TOPOLOGICAL ROBOTICS: SUBSPACE ARRANGEMENTS AND COLLISION FREE MOTION PLANNING

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To S.P. Novikov on the occasion of his 65-th birthday

ABSTRACT. We study an elementary problem of the topological robotics: collective motion of a set of n distinct particles which one has to move from an initial configuration to a final configuration, with the requirement that no collisions occur in the process of motion. The ultimate goal is to construct an algorithm which will perform this task once the initial and the final configurations are given. This reduces to a topological problem of finding the topological complexity $\mathbf{TC}(C_n(\mathbf{R}^m))$ (as defined in [2, 3]) of the configutation space $C_n(\mathbf{R}^m)$ of n distinct ordered particles in $\mathbf{R}^m$. We solve this problem for $m = 2$ (the planar case) and for all odd m, including the case $m = 3$ (particles in the three-dimensional space). We also study a more general motion planning problem in Euclidean space with a hyperplane arrangement as obstacle.

1. INTRODUCTION

In this paper we study an elementary problem of the topological robotics: collective motion of a set of n distinct particles which one has to move from an initial configuration to a final configuration, with the requirement that no collisions occur in the process of motion. This problem is clearly of practical interest. It becomes quite difficult when the number of particles is large. The ultimate goal is to construct an algorithm which will perform this task once the initial and the final configurations are given, see [5], [9].

Any such motion planning algorithm must have instabilities [2], i.e. the motion of the system will always be discontinuous as a function of the initial and final configurations. These instabilities of motion are caused by topological reasons. A general approach to study instabilities of robot motion was suggested recently in [2, 3]. With any path-connected topological space X one associates a number $\mathbf{TC}(X)$, *the topological complexity of* X. This number is of fundamental importance for the motion planning problem: $\mathbf{TC}(X)$ determines the character of instabilities which have all motion planning algorithms in X. See §2 for a brief summary.

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In this paper we tackle the problem of calculating the topological complexity of the complements of subspace arrangements. In particular, we compute the topological complexity of the configuration spaces of n distinct points on the plane $\mathbf{R}^2$ and in the space $\mathbf{R}^3$. Our main results can be stated as follows:

Theorem 1. *Any motion planner for collision free motion of n distinct points on the plane $\mathbf{R}^2$ has order of instability at least $2n-2$. There exist motion planners having order of instability equal $2n-2$.*

Theorem 2. *Any motion planner for collision free motion of n distinct points in the three-dimensional space $\mathbf{R}^3$ has order of instability at least $2n-1$. There exist motion planners having order of instability equal $2n-1$.*

2. The motion planning problem

In this section we recall some definitions and results from [2, 3]. In particular we define the terms which are used in the statements of our main Theorems 1 and 2.

Let X be a topological space, thought of as the configuration space of a mechanical system. Given two points $A, B \in X$, one wants to connect them by a path in X. We always assume that X is a connected CW complex. A solution to this motion planning problem is a rule that takes $(A, B) \in X \times X$ as an input and produces a path from A to B as an output. Let PX be the space of all continuous paths $\gamma : [0,1] \to X$, equipped with the compact-open topology, and let $\pi : PX \to X \times X$ be the map assigning the end points to a path: $\pi(\gamma) = (\gamma(0), \gamma(1))$. The map π is a fibration whose fiber is the based loop space ΩX. The motion planning problem consists of finding a section s of this fibration.

The section s cannot be continuous, unless X is contractible, see [2]. One defines $\mathbf{TC}(X)$, the *topological complexity of* X, as the smallest number k such that $X \times X$ can be covered by k open sets $U_1, \dots, U_k$, so that for every $i = 1, \dots, k$ there exists a continuous section $s_i : U_i \to PX$, $\pi \circ s_i = \mathrm{id}$.

Definition 1. *A motion planner in X is given by finitely many subsets $F_1, \dots, F_k \subset X \times X$ and by continuous maps $s_i : F_i \to PX$, where $i = 1, \dots, k$, such that:*

- *(a) the sets $F_1, \dots, F_k$ are pairwise disjoint ($F_i \cap F_j = \emptyset$, $i \neq j$) and cover $X \times X$;*
- *(b) $\pi \circ s_i = 1_{F_i}$ for any $i = 1, \dots, k$;*
- *(c) each F_i is an ENR.*

The subsets F_i are the *local domains* of the motion planner; the maps s_i are the *local rules*. Any motion planner determines a *motion planning algorithm*: given a pair (A, B) of initial - final configurations, we first determine the index $i \in \{1, 2, \dots, k\}$, such that the local domain F_i contains (A, B); then we apply the local rule s_i and produce the path $s_i(A, B)$ in X as an output.

Definition 2. *The order of instability of a motion planner is defined as the largest integer r such that the closures of some r among the local domains $F_1, \dots, F_k$ have a non-empty intersection:*

$$\bar{F}_{i_1} \cap \bar{F}_{i_2} \cap \dots \cap \bar{F}_{i_r} \neq \emptyset \quad \textit{where} \quad 1 \leq i_1 < i_2 < \dots < i_r \leq k.$$

The order of instability describes character of discontinuity of the motion planning algorithm determined by the motion planner.

Theorem 3. ([3]) *Let X be a connected smooth manifold. Then the minimal integer k, such that X admits a motion planner with k local rules, equals $\mathbf{TC}(X)$. Moreover, the minimal integer $r > 0$, such that X admits a motion planner with order of instability r, equals $\mathbf{TC}(X)$.*

This theorem explains importance of knowing the number $\mathbf{TC}(X)$ while solving practical motion planning problems.

Let us mention now some other results from [2, 3], which will be used later in this paper.

Theorem 4. ([2, 3]) *$\mathbf{TC}(X)$ depends only on the homotopy type of X. One has*

$$\mathrm{cat}(X) \leq \mathbf{TC}(X) \leq 2\ \mathrm{cat}(X) - 1$$

where $\mathrm{cat}(X)$ *is the Lusternik-Schnirelmann category of X. If X is r-connected then*

$$\mathbf{TC}(X) < \frac{2\ \dim(X) + 1}{r+1} + 1. \tag{2.1}$$

The following result provides a lower bound for $\mathbf{TC}(X)$ in terms of the cohomology ring $H^*(X;\mathbf{k})$ with coefficients in a field $\mathbf{k}$. The tensor product $H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k})$ is also a graded ring with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|}\, u_1 u_2 \otimes v_1 v_2$$

where $|v_1|$ and $|u_2|$ are the degrees of the cohomology classes v_1 and u_2. The cohomology multiplication $H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k}) \to H^*(X;\mathbf{k})$ is a ring homomorphism. Let $I \subset H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k})$ be the kernel of this homomorphism. The ideal I is called *the ideal of zero-divisors* of $H^*(X;\mathbf{k})$. The *zero-divisors-cup-length* is the length of the longest nontrivial product in the ideal of zero-divisors.

Theorem 5. ([2]) *The topological complexity $\mathbf{TC}(X)$ is greater than the zero-divisors-cup-length of $H^*(X;\mathbf{k})$.*

The topological complexity $\mathbf{TC}(X)$, as well as the Lusternik-Schnirelmann category $\mathrm{cat}(X)$, are particular cases of the notion of *Schwarz genus* (also known as *sectional category*) of a fibration; it was introduced and thoroughly studied by A. Schwarz in [8].

3. Hyperplane arrangements as obstacles: the main results

In this section we study the topological complexity of complex hyperplane arrangements complements. An important special case, which we mainly have in mind, is given by the configuration space of n distinct points on the plane

$$\{(z_1, z_2, \dots, z_n); z_i \in \mathbf{C}, z_i \neq z_j\}.$$

Let $\mathcal{A} = \{H\}$ be a finite set of hyperplanes in an affine complex space $\mathbf{C}^\ell$. We will denote by $M(\mathcal{A})$ the complement, i.e. $M(\mathcal{A}) = \mathbf{C}^\ell - \cup_{H\in\mathcal{A}} H$. We will study the motion planning problem in $M(\mathcal{A})$. From a different point of view, we may say that we live in $\mathbf{C}^\ell$ and the union of hyperplanes $\cup H$ represent our obstacles.

If $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$ then $\mathcal{A}$ is called *central*, and up to change of coordinates the hyperplanes can be assumed linear.

Suppose that $\mathcal{A}$ is linear. For each $H \in \mathcal{A}$ one can fix a linear functional α_H (unique up to a non-zero multiplicative constant) such that $H = \{\alpha_H = 0\}$. A set of hyperplanes $H_i \in \mathcal{A}$ is called *linear independent* if the corresponding functionals

α_{H_i} are linearly independent. The rank of $\{\alpha_H\}$, i.e., the cardinality of a maximal independent subset, is called the *rank* of $\mathcal{A}$ and denoted by $\mathrm{rk}(\mathcal{A})$. Clearly $\mathrm{rk}(\mathcal{A}) \leq \ell$ and the equality occurs if and only if $\bigcap_H H = 0$.

If $\mathcal{A}$ is not central we define its rank as the rank of a maximal central subarrangement of $\mathcal{A}$.

If $\mathcal{A}_i$ $(i = 1, 2)$ are central arrangements in spaces $\mathbf{C}^{\ell_i}$ respectively then one can define their *product* as the arrangement $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ in the space $\mathbf{C}^{\ell_1} \oplus \mathbf{C}^{\ell_2}$ consisting of $H \oplus \mathbf{C}^{\ell_2}$ for $H \in \mathcal{A}_1$ and $\mathbf{C}^{\ell_1} \oplus H'$ for $H' \in \mathcal{A}_2$. It is easy to see that $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}_1) + \mathrm{rk}(\mathcal{A}_2)$ and $M(\mathcal{A}) = M(\mathcal{A}_1) \times M(\mathcal{A}_2)$.

While dealing with the arrangement complements we will need the following nontrivial result - see [6], Section 5.2: *if $\mathcal{A}$ is an arbitrary arrangement of rank r then $M(\mathcal{A})$ has homotopy type of a simplicial complex of dimension r.*

One of the most interesting series of examples of central arrangements are the complexifications of the sets of mirrors for Weyl groups. They are called reflection arrangements. For instance the reflection arrangement of type A_{m-1} is given in $\mathbf{C}^m$ by the equations $z_i - z_j = 0$ for all $1 \leq i < j \leq m$; it has rank $m - 1$. The complement of this arrangement (i.e. of the union of its hyperplanes) can be identified with the configuration space of all m-tuples of distinct points on the plane $\mathbf{R}^2$ or $\mathbf{C}$. This space appears in Theorem 1 in the Introduction.

Now we state the main theorems about the topological complexity of the arrangement complements:

Theorem 6. *Let M be the complement of a central complex hyperplane arrangement of rank r. Then the topological complexity satisfies $\mathbf{TC}(M) \leq 2r$.*

This is slightly better than the upper bound $2r + 1$ given by (1) mentioning the fact (see above) that the complement $M(\mathcal{A})$ has homotopy type of an r-dimensional complex, (2) using the homotopy invariance of $\mathbf{TC}(X)$, see [2], and (3) invoking Theorem 4.

The above estimate can be improved in the case where the arrangement is a product.

Theorem 7. *If $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$, where for every $i = 1, 2, \ldots, k$, $\mathcal{A}_i$ is a central complex hyperplane arrangement, then*

$$\mathbf{TC}(M(\mathcal{A})) \leq 2\sum_{i=1}^{k} r_i - k + 1 = 2r - k + 1.$$

Here r_i denotes the rank of $\mathcal{A}_i$ and $r = \mathrm{rk}(\mathcal{A})$.

The next theorem gives a necessary condition when the upper bound given by Theorem 6 is exact:

Theorem 8. *Let $\mathcal{A}$ be a central complex hyperplane arrangement of rank r. Assume that there exist $2r-1$ hyperplanes $H_1, H_2, \ldots, H_{2r-1} \in \mathcal{A}$ such that $H_1, H_2, \ldots, H_r$ are independent and for any $j = 1, 2, \ldots, r$ the hyperplanes $H_j, H_{r+1}, \ldots, H_{2r-1}$ are independent. Then*

$$\mathbf{TC}(M(\mathcal{A})) = 2r.$$

There are at least two large classes of arrangements satisfying conditions of the previous theorem:

1. **Generic arrangements of cardinality at least** $2r-1$. These are the central arrangements whose any subset of cardinality r is independent. Then any subset of cardinality $2r-1$ satisfies the condition.

2. **Reflection arrangements for reflection groups of types** A_n, B_n, **and** D_n**.** Since every arrangement of type B_n and D_n contains a subarrangement of type A_{n-1}, it is enough to show that the condition is satisfied for the A_{n-1} arrangement. Recall that the rank of A_{n-1} is $r = n-1$. Denote by H_{ij} the hyperplane given by the equation $z_i - z_j = 0$. Then the set of hyperplanes $\{H_{1j}, H_{2k}; j = 2, 3, \ldots, n, k = 3, 4, \ldots, n\}$ satisfies the condition of Theorem 8.

Corollary 1. *Let $C_n(\mathbf{R}^2)$ denote the configuration space of ordered sequences of n distinct points on the plane. Then the topological complexity of $C_n(\mathbf{R}^2)$ equals $2n-2$, i.e. $\mathbf{TC}(C_n(\mathbf{R}^2)) = 2n-2$.* □

This Corollary follows from Theorem 8 and the above discussion.

Corollary 1 combined with Theorem 3 implies our main Theorem 1.

4. Proofs of Theorems 6 and 7

In order to prove Theorem 6 we need to use the well-known relations between the complement of a central arrangement and that of the projectivization of this arrangement. Let $\{H_1, H_2, \ldots, H_n\}$ be a central arrangement of hyperplanes in $\mathbf{C}^\ell$ of rank r. Since $M = M(\mathcal{A})$ is invariant with respect of the $\mathbf{C}^*$-action on $\mathbf{C}^\ell$ by multiplication, we can consider the factor-space $M^* = M/\mathbf{C}^*$. This space is nothing but the complement in $\mathbf{CP}^{\ell-1}$ to the union of the projectivizations of all H_i. The easiest (although noncanonical) way to represent M^* is to choose the coordinates so that $H_1 = \{z_\ell = 0\}$ and put $z_\ell = 1$ in the equations of $H_2, \ldots, H_n$. We obtain a not necessarily central arrangement of rank $\le r-1$, whose complement is M^*. Hence we see that M^* has homotopy type of a cell complex of dimension $\le r-1$. Using homotopy invariance of $\mathbf{TC}(X)$ and the dimensional upper bound (see Theorems 3 and 4 from [2]) we obtain that $\mathbf{TC}(M^*) \le 2r-1$.

A simple and well-known observation is that the fibration $M \to M^*$ is trivial, i.e. M is homeomorphic to $M^* \times \mathbf{C}^*$ (cf. [6], Proposition 5.1). Now we may apply the product inequality (see [3], Theorem 11) combined with the obvious fact $\mathbf{TC}(\mathbf{C}^*) = 2$. We obtain $\mathbf{TC}(M) \le 2r - 1 + 2 - 1 = 2r$. This proves Theorem 6.

Now we want to prove Theorem 7. Suppose $\mathcal{A}$ is a central arrangement of rank r that can be represented as the product $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$ of central arrangements. Denote by M_i the complement of $\mathcal{A}_i$ and by r_i its rank. Using again the product inequality from [3] and Theorem 6 we have

$$\mathbf{TC}(M) = \mathbf{TC}(M_1 \times M_2 \times \cdots \times M_k) \le \sum_{i=1}^{k} 2r_i - k + 1 = 2r - k + 1.$$

Here r is the rank of $\mathcal{A}$. □

5. Proof of Theorem 8

Monomials and flags of flats. In this section we will use only the combinatorics of a hyperplane arrangement $\mathcal{A}$ coded in its matroid. This means that the only information about $\mathcal{A}$ we need it is what its subsets are independent. In fact we can forget about the arrangement and consider an arbitrary simple matroid (e.g., see [10]).

Let $\mathcal{M}$ be a simple matroid of rank r on a set S and R a commutative ring. Recall that the Orlik-Solomon algebra of $\mathcal{M}$ over R is a graded R-algebra $A = \oplus_{p=1}^{r} A_r$ that is the factor of the exterior algebra of the free R-module with basis $\{e_s | s \in S\}$ over the ideal generated by $d(e_{s_1} \wedge e_{s_2} \wedge \cdots \wedge e_{s_p})$ for all dependent subsets $\{s_1, \ldots, s_p\} \subset S$. Here d is the differential of the exterior algebra of degree -1 satisfying the graded Leibniz condition and sending every e_s to 1. If $\mathcal{M}$ is matroid of an arrangement then A is naturally isomorphic to $H^*(M(\mathcal{A}); R)$ ([6], Section 5.4).

Now we recall relations between nonzero monomials in A_r, ordered bases of $\mathcal{M}$, and maximal flags of flats of $\mathcal{M}$. Here a base of $\mathcal{M}$ is a maximal independent subset of S and a flat is a subset of S closed with respect to the dependence relation. If $\mathcal{M}$ is a matroid of an arrangement, then flats are just the intersections of some hyperplanes ordered opposite to inclusions.

Each (linearly) ordered subset T of S of cardinality p defines a monomial $m(T) = \prod_{t \in T} e_t$ in A_p. Here the product is taken in the order on T. First, $m(T) \neq 0$ if and only if T is independent in $\mathcal{M}$. On the other hand independent T generates a flag $\mathcal{F}(T)$ in the lattice of all the flats of $\mathcal{M}$ ordered by inclusion. If $T = (t_1, t_2, \ldots, t_p)$ then

$$\mathcal{F}(T) = (< t_p > \subset < t_p, t_{p-1} > \subset \cdots \subset < t_p, t_{p-1}, \ldots, t_1 >)$$

where $< U >$ is the flat generated by a subset U of S. Notice that the rank of the i-th flat in the flag is i.

The nonzero monomials in A_p generate A_p as a linear space and are linearly dependent in general. A way to define a monomial basis of A_p is as follows. Fix an order on S and call a subset C of S a broken circuit if it is independent but there is $s \in S - C$ such that $s < s'$ for every $s' \in C$ and $C \cup \{s\}$ is a circuit (i.e., a minimal dependent set). Then a subset $T \subset S$ is an **nbc**-set if it does not contain any broken circuit. Providing all subsets of S with the induced orders we obtain the set of monomials $\{m(T)\}$ in A_p where T is running through all the **nbc**-sets of cardinality p. This set which we denote by $\mathbf{nbc}_p$ form a basis of A_p (cf. [6], sect. 3.1). We emphasize that this set depends on the order in S.

Going back to a flag $F = (X_0 = \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_p)$ of flats one notices that an arbitrary choice $s_{p-i} \in F_i = X_{i+1} - X_i$ $(i = 0, \ldots, p-1)$ produces an ordered independent set $T = (s_1, \ldots, s_p)$ whence a nonzero monomial $m(T)$. We denote the set of all these ordered bases by $\nu(F)$ and the respective set of monomials by $\mu(F)$. Notice that for each T as above we have $\mathcal{F}(T) = F$. A monomial from $\mu(F)$ is in $\mathbf{nbc}_p$ if and only if the chosen elements are the smallest (in a fixed order on S) in each F_i and their order in T coincides with the induced order. In particular for a flag F there is at most one **nbc**-monomial in $\mu(F)$. We call this monomial $m(F)$. If it does not exist then we put $m(F) = 0$. A flag F for which $m(F) \neq 0$ is called standard (cf. [7]).

The flags $F = (X_0 \subset X_1 \subset \cdots \subset X_p)$ of flats with $\mathrm{rk}(X_i) = i$ provide a convenient language in order to describe the decomposition of a monomial $m(T)$ into a linear combination of the **nbc**-monomials. We denote by $\mathcal{U} = \oplus \mathcal{U}_p$ the linear space on the set of all flags as a basis over a field K graded by their length.

The following lemma is probably known to specialists. H.Terao has informed the second author that he has known it since 1995 but we could not find a proof in the literature.

Lemma 9. *Let T be an independent set in $\mathcal{M}$ taken with the order induced by the fixed order on S. Then*

$$m(T) = \sum (\mathrm{sgn}\sigma) m(\mathcal{F}(\sigma T))$$

where σ is running through the symmetric group Σ_p and is acting on T by permuting its elements.

Proof. We will use a small part of the techniques from [6], Section 3.4. We again denote by $A = \oplus A_p$ the graded Orlik-Solomon algebra of $\mathcal{M}$ with coefficients in R. Define the graded linear map $\phi : \mathcal{U} \to A$ via $\psi(F) = m(F)$ for every flag F from $\mathcal{U}$. Also define the graded linear map $\psi : A \to \mathcal{U}$ via $\psi(m(T)) = \sum (\mathrm{sgn}\sigma)\mathcal{F}(\sigma T)$ for every independent subset T of S taken with the induced order. The fact that ψ is well-defined is proved among other things in [6], Lemma 3.107.

Now we claim that ψ is a section of ϕ. Indeed let T be an independent subset of S from $\mathbf{nbc}_p$. Then for every $\sigma \in \Sigma_p$ the minimal element of each flat of the flag $\mathcal{F}(\sigma T)$ is from T. This implies that $m(\mathcal{F}(\sigma T)) \neq 0$ if and only if σ is the identity permutation whence $\phi\psi(m(T)) = m(T)$. Since the monomials $m(T)$ for those T form a basis of A the result follows. □

Let $\mathcal{M}$ be an arbitrary matroid of rank r on S and A its OS algebra. Consider the graded algebra $\bar{A} = A \otimes A$ where the tensor product is in the category of graded algebras over the basic field. In particular, if $b \in A_k$ and $c \in A_\ell$ then $(a \otimes b)(c \otimes d) = (-1)^{k\ell} ac \otimes bd$ for all $a, d \in A$. For each standard generator e_s $(s \in S)$ of A define the element of degree one of $\bar{A}$ via

$$\bar{e}_s = 1 \otimes e_s - e_s \otimes 1.$$

Our goal is to study the product

$$\pi = \prod_{s \in S} \bar{e}_s$$

taken in some order on S. If $|S| > 2r$ then clearly $\pi = 0$. In the rest of this section we assume that $|S| \leq 2r$.

The following lemma is straightforward.

Lemma 10. *(i) Fix an order on S. Then*

$$\pi = \sum_{(T,T')} (-1)^{|T|} \mathrm{sign}(\sigma) \cdot m(T) \otimes m(T'),$$

where (T, T') runs through all the pairs of complementary independent sets taken in the induced orders and σ is the shuffle on S preserving orders inside T and T' and putting every element of T' after all elements of T.

(ii) If an order on S changes via a permutation τ then π gets multiplied by $\mathrm{sign}\tau$. □

Clearly the linear space $A_p \otimes A_q$ is generated by simple tensors $m_1 \otimes m_2$ where m_1 and m_2 are nonzero monomials from A_p and A_q respectively, and a basis in this space is formed by the simple tensors where both monomials are **nbc**.

The following lemma is an immediate corollary of Lemma 9.

Lemma 11. *Fix an order on S and let $\bar{m} = (m_1, m_2)$ be a pair of monomials both* **nbc**. *Order respectively the sets T_1 and T_2 of their elements and consider the flags $F^i = \mathcal{F}(T_i)$ $(i = 1, 2)$. Then any simple tensor of monomials from Lemma 10 having* *in its decomposition $\bar{m}$ with a nonzero coefficient c has the form $m(T)\otimes m(T')$ where T is complementary to T', $T \in \mu(F^1)$, and $T' \in \mu(F^2)$. Moreover $c = \pm 1$.* □

Nonvanishing products in the tensor square. It is not hard to prove directly that for any matroid of rank r the product of any $2r$ elements $\bar{e}_s$ is 0. Later we will get this result for arrangements as a corollary of others. In this section we consider matroids of rank r with the number of elements less than $2r$. For these matroids the product

$$\pi = \prod_i \bar{e}_i$$

can be nonzero in $A \otimes A$.

Proposition 12. *Suppose that S is the disjoint union of two sets T_1 and T_2 with the following properties: T_1 is independent and $T_2 \cup \{e_s\}$ is independent for every $s \in S$. Then $\pi \neq 0$.*

Proof. Put $|S| = n$ and $|T_2| = p$. Clearly $p < r$ and $n - p \leq r$. Consider the flag $F = \mathcal{F}(T_1) = (X_0 = \emptyset \subset X_1 \subset \cdots \subset X_p)$ corresponding to some order on S. Without any loss of generality we can assume that F is standard in this order and put $m = m(F) = m(T_2)$. The condition on T_2 implies that $|X_{i+1} - X_i| = 1$ for all $i = 0, 1, 2, \ldots, p-1$. Thus $m(T_2)$ is the only monomial having m in its decomposition with a nonzero coefficient. If m' is an arbitrary $\mathbf{nbc}_{n-p}$-monomial from the decomposition of $m(T_2)$ then $m(T_1) \otimes m(T_2)$ is the only simple tensor of complementary monomials having $m \otimes m'$ is its decomposition which completes the proof. □

Proof of Theorem 8. Let $\mathcal{A}$ be an arrangement satisfying the condition of Theorem 8. Denote by S the subarrangement consisting of $H_1, \ldots, H_{2r-1}$ and by $\mathcal{M}$ the matroid on S of this subarrangement. By Proposition 12, $\pi = \prod_{s\in S} \bar{e}_s \neq 0$. Since the Orlik-Solomon algebra A' of any subarrangement is a natural subalgebra of the Orlik - Solomon algebra A of the whole arrangement (see [6], Proposition 3.66), the product $\prod_{s\in S} \bar{e}_s \neq 0$ in the cohomology ring of $M(\mathcal{A})$. Applying the cohomological lower bound for the topological complexity (see Theorem 5), we obtain $\mathbf{TC}(M(\mathcal{A})) \geq 2r$. This and Theorem 6 imply the result. □

The proofs of the following corollaries are straightforward.

Corollary 2. *Let $\mathcal{A}$ be an arrangement satisfying the condition of Theorem 8. Then $\mathbf{TC}(M(\mathcal{A})^*) = 2r - 1$.* □

Corollary 3. *For an arbitrary complex central hyperplane arrangement $\mathcal{A}$ of rank r the product of any $2r$ elements from the kernel of the multiplication homomorphism $H^*(M) \otimes H^*(M) \to H^*(M)$ equals zero.* □

6. Example: motion planner for collision free control of three particles

Here we will describe a recipe to obtain an explicit motion planner for moving a triple of ordered points in $\mathbf{C}$ (or $\mathbf{R}^2$) avoiding collisions.

The configuration space of such triples

$$C_3 = \{(z_1, z_2, z_3); z_1 \neq z_2, z_1 \neq z_3, z_2 \neq z_3\}$$

has superfluous coordinates: the first point z_1 may be arbitrary and one has to observe only the relative positions of the second and the third particles with respect to the first. Hence C_3 is homeomorphic (via the map $(z_1, z_2, z_3) \mapsto (z_1, z_2 - z_1, z_3 - z_1))$ to the product $\mathbf{C} \times M$, where

$$M = \mathbf{C}^2 - (H_1 \cup H_2 \cup H_3),$$

with $\mathbf{C}^2 = \{(z_1, z_2),\ z_1, z_2 \in \mathbf{C}\}$, $H_1 = \{z_1 = 0\}$, $H_2 = \{z_2 = 0\}$, and $H_3 = \{z_1 = z_2\}$.

One may perform another change of coordinates and represent M as a product $M^* \times \mathbf{C}^*$, where $M^* = \mathbf{C} - \{0, 1\}$ and $\mathbf{C}^* = \mathbf{C} - \{0\}$. We assign to a pair $(z_1, z_2) \in M$,

$$h(z_1, z_2) = \left(\frac{z_1}{z_2}, z_2\right) \in M^* \times \mathbf{C}^*.$$

The obtained map $h : M \to M^* \times \mathbf{C}^*$ is a homeomorphism.

Now we see that M^* is homotopy equivalent to a one-dimensional complex, hence it admits a motion planner with 3 sets, see Theorems 3 and 4 in [2]. A motion planner on M^* can be easily described explicitly. $\mathbf{C}^* = \mathbf{C} - \{0\}$ admits a motion planner with two sets, see [2]; such motion planner may also be explicitly constructed. The explicit construction of a motion planner on the product $M^* \times \mathbf{C}^*$ may be now obtained by repeating the arguments used in the proof of the product inequality (see Theorem 11 of [2], or the construction after Theorem 12.1 in [3]). This gives a motion planner in M having 4 local domains. Making this recipe precise is straightforward.

7. Motion planning for collision free motion of n particles in the 3-dimensional space

In this section we study the problem of constructing a collision free control for n distinct particles in $\mathbf{R}^3$. Our main result is:

Theorem 13. *Let $M = C_n(\mathbf{R}^3)$ be the configuration space of n distinct ordered particles in $\mathbf{R}^3$. Then $\mathbf{TC}(M) = 2n - 1$.*

Proof. Similarly to the plane case, the configuration space of n ordered distinct points in $\mathbf{R}^3$ can be viewed as the complement to an arrangement of linear subspaces. Indeed the space of all n-tuples of points from $\mathbf{R}^3$ can be identified with $\mathbf{R}^n \otimes \mathbf{R}^3 = \mathbf{R}^{3n}$ with the coordinates x_{ik}, where $i = 1, 2, \ldots, n$ and $k = 1, 2, 3$. Here the numbers x_{j1}, x_{j2}, x_{j3} are the space coordinates of the particle number j. To exclude coincidences one needs to remove from $\mathbf{R}^{3n}$ the union of the subspaces H_{ij} $(1 \le i < j \le n)$, where H_{ij} is given by the system of 3 equations

$$x_{i1} - x_{j1} = 0, \quad x_{i2} - x_{j2} = 0, \quad x_{i3} - x_{j3} = 0.$$

Hence the configuration space we are interested in is $M = \mathbf{R}^{3n} - \bigcup H_{ij}$.

It follows immediately from this representation of M that it is simply connected. Indeed, M is obtained from the Euclidean space by removing finitely many subspaces of codimension 3.

To find a low and upper bounds for the topological complexity $\mathbf{TC}(M)$ it is convenient to start with its cohomology ring $H^*(M)$ with the integral coefficients. For a general subspace arrangement this ring is much more complicated than Orlik-Solomon algebras and is not even defined by the combinatorics of the arrangement. The property of our arrangement that simplifies the matter is the fact that the dimensions of all our subspaces and their intersections are divisible by 3. Moreover the intersection lattice of the arrangement in $\mathbf{R}^{3n}$ coincides with the lattice of the braid arrangement corresponding to n distinct points in $\mathbf{C}$. Under these conditions the cohomology algebra $A = H^*(M)$ is defined similarly to the Orlik-Solomon algebra with the only difference that the degree of the natural generators e_{ij} is 2, see [11], Section 7. This implies that A is a commutative algebra. More precisely $A = \mathbb{Z}[e_{ij}]/I$, where the ideal I is generated by the polynomials e_{ij}^2 and

$$e_{ij}e_{ik} - e_{ij}e_{jk} + e_{ik}e_{jk}$$

for every triple $1 \le i < j < k \le n$.

Except being commutative, A has properties very similar to the respective Orlik-Solomon algebra. For instance, $A = \oplus_{p=0}^{n-1} A_p$ although here degree of A_p is $2p$. A monomial of the form $e_{i_1j_1}e_{i_2j_2}e_{i_kj_k}$ is nonzero in A if and only if the respective linear functionals $x_{i_\alpha} - x_{j_\alpha}$, where $\alpha = 1, \dots, k$, are linearly independent.

In particular, we find that the top dimension, where M has a nontrivial cohomology, is $2(n-1)$. Since M is simply connected, it follows that M is homotopy equivalent to a cell-complex of dimension $\le 2(n-1)$ (see for example, [1]). Then using inequality (2.1) we obtain

$$\mathbf{TC}(M) < \frac{4(n-1)+1}{2} + 1,$$

i.e. $\mathbf{TC}(M) \le 2n - 1$.

Now we will use the cohomological lower bound for the topological complexity given by Theorem 5. We will find a non-zero product in $A \otimes A$ of $2n-2$ elements of the form

$$\bar{e}_{ij} = 1 \otimes e_{ij} - e_{ij} \otimes 1,$$

which are zero-divisors. Consider the following product

$$\pi = \prod_{i=2}^{n} (\bar{e}_{1i})^2 \in A \otimes A.$$

An easy computation gives $(\bar{e}_{ij})^2 = -2e_{ij} \otimes e_{ij}$ for arbitrary i, j. Hence we find $\pi = (-2)^{n-1} m \otimes m$, where

$$m = \prod_{i=2}^{n} e_{1i}.$$

Since the linear functionals $\{x_1 - x_i | i = 2, 3, \dots, n\}$ are linear independent, the monomial $m \neq 0$ is nonzero in A, and hence the product π is nonzero. Thus we obtain the opposite inequality $\mathbf{TC}(M) \ge 2n - 1$. □ □

Theorem 2 follows from Theorem 13 combined with Theorem 3.

Remark. Let $C_n(\mathbf{R}^m)$ denote the configuration space of n ordered distinct points in $\mathbf{R}^m$. Repeating the argumant of Theorem 13 we find $\mathbf{TC}(C_n(\mathbf{R}^m)) = 2n - 1$ for m odd.

Conjecture. *For m even* $\mathbf{TC}(C_n(\mathbf{R}^m)) = 2n - 2$.

This would generalize our Corollary 1 where $m = 2$. Our arguments show that for any even m the topological complexity $\mathbf{TC}(C_n(\mathbf{R}^m))$ equals either $2n - 2$ or $2n - 1$.

References

[1] S. Eilenberg and T. Ganea, *On the Lusternik - Schnirelmann category of abstract groups*, Annals of Mathematics **65**(1957), 517 - 518.

[2] M. Farber, *Topological complexity of motion planning*, math.AT/0111197, to appear in "Discrete and Computational Geometry".

[3] M. Farber, *Instabilities of robot motion*, cs.RO/0205015, to appear in "Topology and its Applications".

[4] I. James, *Lusternik-Schnirelmann category*, Handbook of Algebraic Topology, 1293 - 1310, Elsevier, 1995.

[5] J.-C. Latombe, *Robot motion planning*, Kluwer, Boston, 1991

[6] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, 1992.

[7] V. Schechtman and A. Varchenko. *Arrangements of hyperplanes and Lie algebra homology*, *Invent. Math.*, **106**(1991), 139–194.

[8] A. Schwarz, *The genus of a fiber space*, A.M.S. Transl. **55**(1966), 49 - 140.

[9] M. Sharir, *Algorithmic motion planning*, Handbook of Discrete and Computational Geometry, J.E. Goodman, J.O'Rourke eds, 1997

[10] D. J. A. Welsh, *Matroid Theory*, L. M. S. Monographs, No. 8. Academic Press, London-New York, 1976.

[11] S. Yuzvinsky, *Small rational model of subspace complement*, Trans. AMS **354**(2002), 1932-1945.

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