension of $Z(a)$ is defined by $\mathcal{X}(a)$. If the collection $\mathcal{X}(a)$ is affine (in particular if $a \in R_1$) then the space $Z(a)$ itself is defined by $\mathcal{X}(a)$. If besides $a \neq 0$ then for every $b \in Z(a) \setminus \{0\}$ we have $Z(b) = Z(a)$.*

Proof. Only the last statement needs a proof. Let $b \in Z(a) = V(\mathcal{X}(a))^*$ and $b \neq 0$. Then there exists $K \in \Pi_1(\mathcal{X}(a))$ such that $b_i \neq 0$ for every $i \in K$. Take $X \in \mathcal{X}(a)$. We have $\sum_{i \in X} b_i = 0$. Besides by Proposition 2.5, since $a \neq 0$ we have $X \cap K \neq \emptyset$ whence $b(X) \neq 0$. This implies that $X \in \mathcal{X}(b)$ and $\mathcal{X}(a) \subset \mathcal{X}(b)$. By symmetry $\mathcal{X}(a) = \mathcal{X}(b)$ and the result follows. □

Corollary 3.6 *(cf. [9, 3]) All the irreducible components of R_1 are linear.*

Proof. Due to Theorem 3.4, R_1 is the union of finite number of linear spaces. The result follows. □

The previous corollary leaves open the question what the linear components of R_1 are. The answer turns out to be simple.

Corollary 3.7 *For every $a \in R_1 \setminus \{0\}$, the space $Z(a) = V(\mathcal{X}(a))^*$ is a maximal linear space in R_1 whence an irreducible component of R_1.*

Proof. Theorem 3.4 and Corollary 3.5 imply that

$$R_1 = \bigcup_{a \in R_1 \setminus \{0\}} Z(a)$$

and each two distinct $Z(a)$ and $Z(b)$ intersect only at 0. Since there is only finite number of these spaces and $|F| = \infty$, each one of them is a maximal linear subspace in R_1. □

Corollary 3.8 *The pairwise intersection of the irreducible components of R_1 is 0.*

Corollary 3.7 gives a description of the irreducible components of R_1 in terms of elements of A_1^*. One can give a description by properties of subsets of $L'(2)$. For every $\mathcal{X} \subset L'(2)$ put $I(\mathcal{X}) = \cup_{X \in \mathcal{X}} X$ and as above denote by $V(\mathcal{X})^*$ the subspace of A_1^* corresponding to $\mathcal{X}$ on $I(\mathcal{X})$.

Theorem 3.9 *Let $\mathcal{X} \subset L'(2)$ satisfy the following conditions. (i) It is affine on $I = I(\mathcal{X})$, (ii) $|\Pi_1(\mathcal{X})| \geq 3$ and (iii) no $X \in \mathcal{X}$ lies in $\bigcup_{K \in \Pi \setminus \Pi_1} K$. Then $V(\mathcal{X})$ is an irreducible component of R_1. The map $\mathcal{X} \mapsto V(\mathcal{X})$ defines a one-to-one correspondence of the set of collections satisfying conditions (i)-(iii) with the set of irreducible components of R_1.*

Proof. Suppose $\mathcal{X}$ satisfies the conditions (i)-(iii). Put $V = V(\mathcal{X})^*$ and fix $a \in V \setminus \{0\}$. If $X \in \mathcal{X}$ then $\sum_{i \in X} a_i = 0$. Besides since $a \neq 0$ and $\mathcal{X}$ satisfies (iii), Proposition 2.5 implies that $a(X) \neq 0$. Thus $X \in \mathcal{X}(a)$ and $\mathcal{X} \subset \mathcal{X}(a)$.

On the other hand, let $X \in \mathcal{X}(a)$. Since $a(X) \neq 0$ there exists $K \in \Pi_1(\mathcal{X})$ such that $X \cap K \neq \emptyset$. Since $\sum_{i \in X \cap I} a_i = 0$ and $a(K) > 0$ or $a(K) < 0$ there exist distinct $K_1, K_2 \in \Pi_1(\mathcal{X})$ intersecting with X. This means that $X = X_{ij}$ for some $i \in K_1$ and $j \in K_2$. Since by definition of $\Pi(\mathcal{X})$ we have $X_{ij} \in \mathcal{X}$ it follows that $X \in \mathcal{X}$ whence $\mathcal{X}(a) = \mathcal{X}$.

Now Theorem 3.4 imply that $V = V(\mathcal{X}(a))^* = Z(a)$. By condition (ii) $\dim Z(a) \geq 2$ whence $a \in R_1$. Then Corollary 3.7 implies that $V(\mathcal{X})$ is an irreducible component of R_1.

Conversely let V be an irreducible component of R_1 and $a \in V \setminus \{0\}$. By Corollaries 3.7 and 3.8, $V = V(\mathcal{X}(a))^*$. Clearly $\mathcal{X}(a)$ satisfies the conditions (i)-(iii) above. Since $\mathcal{X}(a) \mapsto V$, the map is surjective.

Finally suppose that $V = V(\mathcal{X}_1) = V(\mathcal{X}_2)$ where X_i $(i = 1, 2)$ satisfy the conditions (i)-(iii). Choose a non-zero $a \in V$. By the first part of the proof, $\mathcal{X}_1 = \mathcal{X}(a) = \mathcal{X}_2$ whence the above map is injective. This completes the proof. $\square$

Remark 3.10 *Let us compare the description of the irreducible components of R_1 from Theorem 3.9 and that from [5]. Fix an $\mathcal{X}$ as in the above theorem and consider the submatroid of L on $I' = \bigcup_{K \in \Pi_1(\mathcal{X})} K$. Then it is easy to see that the partition π generated by $\Pi_1(\mathcal{X})$ on I' is neighborly (using the terminology from [5], p.144) and the set of all polychrome flats of rank 2 coincides with $\mathcal{X}$ (moreover every polychrome X intersects every element of π). Thus L_π from [5] coincides with $V(\mathcal{X})$ which implies $L_\pi = V_\pi$ (cf. [5], Remark 3.15).*

4 $H^*(A^*, a)$ and cohomology of local systems

For the rest of the paper we consider only complex arrangements and $F = \mathbb{C}$. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^\ell$ and A its Orlik-Solomon algebra. According to the Orlik-Solomon theorem [13], A is isomorphic to the algebra of closed differential forms on the complement M' of $\mathcal{A}$ generated by the forms $\omega_i = \frac{d\alpha_i}{\alpha_i}$ where α_i is any linear functional on $\mathbb{C}^\ell$ with kernel H_i. The isomorphism is given by $e_i \mapsto \omega_i$. Due to the Brieskorn theorem [2] this algebra of forms is isomorphic to $H^*(M', \mathbb{C})$ under the de Rham homomorphism. If we projectivize $\mathcal{A}$ and denote by M its complement in $\mathbf{P}\mathbb{C}^{\ell-1}$ then $H^*(M, \mathbb{C})$ is isomorphic to the subalgebra A^* of A generated by A_1^*.

Denote the form on M corresponding to $a \in A_1^*$ by $\omega(a)$. Since $\omega(a)$ is closed and $\omega(a) \wedge \omega(a) = 0$ it defines a local system on M that we denote by $\mathcal{L}(a)$. More explicitly $\mathcal{L}(a)$ is associated with the one-dimensional representation of $\pi_1(M)$ sending its generator corresponding to H_k to $\exp(2\pi i a_k)$.

We will need the following theorem.

Theorem (STV) [12]. *Suppose that $a \in A_1^*$ and for all $X \in L$ such that $\mathcal{A}_X$ is indecomposable, the sum $\sum_{i \in X} a_i$ is not a positive integer. Then*

$$H^p(M, \mathcal{L}(a)) = H^p(A^*, a)$$

for every p.

Let us record a simple but important observation for the rest of this section.

Lemma 4.1 *For every $a \in A_1^*$*

(i) $H^*(A^*, a) = H^*(A, \lambda a)$ for every $\lambda \in \mathbb{C}^*$;

(ii) $H^*(M, \mathcal{L}(a)) = H^*(M, \mathcal{L}(a + N))$ for every $N = (N_i) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n N_i = 0$.

The main result of this section is as follows.

Proposition 4.2 *For every $a \in A_1^*$ and every p*

$$\dim H^p(M, \mathcal{L}(a)) \geq \dim H^p(A^*, a).$$

Proof. Let $a \in A_1^*$. Fix a real positive number ϵ such that for every $X \in L$ and $0 < t < \epsilon$ the number $(1-t)\sum_{i \in X} a_i$ is not a positive integer. Then by Theorem (STV) and Lemma 4.1(i) we have for every p

$$H^p(M, \mathcal{L}((1-t)a) = H^p(A^*, (1-t)a) = H^p(A^*, a).$$

Using the upper semicontinuity of the dimension of cohomlogy for $t \to 0$ we obtain the result. $\square$

Corollary 4.3 *For every $a \in A_1^*$, $N \in \mathbb{Z}^n$ and p*

$$\dim H^p(M, \mathcal{L}(a)) \geq \dim H^p(A^*, a+N).$$

Corollary 4.3 and examples justify the following conjecture.

Conjecture 4.4 *For every p and almost all $a \in A_1^*$ among those with $H^p(M, \mathcal{L}(a)) \neq 0$ one has*

$$\dim H^p(M, \mathcal{L}(a)) = \sup_{N \in \mathbb{Z}^n} \dim H^p(A^*, a+N).$$

See Introduction for a discussion of "almost all". A version of this conjecture will be proved for $p = 1$ in the next section.

5 Characteristic varieties

Our first goal in this section is to give a more precise statement of the Proposition 1.7 from [1]. Since we work here in more generality than in the previous sections we start by fixing notation.

Let M be an arbitrary quasiprojective variety. We identify the torus $H^1(M, \mathbb{C}^*) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ with the set of all (rank one) local systems on M. We also call $H^1(M, \mathbb{C}^*)$ the torus of characters $\mathrm{Char}(\pi_1(M))$ of $\pi_1(M)$. Let $\Sigma^k(M) \subset \mathrm{Char}(\pi_1(M))$ be the subset consisting of those local systems $\mathcal{L}$ for which $\dim H^1(M, \mathcal{L}) \geq k$. Then $\Sigma^k(M)$ is an algebraic subvariety of $\mathrm{Char}(\pi_1(M))$. We want to study the structure of irreducible components of $\Sigma^k(M)$ of positive dimension. According to [1], such a component belongs to a component of $\Sigma^1(M)$. The latter defines a map $f : M \to C$ to a smooth curve C and has a form $\rho f^*(H^1(C, \mathbb{C}^*))$ for some torsion point ρ in $\mathrm{Char}(\pi_1(M))$. We assume that C *isn't* compact (which is always the case if M is a complement to an arrangement, cf. 5.2), i.e. $\pi_1(C)$ is a free group with, say, n generators. We shall denote such a group below as F_n. Notice that for a local system $\mathcal{L}$ on C we have $\dim H^1(C, \mathcal{L}) = n - 1$ if $\mathcal{L}$ is not trivial and for the trivial local system it is n (indeed F_n has cohomological dimension equal to 1, the euler characteristic is independent of a local system and is equal to $1 - n$ and $\dim H^0(C, \mathcal{L})$ is 1 or zero depending on whether the system is trivial or not).

Theorem 5.1 *Let V be a component of $\Sigma^k(M)$ containing the identity character. Then there is a finite set of local systems $E \subset V$ (possibly empty) and an integer s such that for any local system $\mathcal{L}$ in $V \backslash E$ we have* $\dim H^1(M, \mathcal{L}) =$

s (i.e. $max\{k|V \subset \Sigma^k(M)\} = s$). For such a component and the corresponding map $f : M \to C$ such that $V = f^*(H^1(C, \mathbb{C}^*))$ the number of generators of the fundamental group of C is $s+1$.

Proof. By theorem 1.6 in [1] any local system in V has form $f^*(\mathcal{L})$ for an appropriate local system on C. The main step in the argument is the following statement implicitly, as D.Arapura pointed out, contained in [1]. For all but finitely many local systems in V we have

$$H^1(M, f^*(\mathcal{L})) = H^1(C, \mathcal{L}). \tag{5.1}$$

Using that C has cohomological dimension 1, the projection formula $R^q f_*(f^*(\mathcal{L})) = R^q f_* \mathbb{C} \otimes \mathcal{L}$ and the Leray spectral sequence

$$E_2^{p,q} = H^p(C, R^q f_*(f^*(\mathcal{L}))) \Longrightarrow H^{p+q}(M, f^*(\mathcal{L}))$$

we see that (5.1) is a corollary of the vanishing of $H^0(C, R^1 f_*(\mathbb{C}) \otimes \mathcal{L})$ for all but finitely many local systems $\mathcal{L}$ on C. As it is shown in [1], if U is the subset of C over which f is a locally trivial fibration, then

$$H^0(C, R^q f_* \mathbb{C} \otimes \mathcal{L}) = H^0(U, R^q f_* \mathbb{C} \otimes \mathcal{L}).$$

let g_i $(i = 1, ..., s)$ be a system of generators of $\pi_1(U)$ and $\lambda_i, i = 1, ..., s$ (resp. $\mu_{i,j}, i = 1, .., s, j = 1, \dots, rkR^1 f_* \mathbb{C})$) the eigenvalues of the action of g_i on $\mathcal{L}$ (resp. on $R^1 f_*(\mathbb{C})$). Then $H^0(U, R^q f_* \mathbb{C} \otimes \mathcal{L}) = 0$ unless for each i there is j such that $\lambda_i \cdot \mu_{i,j} = 1$. Hence (5.1) is valid for all but at most $\dim(R^q f_* \mathbb{C}) \cdot s$ local systems. $\square$

Now let $\mathcal{A}$ be an arrangement of n lines in $\mathbf{P}\mathbb{C}^2$, $M = \mathbf{P}\mathbb{C}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ and $A^* = H^*(M, \mathbb{C})$ the subalgebra of the Orlik-Solomon algebra generated by A_1^*. We have the following extension of results of [9] to the case of general arrangements (not necessarily containing a line transversal to the rest of the lines of the arrangement) (cf. also [3]).

Theorem 5.2 *Let $V^k \subset A_1^*$ be an irreducible component of the variety of those $a \in A_1^*$ for which $\dim H^1(A^*, a) \geq k$. Then the image of V^k under the map $\exp : a \mapsto \mathcal{L}(a)$ is an irreducible component of $\Sigma^s(M)$ and exp is a universal covering of this component.*

Proof. By Theorem 3.4 and Corollary 3.7 V^k is a linear space. By Proposition 4.2 we have $\dim H^1(A^*, a) \leq \dim H^1(M, \mathcal{L}(a))$. Thus $\exp V^k \subset \mathcal{V}$ where $\mathcal{V}$ is an irreducible component of Σ^s with $s \geq k$. In this case $\mathcal{V}$ is a subtorus of $H^1(M, \mathbb{C}^*)$ (i.e. no translation is needed) since V^k contains the origin. Let $f : M \to C$ be a map as in Theorem 5.1. Then C is a rational curve, i.e., the complement in $\mathbf{P}^1$ to a set $\{p_1, \dots, p_{s+1}\}$. Indeed, according to H.Hironaka, f is a restriction of $\bar{f} : \bar{M} \to \bar{C}$ where $\bar{M}$ and $\bar{C}$ are compactifications of M and C respectively. If $\bar{C}$ had a positive genus it would admit a non trivial holomorphic 1-form and its pull back via $\bar{f}$ to $\bar{M}$ would yield a non trivial holomorphic 1-form on $\bar{M}$ contradicting the rationality of $\bar{M}$.

Denote by ω_i a 1-form on $\mathbf{P}^1$ that generates $H^1(\mathbf{P}^1 \setminus \{p_i\}, \mathbb{C})$ $(i = 1, \ldots, s+1)$. The pull backs of these forms span an s-dimensional space of $Z(a)$. Since exp is a local homeomorphism and $\dim V^k \leq s$ the result follows. $\square$

Finally we can prove a version of Conjecture 4.4 for $p = 1$. Recall that for every $a \in A_1^*$ we have $H^1(A^*, a) = H^1(A, a)$.

Theorem 5.3 *For all but finitely many cosets mod* $\mathbf{Z}^n$ *of* $a \in R_1$

$$\dim H^1(M, \mathcal{L}(a)) = \sup_{N \in \mathbb{Z}^n} \dim H^1(A, a + N).$$

Proof. The arguments are similar to the proof of Theorem 5.2. We first notice that since $\dim H^1(A, a) > 0$, the local system $\mathcal{L}(a)$ belongs to a component of the characteristic variety of positive dimension containing the identity character. Indeed, since $\dim Z(a) > 1$, one can find a deformation $a(t)$ of a $(a(0) = a)$ for $0 \leq t \leq 1$ such that $a(t)$ is not proportional to $a(t')$ for $t \neq t'$. Then $a(t) \in R_1$ and $\mathcal{L}(a(t))$ provide a deformation of $\mathcal{L}(a)$.

Next let us consider the map $\pi : M \to \mathbf{P}^1 \setminus \{p_1, .., p_m\}$ corresponding to a component of the characteristic variety containing $\mathcal{L}(a)$. Then $\mathcal{L}(a)$ is a pullback of a local system $\mathcal{L}(\omega)$ on $\mathbf{P}^1 \setminus \{p_1, .., p_m\}$ corresponding to a differential form ω. Let $\tilde{\omega}$ be the form with constant coefficients on M representing the cohomology class of the pullback of ω (such a form exists according to Brieskorn's theorem [2]). Then $\tilde{\omega}$ and $\omega(a)$ represent the same local system and hence $\omega(a) = \tilde{\omega} + \omega(N)$ for some $N \in \mathbb{Z}^n$. Equivalenlty $a = \tilde{a} + N$ where $\tilde{a} \in A_1^*$ and $\tilde{\omega} = \omega(\tilde{a})$.

Now we claim that $H^1(A, \tilde{a}) = H^1(M, \mathcal{L}(a))$. We need to show that $\dim H^1(A, \tilde{a}) \geq m-1$ since $\dim H^1(M, \mathcal{L}(a)) = m-1$ with only finitely many exceptions by [1]. Indeed the map $H^1(\mathbf{P}^1 \setminus \{p_1, .., p_m\}) \to A_1^*$ is injective (by Leray Spectral sequence) and its image lies in $Z(\tilde{a})$. This gives us an $(m-1)$-dimensional subspace in $H^1(A, \tilde{a})$ that completes the proof. $\square$.

Exceptional cosets, for which the equality of the theorem fails may indeed exist. It follows from Example 4.5 in [3]. The arrangement in this example consists of seven lines that can be represented in three different ways as the union of braid arrangement with a line connecting two double points. For this arrangement Σ^1 contains non empty Σ^2 consisting of one point outside of the origin.

Finally let us note that theorems 5.2 and 5.3 are valid for arbitrary arrangements in $\mathbf{P}(\mathbb{C}^n)$. Indeed by Lefschetz theorem (cf. [10]) for a generic plane H the cohomology of the complement to arrangement in P^n and its intersection with H are the same in dimension 1 and injects in dimension 2. Hence the cohomology of the Orlik-Solomon algebra in dimension 1 is the same for both arrangements. On the other hand characteristic varieties also are the same since they depend only on the fundamental group (cf. definition in section 1 of [9]). This yields 5.2 for arrangements in $\mathbf{P}^n$. The theorem 5.3 follows from this comparison and the Lefschetz theorem for cohomology with coefficients in local system (which is a consequence of the same argument as in the case of constant coefficients, i.e. that the complement in $\mathbf{P}^n$ obtained from the complement in H by attaching cells of dimension greater than 2 cf. [10],[16])

6 Types of affine matrices, realizations and examples

In this section, we exhibit examples of types of matrices Q with affine components and discuss the problem of their realizations by matroids and arrangements.

Recall that for every subset $\mathcal{X} \subset L'(2)$ we put $I = I(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$ and $n = |I|$. Then $Q = Q(\mathcal{X}) = (q_{ij})$ is a symmetric $n \times n$-matrix with $m_i = q_{ii} \in \mathbb{Z}_+$ and $q_{ij} = -1$ or 0 for $i \neq j$. To parametrize matrices Q we write $Q = Q(m, \Gamma)$ where Γ is the graph on n vertices $v_i, \ldots, v_n$ with edges $\{v_i, v_j\}$ for $q_{ij} = -1$. The vector $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ can be regarded as a labeling of the vertices of Γ. The decomposition of Q into the direct sum of indecomposable components corresponds to the partition of Γ into connected components.

The section is organized in the following way. First we collect some information and series of examples of pairs (Γ, m) with $Q(m, \Gamma)$ affine indecomposable. Then we attend the question which of the affine matrices and more importantly matrices Q having at least three affine components can be realized by matroids. These (especially the latter) are more restrictive conditions which allows us to classify some classes of examples. The final question is whether the matroids appearing in this way can be represented by arrangements of complex lines. We give some examples of representations leaving negative results till the next section where we will apply different methods.

6.1. Affine indecomposable matrices.

We start with the following two general observations.

Proposition 6.1 *Let Γ be an arbitrary (finite) connected graph. Then the set $\mathcal{M}(\Gamma)$ of vectors $m \in \mathbb{Z}_+^n$, for which $Q(m,\Gamma)$ is affine, is finite.*

Proof. Suppose that Γ has n vertices. For each $m \in \mathcal{M}(\Gamma)$ denote by Q_k the principal submatrix of $Q = Q(m,\Gamma)$ such that the involved rows have k smallest m_i and by D_k the determinant of Q_k. Without loss of generality assume that $m_1 \le m_2 \le \cdots \le m_n$. [7], Lemma 4.4 implies that for $k < n$ matrix Q_k is a direct sum of finite matrices whence $D_k > 0$. Thus for fixed $k < n$ and Q_k determinant D_n can be viewed as a polynomial in $m_{k+1}, \dots, m_n$ with a positive coefficient D_k of the highest degree term $m_{k+1} \cdots m_n$. Since $D_n = 0$ the possible value of m_{k+1} is bounded from above whence m_{k+1} can afford only finite set of values. The result follows by induction on k. $\square$

Proposition 6.2 *Let Γ be a connected graph on n vertices with a set E of edges. Let $\mathcal{N}$ be the set of all positive integer column vectors u ($u^t = (u_1, \dots, u_n)$) such that u_i are mutually relatively prime and each u_i divides $u^{(i)} = \sum_{\{i,j\} \in E} u_j$. Define $m = f(u)$ via $m_i = u^{(i)}/u_i$. Then (i) $m \in \mathcal{M}(\Gamma)$, (ii) u lies in the null space of $Q(m,\Gamma)$ and (iii) $f : \mathcal{N} \to \mathcal{M}(\Gamma)$ is a bijection.*

Proof. Fix $u \in \mathcal{N}$ and consider the respective matrix $Q = Q(m,\Gamma)$. Clearly $Qu = 0$ whence Q is affine (see [7], Corollary 4.3) which proves (i) and (ii). Now let Q be an arbitrary integer affine matrix. Then its null space contains a unique positive integer vector with mutually relatively prime coordinates which implies (iii). $\square$

Examples of affine indecomposable matrices.

1. *Laplace matrices of graphs.* Here G is an arbitrary connected graph and Q is its Laplace matrix, i.e., $Q = Q(d,\Gamma)$ where d_i is the degree of v_i. It is easy to check that Q is affine, in particular a column vector u generating its null-space can be taken as $u_i = 1$ for all i.

2. *Affine generalized Cartan matrices.* Here Γ is an extended Dynkin diagram with single edges and $m_i = 2$ for all i. Notice that for the diagrams of type other than $A_\ell^{(1)}$ the labeling is different from that in the previous example.

3. *Full graphs.* Let Γ be the full graph on n vertices. Then it is not hard to describe all the labeling m so that $Q = Q(m,\Gamma)$ is affine.

Proposition 6.3 *For the full graph Γ the matrix $Q = Q(m,\Gamma)$ is affine if and only if*

$$\sum_{i=1}^{n} \frac{1}{m_i + 1} = 1. \tag{6.1}$$

Proof. Since the null space of Q should contain a positive vector the following system should have a positive solution:

$$(m_i+1)x_i = \sum_{j=1}^{n} x_j \ , 1 \leq i \leq n. \tag{6.2}$$

Clearly (6.1) is equivalent to it. Conversely (6.1) implies that any solution of (6.2) is proportional to

$$x_i = \frac{1}{m_i+1}, \ i = 1, \dots, n.$$

□

Notice that the Laplace matrix of the full graph has $m_i = n-1$ for all i.

4. *Bushes.* Let Γ be the bush on $n+1$ vertices. By that, we mean that Γ is a tree whose one vertex v_0 has degree n (the root) and all the other vertices $v_1, \dots, v_n$ (changing enumeration slightly) are leaves. An argument similar to that of Proposition 6.3 shows that Q is affine if and only if

$$m_0 = \sum_{i=1}^{n} \frac{1}{m_i}. \tag{6.3}$$

If (6.3) holds then the null space of Q is generated by the vector $u^t = (1, 1/m_1, \dots, 1/m_n)$. As a concrete example, the complete list of admissible vectors m for the extended Dynkin diagram $D_4^{(1)}$ consists of the following 20 vectors (up to permutations of leaves): (4,1,1,1,1) (the Laplace matrix), (3,2,2,1,1), (2,2,2,2,2) (the Cartan matrix), (2,3,3,3,1), (2,6,3,2,1), (2,4,4,2,1), (1,4,4,4,4), (1,3,4,4,6), (1,12,4,3,3), (1,6,6,3,3), (1,12,12,3,2), (1,15,10,3,2), (1,18,9,3,2), (1,24,8,3,2), (1,42,7,3,2), (1,8,8,4,2), (1,12,6,4,2), (1,20,5,4,2), (1,10,5,5,2), (1,6,6,6,2).

6.2. Realization of matrices Q by matroids.

Here we study the problem about a realization of Q by a matroid of rank at most three. A matrix Q is indecomposable affine in the first part of the subsection and then will have several affine components. We describe matroids as in section 3 by the incidence matrices J of the collections of dependent sets. More precisely the columns of J are parametrized by the points of the matroid and the rows by its dependent sets. If the matroid is representable over a field $\mathbb{C}$ by an arrangement of lines in the projective plane then the columns of J correspond to the lines and its rows to the multiple points of their intersections. Abstractly J is a 0-1 matrix whose any two columns both have entries 1 in at most one common row. The decomposition of Q into indecomposables defines a partition Π of columns of J hence for each $K \in \Pi$ defines a matrix J_K of the respective columns. We will always assume that J_K does not have 0-rows for every K. A realization of Q by J is the identity $Q = J^tJ - E$.

Not all affine matrices are realizable.

Example 6.4 *Let Q be the Laplace matrix of the graph Γ on vertices $\{v_1, \ldots, v_5\}$ which is the full graph on $\{v_2, \ldots, v_5\}$ extended by the single edge $\{1, 5\}$. We want to observe that Q is not realizable. Moreover consider the indecomposable component S (on rows 1 to 4) of $Q + E$:*

$$S = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}.$$

Suppose $S = J^tJ$ for a 0-1 matrix J. Then the first column of J has precisely two 1's in some rows and each of the following three columns has 1 in one of these rows. Thus some two of these three columns have 1 in common row which is a contradiction. Therefore S and Q are not realizable.

If an affine matrix is realizable then it could have several different (up to permutations of rows in J) realizations. Clearly the number of realizations is finite. For instance, the Cartan matrix of type $D_4^{(1)}$ has 3 different realizations. For certain ordering on the vertices they are

$$J_i \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where

$$J_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, J_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

More interesting problem for us is the realization of decomposable matrices with at least 3 affine components. This class turns out to be more restrictive. While studying this class we sometimes apply the dual point of view to a matrix J realizing Q. Each column C_i of it can be considered as a subset of the set R of all rows of J. Since there are no 0-rows in J_K, the system $\mathcal{C}(K) = \{C_i\}_{i\in K}$ is a covering of R for each $K \in \Pi$. If $K' \in \Pi$ and $K' \neq K$ then C_j for every $j \in K'$ is a *transversal* of $\mathcal{C}(K)$. By that we mean that $C_j \cap C_i \neq \emptyset$ for all $i \in K$ whence $|C_i \cap C_j| = 1$. This immediately implies the inequalities

$$|\bar{K}| \leq |C_j| \leq |K| \tag{6.4}$$

where $j \in K'$ and $\bar{K}$ is a subset of K such that $C_i \cap C_k = \emptyset$ for every $i, k \in \bar{K}$.

The inequalities (6.4) allow us to classify in some sense a series of realizable matrices with at least 3 affine components.

Theorem 6.5 *Suppose $Q = Q(m,\Gamma)$ is a realizable matrix whose graph Γ has at least two connected components that are full graphs with at least two vertices each and the matrices corresponding to the components are affine. Then there exists an integer $n \geq 2$ such that all the diagonal elements of Q are $n-1$, $|\Pi| \leq n+1$ and $|K| = n$ for each $K \in \Pi$. If $|\Pi| = n+1$ (i.e. maximal) then the realizations of Q are parametrized by latin $n \times n$ squares with entries $1, \dots, n$.*

Proof. First notice that for any matrix corresponding to a full graph all pairs $\{C_i, C_j\}$ in any realization J are disjoint. Then (6.4) implies that $|C_i| = |K|$ for each i and each $K \in \Pi$. Put $n = |K|$. Since each Q_K is at least 2×2-matrix we have $n \geq 2$. Clearly all the diagonal elements of Q are $n-1$.

Now order all the elements of Π as $K_0, K_1, \dots, K_{\ell-1}$. Ordering the elements inside K_s $(s = 0, 1)$ and the rows of J appropriately we can assume without loss of generality that (i) $C_i = \{n(i-1)+1, \dots, ni\}$ for $i \in K_0$, i.e., $1 \leq i \leq n$, (ii) $C_j = \{j-n, j, \dots, j+(n-2)n\}$ for $j \in K_1$, i.e., $n+1 \leq j \leq 2n$, and (iii) for every $r > 0$ the first n rows of $J_r = J_{K_r}$ form the identity matrix. Now break all the rows of J into blocks of n subsequent rows from $jn+1$ to $(j+1)n$ $(j = 0, 1, \dots, n-1)$. All the blocks of J_r $(r = 1, \dots, \ell-1)$ are in bijection with the set of n-permutations $\{\sigma(i,j)\}$ $(1 \leq i \leq n,\ 1 \leq j \leq \ell-1)$ with the following properties:

(i) $\sigma(1,j) = \sigma(i,1) = e$ (the identity permutation) for all i, j;

(ii) every two permutations $\sigma(i_1, j)$ and $\sigma(i_2, j)$ for $j > 1$ are disjoint, i.e., the images of any element are distinct under these permutations;

(iii) similar condition holds for $\sigma(i, j_1)$ and $\sigma(i, j_2)$ for $i > 1$.

Both yet unproved statements of the theorem follow. □

It is intersting to notice that in terms of line arrangements the above full-graph connected components of Γ correspond to sets of lines such that any point from $\mathcal{X}$ lies on exactly one line of the set. We will discuss the existence of such arrangements in 6.3.

We obtain another series of representable matrices Q using the affine matrices corresponded to bushes. Let Γ be the graph of three connected components, each one being the bush on $r+1$ vertices. Let the restriction of the labeling vector m to each connected component be $(1, r, \ldots, r)$ where 1 labels the root of the bush. Then $Q(m, \Gamma)$ has three affine components and is realizable. As a matroid one can take that of the complex arrangement of lines of type $\mathcal{A}_{r,1,3}$ given by $xyz(x^r - y^r)(x^r - z^r)(y^r - z^r)$. It follows immediately from [3], Remark 6.3.

In the rest of this subsection we classify all the realizable matrices with several affine components such that at least three of the components are Cartan matrices. Our argument is straightforward: given an indecomposable matrix we consider all possible realizations of it and list all transversal columns for each realization. Going through this list we can see what other components can be realized jointly with the initial matrix. In order to reduce amount of case by case inspections the following simple observation is useful.

Lemma 6.6 *Let Γ be a graph on vertices $\{v_1, \ldots, v_n\}$, $m = (m_1, \ldots, m_n)$ a labeling of its vertices and $Q = Q(m, \Gamma)$. Fix $K \in \Pi$ and $i \in K$. Suppose J is a realization of Q and J' is obtained from J by deleting the ith column and all the rows whose restrictions to $K \setminus \{i\}$ are 0. Then J' is a realization of $Q' = Q(m', \Gamma')$ where Γ' is the subgraph of Γ on $\{v_1, \ldots, \hat{v}_i, \ldots, v_n\}$ and $m'_j \leq m_j$ for all $j \neq i$ with equality on $K \setminus \{i\}$.*

Since we consider Cartan matrices the labels m have $m_i = 2$ for all i. Notice that in terms of arrangements of lines, this means that we consider only systems of multiple points such that any line passing through them contains precisely 3 of them. Under this condition, the indecomposable affine matrices are Cartan matrices of extended Dynkin diagrams with singular edges, i.e., the types $A_\ell^{(1)}$ ($\ell \geq 2$), $D_\ell^{(1)}$ ($\ell \geq 4$) and $E_\ell^{(1)}$ ($\ell = 6, 7, 8$). Notice that all these graphs except $A_2^{(1)}$ contain $\Gamma_0 = A_3$ as a subgrpah. Thus the first graph to analyze is Γ_0.

The following facts about Γ_0 can be obtained by simple inspection.

(i) The Cartan matrix of Γ_0 has the unique (up to permutations) realization by

$$J_0 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) There are precisely two (up to permutations of columns and rows) maximal sets of columns transversal to the collection of columns of J_0 such that every two of them intersect at no more than one element. Written as matrices they are

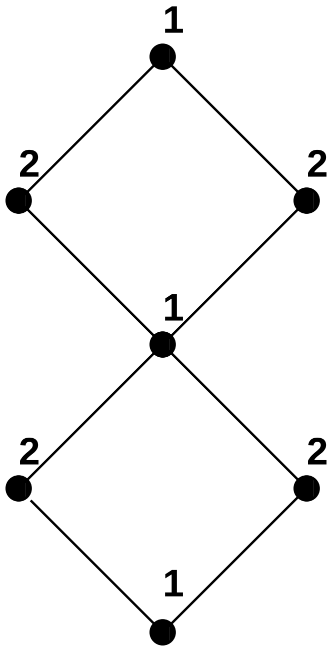
$$T_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The respective symmetric matrices $R_i = T_i^t T_i - E$ $(i = 1, 2)$ are

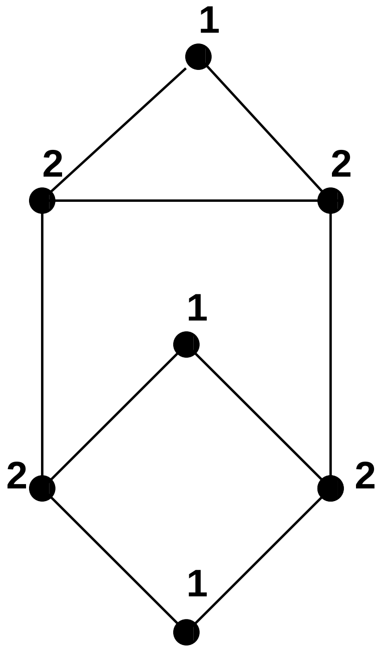
$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

These matrices correspond to the labeled graphs Γ_1 and Γ_2.



Γ_1



Γ_2

(iii) Now suppose that a graph Γ has (at least three) extended Dynkin diagrams with single edges as its connected components and at least one of them, say Γ', contains A_3 as a subgraph (i.e., this component is not $A_2^{(1)}$). Suppose the respective Cartan matrix $Q = Q(m, \Gamma)$ $(m = (2, \ldots, 2))$ is realizable by J. Denote by J' the submatrix of J that consists of the columns of J corresponding to vertices of Γ' and by J'' the matrix of the other columns. We can apply Lemma 6.6. It implies that J'' can be obtained from either of J_1 and J_2 by adding three new rows that form the identity block with the first three rows of J_i and have only 0 in the other positions and then omitting some columns (maybe none). The addition of three new rows does not change Γ_i, just makes the labeling $(2, 2, \ldots, 2)$. Thus one should be able to delete some vertices from one of these graphs and obtain a graph having at least two extended Dynkin diagrams as its connected components. It is quite obviously impossible and we get a contradiction. Thus Q cannot be realized.

(iv) It is left to consider Q whose each component is the Cartan matrix $A_2^{(1)}$. This case is covered by Theorem 6.5. Moreover since the permutation group on $\{1, 2, 3\}$ has only two elements without a fixed point there is the unique realization of Q (up to permutations of rows and columns). The maximum number of components Q can have is 4 and if it has 4 affine components it is realized by the matrix J_C

$$J_C = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If it has three affine components then it is realized by either 9 or 10 or 11 first columns of J_C.

Summing up, we obtain the following result.

Proposition 6.7 *Let Q have at least three affine components that are Cartan matrices. Then Q is realizable if and only if Q has at most four components such that each affine component is $2I_3$ (where I_3 is the 3×3 identity matrix) and if a finite component exists then it is a principle submatix of $2I_3$.*

Similarly it is possible to study the case where $m_i \leq 2$ for every i. The only extra possibility here is given by three components each being the affine matrix corresponding to the bush with 2 leaves.

6.3. Representations of matroids.

If a matrix Q is realizable then we can ask if the respective matroid can be represented by an arrangement $\mathcal{A}$ of lines in a projective plane over a given field. If it is possible then the set $\mathcal{X}$ of all multiple points of $\mathcal{A}$ define component $V(\mathcal{X})$ of R_1 with the associated matrix Q.

Here we give the positive answers about the representability of certain matroids from this section.

For Q from Theorem 6.5 representations over $\mathbb{C}$ exist for $n = 2$ and $n = 3$. For $n = 2$ it is the braid arrangement given by $xyz(x-y)(x-z)(y-z)$ and for $n = 3$ the Hessian arrangement of 12 lines containing 9 flexes of a non-singular cubic (e.g., given by $xyz\prod_{i,j=0,1,2}(x+\zeta^i y+\zeta^j z)$ where $\zeta = \exp(2\pi\sqrt{-1}/3)$). In Section 7, we will prove that for $n > 3$ these matroids cannot be represented over $\mathbb{C}$.

The arrangement of 9 lines projectively dual to the Hessian arrangement (that is a representation of the plane of order 3) gives a different example. All the 12 points of this arrangement do not define any component of R_1

since all components of the respective matrix Q are finite. Any subset of 9 points such that the left out 3 points have all 9 lines passing through them defines a component of R_1 of the above type with $n = \ell = 3$.

Let us remark about matrices Q from Theorem 6.5 that they completely cover the case of arrangements of lines with only triple multiple points and $R_1 \neq 0$. Indeed if any row of matrix J has exactly three 1's then Q has at most three components. Since $R_1 \neq 0$ the number of components is exactly three and each one of them corresponds to a full graph. It follows from the above theorem that there are $3n$ lines (for some n) partitioned in 3 groups of equal size n such that lines in each group are in general position and each multiple point is on a line from each group. In section 7, we will show that these arrangements (equivalently their duals) do not exist for $n > 5$.

Another series of representable (by definition) matroids come from realizations of matrices having components corresponding to bushes. The respective arrangements are of types $\mathcal{A}_{r,1,3}$.

Finally Proposition 6.7 gives the following result for arrangements.

Theorem 6.8 *Let $\mathcal{A}$ be a projective arrangement of lines with a set $\mathcal{X}$ of points from L such that $V(\mathcal{X})$ is a component of R_1 of positive dimension. Let any line from L having a point from $\mathcal{X}$ have exactly three of them. Then $L(2)$ for this arrangement is given by the incidence matrix that is a submatrix of J_C from subsection 6.2 that includes the first 9 of its columns.*

7 Direct sum decompositions and pencils.

In the case where $\mathcal{A}$ is an arrangement over $\mathbb{C}$ the matrix $Q = Q(\mathcal{X})$ assigned to a subset $\mathcal{X}$ of $L'(2)$ has a natural geometric interpretation as follows. Let us consider the arrangement of lines induced by $\mathcal{A}$ in a generic plane P in $\mathbf{P}(\mathbb{C})^{\ell-1}$. Let P' be the blow up of P at the intersection points of the images of $X \in \mathcal{X}$ with P. Let Σ be a free abelian group generators of which correspond to the proper preimages in P' of intersections P with the hyperplanes from $\mathcal{A}$.

Proposition 7.1 *Intersection form on P' induces a symmetric bilinear form on Σ which has $-Q$ as its matrix.*

Proof. The entries of the bilinear form induced on Σ by the intersection form of P' are the following. If the lines in P intersect at a point from $\mathcal{X}$ then the entry corresponding to these lines is 0, if the lines intersect at a point outside of $\mathcal{X}$ then the entry is 1 and the diagonal entry is $1 - m$ where

m is the number of points from $\mathcal{X}$ on this line (all this since a blow up at a point decreases the intersection index by 1). Comparing with the definition of Q in section 2 we conclude the proof. $\square$

Now first part of Proposition 2.2 (ii) from this point of view is a consequence of the Hodge index theorem claiming that the signature of the intersection form on P' is $(1, rkH_2(P') - 1)$

Next let us assume that $\dim R_1 > 0$ and the corresponding component S of the characteristic variety (i.e. the image of a component of R_1 under the exponential map) is essential (cf. [9]). The latter means that $\mathcal{A}$ does not have a subarrangement $\mathcal{A}'$ such that for some component S' of the characteristic variety of $\mathcal{A}'$ one has $S = i^*(S')$ where the map i^* of the character groups $i^* : H^1(\mathbf{P}^2(\mathbb{C}) \setminus \mathcal{A}', \mathbf{C}^*) \to H^1(\mathbf{P}^2(\mathbb{C}) \setminus \mathcal{A}, \mathbf{C}^*)$ is induced by inclusion. Recall also that S corresponds to a choice of a subset $\mathcal{X} \subset L'(2)$ and to emphasize this we sometimes write $S(\mathcal{X})$ for S.

Let $\pi : P \setminus \mathcal{A} \to \mathbf{P}^1 \setminus \{p_1, \ldots, p_m\}$ be the map corresponding to the component S (cf. section 5). The map π defines a rational map $\tilde{\pi} : P \to \mathbf{P}^1$ regular outside of a set of codimension 2 (cf. [11]), i.e. a finite set of points. Since π is regular outside of $\mathcal{A}$ we have

$$\bigcup \tilde{\pi}^{-1}(p_i) \subset \mathcal{A}. \tag{7.1}$$

The pencil of curves consisting of the closures of fibres of $\tilde{\pi}$ does not have common components. Indeed, if we assume that ℓ is such an irreducible component, then from (7.1) ℓ must be a line and S is the image of a component in the arrangement obtained from $\mathcal{A}$ by deleting ℓ in contradiction with assumption that S is essential. This implies that the pull back of the pencil corresponding to $\tilde{\pi}$ has no base points on P' (i.e. the blow up of P at $\mathcal{X}$). Let $\pi' : P' \to \mathbf{P}^1$ be the corresponding map. It follows from Bertini's theorem that a generic fibre of π' is non singular. Now let us consider a free abelian group D_i generated by the classes of lines from $\mathcal{A}$ which belong to the fibre $\pi'^{-1}(p_i)$. Clearly subgroups D_i and D_j of Σ are orthogonal with respect to the intersection form. On the other hand, since neither of the exceptional curves in P' belongs to a fibre of π' and by Zariski's connectedness theorem the fibres of π' are connected (cf. [11] (3.24)), it follows that the intersection form is irreducible on D_i.

Next recall ([6], exp. X) that for each fibre of a fibration of a surface there is unique linear combination z_0 of components of this fibre with positive integer coefficients such that

a) for any combination z of the components of this fibre $(z, z) \leq 0$ with equality if and only if z is an integer multiple of z_0.

b) $(z, z_0) = 0$ for any z supported on the fibre.

A consequence of this is that the intersection form on each D_i is affine.

We can summarize this as follows.

Proposition 7.2 *Let $\mathcal{A}$ be an arrangement such that* $\dim R_1 > 0$ *and* $S = S(\mathcal{X})$ *an essential component of a characteristic variety containing 1. Then there is a one-to-one correspondence between the blocks of matrix* $Q(\mathcal{X})$ *from section 2 and the fibres of the pencil corresponding to the component* $S(\mathcal{X})$ *.*

Existence of a pencil for which given arrangement is a union of several elements imposes severe restrictions on the arrangement. For example one can show the following (cf. 6.3).

Proposition 7.3 *Let $\mathcal{A}$ be an arrangement of $n \cdot k$ ($n \geq 2$) lines for which:*

a) the corresponding matrix Q is sum of k equal blocks so that the lines corresponding to each block are in general position;

b) any multiple point contains a line from each group;

c) each line contains n multiple points of the arrangement.

Then $k \leq 5$.

Proof. It follows from c) that each group is affine and hence $\dim R_1 > 0$. Hence there is a pencil such that the arrangement forms a union of its singular fibres. Let us calculate the euler characteristic of P', i.e. the plane blown up at the multiple points of the arrangement. According to c) and a) there are n^2 multiple points and hence the euler characteristic of P' is $E_1 = 3 + n^2$. On the other hand, since P' is fibered over $\mathbf{P}^1$ with k fibers consisting of n lines in general position (which has the euler characteristic equal to $2n - \frac{n(n+1)}{2}$) and with generic fibre a non singular plane curve of degree n (which has the euler characteristic $3n - n^2$), we see that $e(P') \geq E_2$ where $E_2 = (2-k)) \cdot (3n - n^2) + k \cdot (2n - \frac{n(n+1)}{2})$ (inequality is due to possible presence of other singular fibres and the semicontinuity of euler characteristic). But $E_2 > E_1$ for $k \geq 6$ which yields (i). Also in the case $k = n+1$ we see that $E_1 = E_2$ if and only if $n = 2, 3$, i.e. in these cases we may and indeed do have a fibration with no degenerate fibres other than those composed of the lines of arrangement. □

Similar argument allows to obtain a restriction on arrangements having a given number of multiple point on each line which bounds the number of arrangements having large $\dim R_1$.

Proposition 7.4 *Let $\mathcal{A}(r,k)$ be a class of arrangements with each line having at most k points and such that $dim R_1 = r$. Then there is a function $F(r,k) \leq 2k+1$ such that the number of lines in each of $r+1$ groups in which the arrangement splits does not exceed $F(r,k)$. In particular the number of the lines in the arrangement is at most $(r+1)F(r,k) < (r+1)(2k+1)$.*

Proof. The argument is similar to the one used in the previous proposition. Let d be the maximum of the numbers of lines in each group of lines forming a singular fibre of the pencil. Then the euler characteristic of the total space of the fibration is greater than $E_2 = (2-r)(3d-d^2)+r(2d-d(d-1)/2)$ (use semicontinuity of euler characteristic). On the other hand the euler characteristic of the blow up is greater than or equal to $E_1 = 3 + d \cdot k \cdot r$. One has $E_2 > E_1$ and hence a contradiction unless $E_2 - E_1$ has a positive root $F(r,k)$ which is the bound on d. Direct calculation shows that when $r \to \infty$ the positive root of $E_2 - E_1$ is an increasing function of r and its limit is $2k+1$. $\square$

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