# Cohomology of the Brieskorn-Orlik-Solomon algebras

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## 1 Introduction

Let $V$ be an affine space of dimension $\ell$ over a field $F$ and let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a non-empty arrangement of hyperplanes of $V$. For each $H \in \mathcal{A}$ fix an affine functional $\alpha_H$ such that $\ker \alpha_H = H$ and put $\alpha_i = \alpha_{H_i}$. The main character of the paper is the graded $F$-algebra $A = A(\mathcal{A}) = \oplus_{p=0}^{\ell} A_p$ generated by the differential forms $\omega_i = d\alpha_i/\alpha_i \in A_1$. If $F = \mathbb{C}$ then, according to Brieskorn's theorem [2], this algebra is isomorphic under the de Rham map to the cohomology algebra of $M = V \setminus \bigcup_{i=1}^{n} H_i$. Explicit and pure combinatorial description of this algebra has been given by Orlik and Solomon [6] and is presented in detail in Section 3 of [7]. For every $\lambda = (\lambda_1, \ldots, \lambda_n) \in F^n$ the left multiplication $d_\lambda$ by $\omega_\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$ defines a cochain complex $(A, d_\lambda)$

$$0 \to A_0 \xrightarrow{d_\lambda} A_1 \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} A_\ell \to 0.$$

The goal of this paper is to study the cohomology $H^p = H^p(A, d_\lambda)$ of this complex.

The study of $H^p$ is motivated by [4] and [5]. These papers are concerned with $H^*(M, \mathcal{L})$ for $F = \mathbb{C}$ where $\mathcal{L}$ is a local system on $M$. The cohomology is used in theory of hypergeometric functions and Knizhnik-Zamolodchikov equations. Kohno [5] proved that if $\mathcal{L}$ is the local system of flat sections of the trivial bundle with respect to the connection $d + \omega_\lambda$, then under a certain genericity condition on $\lambda$, $H^p(M, \mathcal{L}) = 0$ for $p < \ell$. Also if $\mathcal{A}$ is real and transverse to the hyperplane at infinity, he found a basis of $H^\ell(M, \mathcal{L})$ that does not depend on $\lambda$. Then Esnault, Schechtman, and Viehweg [4] proved that under a weaker genericity condition on $\lambda$

$$H^p(M, \mathcal{L}) = H^p(A, d_\lambda)$$

for every $p$.

In this paper we give an algebraic proof that for every field $F$ under a general position condition on $\lambda$ (weaker than in [5]) the cohomology $H^p(A, d_\lambda)$ vanish for $p < \ell$ (Theorem 4.1). For $F = \mathbb{C}$ Theorem 4.1(i) can be deduced from Theorem 3.7 and Corollary 2.8 of [8]. For certain particular cases we give a necessary and sufficient conditions for the vanishing of low cohomology. For instance, for general position arrangements, the condition is just $\lambda \neq 0$ (Proposition 5.1). We consider a class of arrangements in general position to infinity generalizing the condition of transversality to the hyperplane at infinity from [5]. For these arrangements we exhibit a subspace of $A_\ell$ that does not depend on $\lambda$ and is isomorphic to $H^\ell(A)$ under the natural projection (Theorem 6.4). We also give a combinatorial interpretation of this subspace.

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# 2 Cohomology of $A$ for an affine arrangement and its cone

In this section we study relations between cohomology of $A$ for an affine arrangement and its cone.

For convenience of the reader we recall here some relevant facts about $A$ from [7]. Let $E_1$ be a linear space with basis $(e_1, \ldots, e_n)$ and denote by $E = E(\mathcal{A})$ the exterior algebra $\Lambda(E_1)$ of $E_1$. For each $p$, $0 \leq p \leq n$, put $E_p = \Lambda^p(E_1)$. Then every ordered subset $S = \{H_{i_1}, \ldots, H_{i_k}\}$ of $\mathcal{A}$ defines the element $e_S = e_{i_1} \cdots e_{i_k} \in E_k$. Call $S$ dependent if the respective set of functionals $\alpha_i$ is linearly dependent. Also define the linear map $\partial = \partial_E : E \to E$ via $\partial 1 = 0$, $\partial e_i = 1$, and if $k \geq 2$ then

$$\partial e_S = \sum_{j=1}^{k} (-1)^{j-1} e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_k}.$$

Now let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ generated by $\partial e_S$ for all dependent $S$ and by all $e_S$ with $\cap_{i \in S} H_i = \emptyset$. Then

$$A = A(\mathcal{A}) \cong E(\mathcal{A})/I(\mathcal{A}).$$

We will put $a_S = e_S + I(\mathcal{A})$ identifying $a_{\{H_i\}}$ with $\omega_i$.

Let us consider now the case where $\mathcal{A}$ is central, that is $\bigcap_{i=1}^n H_i \neq \emptyset$. Then $I(\mathcal{A})$ is invariant with respect to $\partial$ whence the latter induces the linear map $\partial_A : A \to A$ of degree -1. Besides this map has the following properties:

(1) $\partial_A^2 = 0$,

(2) $\partial_A(ab) = (\partial_A a)b + (-1)^p a(\partial_A b)$ for $a \in A_p$ and $b \in A$.

This gives immediately the following result about the cohomology of $A$.

**Proposition 2.1** *If $\mathcal{A}$ is central and $\sum_{i=1}^n \lambda_i \neq 0$ then $(A, d_\lambda)$ is exact.*

**Proof.** The properties of $\partial_A$ imply

$$(\partial_A d_\lambda + d_\lambda \partial_A)(a) = \partial_A(\omega_\lambda)a = (\sum_{i=1}^n \lambda_i)a$$

for every $a \in A$. Thus the complex $(A, d_\lambda)$ is homotopy equivalent to 0 that proves the result. $\square$

The above Proposition allows us to focus our attention on central arrangements with $\sum \lambda_i = 0$. This case is tightly related to the case of affine arrangements.

Let $\mathcal{A}$ be an arbitrary (affine) arrangement given by a polynomial $Q = \prod_{i=1}^n \alpha_i$ in the affine space $V = \{(x_1, \ldots, x_\ell)\}$. Put $cV = \{(x_0, x_1, \ldots, x_\ell)\}$. It is useful to regard $V$ as the affine hyperplane of $cV$ defined by $x_0 = 1$. Let $S = F[x_1, \ldots, x_\ell]$ and $S^c = F[x_0, x_1, \ldots, x_\ell]$ be the coordinate rings of $V$ and $cV$ respectively. For each $g \in S$ define the homogenization of $g$ as

$$g^h = x_0^{\deg g} g(x_1/x_0, \ldots, x_\ell/x_0)$$

where $\deg g$ is the degree of $g$. Clearly, $g^h \in S^c$ and it is homogeneous of degree $\deg g$. Then the cone $c\mathcal{A}$ of $\mathcal{A}$ is the central arrangement in $cV$ defined by the polynomial $Q^c = x_0 Q^h = x_0 \prod_{i=1}^n \alpha_i^h$. Notice that the arrangement $c\mathcal{A}$ consists of the hyperplane $H_0 = \ker(x_0)$ and the hyperplanes $cH$ for $H \in \mathcal{A}$ where $cH$ is the cone over $H$. For each ordered subset $S$ of $\mathcal{A}$ we denote by $cS$ the set $\{cH | H \in S\}$ taken in the same order.

The algebras $A = A(\mathcal{A})$ and $cA = A(c\mathcal{A})$ are connected by two linear maps:

$$t : A \to cA \text{ via } t(a_S) = (-1)^p \omega_0 a_{cS}, \text{ where } \omega_0 = \frac{dx_0}{x_0} \text{ and } p = |S|,$$

and

$$s : cA \to A \text{ via } s(\omega_0 a_{cS}) = 0, \; s(a_{cS}) = a_S.$$

These maps are well defined and for each $p$ form the exact sequence

$$0 \to A_{p-1} \xrightarrow{t} cA_p \xrightarrow{s} A_p \to 0 \tag{2.1}$$

(see Lemmas 3.49, 3.50, and Corollary 3.57 of [7]).

**Remark 2.2** *The sequence (2.1) is the restriction of the exact sequence (3.3) from [9] taking into consideration the homomorphisms $\rho$ from Proposition 4.3 of [9].*

Let us state several other properties of $s$ and $t$ where for each $i$, $1 \leq i \leq n$, we put $\bar{\omega}_i = d\alpha_i^h / \alpha_i^h (= a_{\{cH_i\}})$.

$$s(ab) = s(a)s(b) \text{ for every } a, b \in cA. \tag{2.2}$$

$$t(\omega_i a) = \bar{\omega}_i t(a) \text{ for every } i = 1, \ldots, n, \ a \in A. \tag{2.3}$$

$$s\partial_{cA} t(a) = (-1)^p a \text{ for every } a \in A_p. \tag{2.4}$$

It suffices to check these properties for $e_S$ and $e_{cS}$. The check is straightforward and we leave it to the reader.

Now we want to introduce a coboundary map on $cA$ closely related to $d_\lambda$. For that we put $\lambda_0 = -\sum_{i=1}^n \lambda_i$ and $\bar{\omega}_\lambda = \sum_{i=1}^n \lambda_i \bar{\omega}_i + \lambda_0 \omega_0$. Then the left multiplication by $\bar{\omega}_\lambda$ is a coboundary map $\bar{d}_\lambda$ of $cA$ converting it into the cochain complex $(cA, \bar{d}_\lambda)$.

**Lemma 2.3** *The maps $s$ and $t$ form the maps of complexes and thus define the exact sequence*

$$0 \to (A, d_\lambda) \xrightarrow{t} (cA, \bar{d}_\lambda) \xrightarrow{s} (A, d_\lambda) \to 0 \tag{2.5}$$

**Proof.** It suffices to check that $d_\lambda s = s\bar{d}_\lambda$ and $td_\lambda = \bar{d}_\lambda t$ on $A(c\mathcal{A})$ and $A(\mathcal{A})$. These equalities follow from definitions of $t$ and $s$ and properties (2.2) and (2.3). $\square$

The exact sequence (2.5) generates the exact sequence of the cohomology spaces

$$\cdots H^{p-1}(A) \xrightarrow{\delta} H^{p-1}(A) \xrightarrow{t^*} H^p(cA) \xrightarrow{s^*} H^p(A) \xrightarrow{\delta} H^p(A) \to \cdots. \tag{2.6}$$

**Proposition 2.4** *For every $p$ the connecting homomorphism $\delta$ in* (2.6) *is 0.*

**Proof.** Let $a \in A_p$ and $\omega_\lambda a = 0$. Due to (2.4), we can take $(-1)^p \partial_{cA} t(a)$ as a preimage of $a$ under $s$. Then, using properties of $\partial = \partial_{cA}$ and (2.3), we have

$$\bar{d}_\lambda(\partial t(a)) = \partial(\bar{\omega}_\lambda)t(a) - \partial(\bar{\omega}_\lambda t(a)) = (\sum_{H \in \mathcal{A}} \lambda_H + \lambda_0)t(a) - \partial t(\omega_\lambda a) = 0.$$

This implies that $\delta(a) = 0$ and completes the proof. $\square$

**Corollary 2.5** *For every $p$ there exists a short exact sequence*

$$0 \to H^{p-1}(A) \to H^p(cA) \to H^p(A) \to 0. \tag{2.7}$$

**Remark 2.6** *If one uses for the coboundary maps of $cA$ and $A$ the multiplication by an arbitrary $\omega \in cA_1$ and $s(\omega)$ respectively then the sequence* (2.6) *is still exact. In this more general case $\delta$ is the multiplication by $\partial_{cA}(\omega)$. Thus if $\partial_{cA}(\omega) \neq 0$ we obtain another proof that $cA$ is acyclic but do not get any information about $H^p(A)$.*

Now we consider briefly the inverse construction to the cone. In the rest of this section we suppose that $\mathcal{A}$ is a central arrangement in $V$. Without loss of generality we can assume that $\alpha_1 = x_1$. Then putting $x_1 = 1$ in every $\alpha_i$ $(i \neq 1)$ we obtain affine functionals $\overline{\alpha}_i$ that define an affine arrangement $\bar{\mathcal{A}}$ in the affine subspace of $V$ given by $x_1 = 1$. Clearly the cone $c\bar{\mathcal{A}}$ can be identified with $\mathcal{A}$. Notice that $\bar{\mathcal{A}}$ and $A(\bar{\mathcal{A}})$ depend on the choice of $H_1$ in $\mathcal{A}$.

In order to find an invariant of $\bar{\mathcal{A}}$ that does not depend on that choice we need more notation. For the algebra $A$ of any arrangement we put

$$\mathrm{Poin}(A,t) = \sum_{p\geq 0} \dim A_p t^p$$

and

$$\chi(A) = \mathrm{Poin}(A,-1) = \sum_{p\geq 0} (-1)^p \dim A_p.$$

If the arrangement is central then $1+t$ divides the polynomial $\mathrm{Poin}(A,t)$ (cf. Corollary 3.58 of [7]) whence $\chi(A) = 0$. Then we have

**Proposition 2.7** $\mathrm{Poin}(A(\bar{\mathcal{A}}),t)$ *and thus* $\chi(A(\bar{\mathcal{A}}))$ *depend only on* $\mathcal{A}$.

**Proof.** Due to Corollary 3.58 of [7] $\mathrm{Poin}(A(\bar{\mathcal{A}}),t) = \frac{1}{1+t}\mathrm{Poin}(A(\mathcal{A}),t)$ that implies the statement. □

We put $\bar{\chi}(\mathcal{A}) = \bar{\chi}(A(\mathcal{A})) = \chi(A(\bar{\mathcal{A}}))$ and will use it in Section 4. This number is equal (up to a sign) to Crapo's beta invariant of the underlying matroid [3]. Notice that if $n = |\mathcal{A}| = 1$ then $\bar{\chi}(\mathcal{A}) = 1$. It is convenient to know a sufficient condition on $\mathcal{A}$ for the vanishing of $\bar{\chi}$.

**Proposition 2.8** *Suppose that $\mathcal{A}$ is a central arrangement with more than one element and there exists $H \in \mathcal{A}$ that is independent of the subarrangement $\mathcal{A}' = \mathcal{A} \setminus \{H\}$, i.e., $\alpha_H$ does not belong to the linear subspace of $V^*$ generated by the functionals of all other hyperplanes from $\mathcal{A}$. Then $\bar{\chi}(\mathcal{A}) = 0$.*

**Proof.** Suppose $H = H_1$ and $\alpha_1 = x_1$. Putting $x_1 = 1$ in every form $\alpha_i$ $(i = 2, \ldots, n)$ we obtain the affine functionals $\bar{\alpha}_i$ defining $\bar{\mathcal{A}}$. If $\bigcap_{\bar{H}\in\bar{\mathcal{A}}} \bar{H} = \emptyset$ then some linear combination of $\bar{\alpha}_i$ equals 1. But then the respective linear combination of $\alpha_i$ equals $x_1$ which contradicts the condition of the proposition. Thus $\bar{\mathcal{A}}$ is central whence $\bar{\chi}(\mathcal{A}) = \chi(A(\bar{\mathcal{A}})) = 0$. □

**Remark 2.9** *According to the terminology of [4] for $n > 1$ the intersection of all the hyperplanes of an arrangement satisfying the condition of Proposition 2.8 (or the arrangement itself) should be called not* **Bad**. *For 3-arrangements this condition is even equivalent to $\bar{\chi}(\mathcal{A}) = 0$. In higher dimensions there are* **Bad** *arrangements with $\bar{\chi}(\mathcal{A}) = 0$. E.g., such is the arrangement $x, y, x - y, z, w, z - w$ in the space $\{(x, y, z, w)\}$ (cf. Erratum of [4]).*

# 3 A complex of sheaves on the intersection poset

In this section we represent the complex $(A, d_\lambda)$ as the complex of global sections of a complex of sheaves. The topological space on which these sheaves are defined is the poset $L = L(\mathcal{A})$ of all the nonempty intersections of hyperplanes from $\mathcal{A}$ ordered *by inclusion*. This poset has the unique maximal element, $V$, but it has a unique minimal element if and only if $\mathcal{A}$ is central. Each $X \in L$ defines the increasing set $L_X = \{Y \in L | Y > X\}$ and the subarrangement $\mathcal{A}_X = \{H \in \mathcal{A} | X \subset H\}$ of $\mathcal{A}$. The standard (*order*) topology on $L$ is given by all the increasing subsets of $L$.

Sheavs on $L$ can be identified with covariant functors. If $\mathcal{F}$ is a covariant functor from $L$ to an additive category then for every subset $U \subset L$ we have

$$\Gamma(U, \mathcal{F}) = \{(a_X) \in \oplus_{X \in U} \mathcal{F}(X) | \mathcal{F}(X \leq Y)(a_X) = a_Y,\ X, Y \in U\}.$$

Fix $p$, $0 \leq p \leq \ell$, and for each $X \in L$ put $\mathcal{F}^p(X) = A_p(\mathcal{A}_X)$. If $Z \in L$ and $\dim Z = \ell - p$ then put $A_Z = A_p(\mathcal{A}_Z)$. The spaces $A_Z$ serve as building blocks, namely $A_p(\mathcal{A}_X) = \oplus A_Z$ where the summation is taking over all $Z \in L$ of dimension $\ell - p$ such that $Z \geq X$ (see Corollary 3.73 of [7]). Thus if $Y \geq X$ then $\mathcal{F}^p(Y)$ is a direct summand of $\mathcal{F}^p(X)$ and there is a projection $\mathcal{F}^p(X) \to \mathcal{F}^p(Y)$. These projections define a sheaf $\mathcal{F}^p$ on $L$. Notice that $\mathcal{F}^p$ is supported on the subposet of $L$ consisting of all elements of dimension less then or equal to $\ell - p$.

**Proposition 3.1** *For every $p$ we have*

$$\Gamma(L, \mathcal{F}^p) = A_p.$$

**Proof.** Fix $p$. Since $A_p = \oplus_{\dim Y = \ell - p} A_Y$ and for every $X \in L$ we have

$$A_p(\mathcal{A}_X) = \oplus_{\dim Y = \ell - p, Y \geq X} A_Y,$$

there is the canonical projection $\rho_X : A_p \to \mathcal{F}^p(X)$. Clearly $\rho = (\rho_X)_X$ is an injective linear map $A_p \to \Gamma(L, \mathcal{F}^p)$. It suffices to check that $\rho$ is surjective. Let $(a_X)_{X\in L} \in \Gamma(L, \mathcal{F}^p)$. Here $a_X \in A_p(\mathcal{A}_X)$ whence $a_X = (a_{X,Y})_{\dim Y=\ell-p, Y\geq X}$. For every $Y$ from this set the restriction of $a_X$ to $Y$ is $a_{X,Y}$. Thus since $(a_X)$ is a section, $a_{X,Y}$ does not depend on $X$ (for every $X \leq Y$) and can be denoted just by $a(Y)$. Now taking $a = (a(Y))_{\dim Y=\ell-p} \in A_p$ we see that $\rho(a) = (a_X)$ that completes the proof. $\square$

**Proposition 3.2** *For every $X \in L$ and $p < \ell - \dim X$ we have*

$$\Gamma(L_X, \mathcal{F}^p) = A_p(\mathcal{A}_X).$$

The proof is similar to the proof of Proposition 3.1.

**Proposition 3.3** *For every $p$, the sheaf $\mathcal{F}^p$ is flasque.*

**Proof.** Let $U \subset L$ be an open (i.e., increasing) nonempty proper subset and $a = (a_X)_{X\in U} \in \Gamma(U, \mathcal{F}^p)$. Let $Z$ be one of the maximal elements of $L \setminus U$. It suffices to prove that $a$ can be extended to $Z$. Consider the nonempty set $U_Z = L_Z \cap U$. If $\dim Z \geq \ell - p$ then the restriction $b$ of $a$ to $U_Z$ is 0 and we put $a_Z = 0$. If $\dim Z < \ell - p$ then due to Proposition 3.2 there exists (and unique) an element $c \in \mathcal{F}^p(Z)$ that extends $b$ to $Z$. Putting $a_Z = c$ we complete the proof. $\square$

Now we want to organize the sheaves $\mathcal{F}^p$ in a complex of sheaves. Recall that for $\lambda \in F^n$ we put $\omega_\lambda = \sum_{i=1}^n \lambda_i\omega_i$. Now for every $X \in L$ put $\omega_\lambda(X) = \sum \lambda_i\omega_i$ where summation is taking over all indexes $i$ such that $H_i \in \mathcal{A}_X$. Denote by $d_\lambda(X)$ the left multiplication by $\omega_\lambda(X)$. Clearly for every $p$, the set $(d_\lambda(X))_{X\in L}$ is a homomorphism $d_\lambda(p) : \mathcal{F}^p \to \mathcal{F}^{p+1}$ and these homomorphisms form the complex of sheaves

$$\mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^\ell \to 0. \tag{3.2}$$

Moreover $\mathcal{E} = \ker(d_\lambda) \subset \mathcal{F}^0$ is the skyscraper sheaf whose only nonzero stalk is at $V$ and it equals $F$. Thus we can augment the complex (3.2) on the left and obtain the complex

$$0 \to \mathcal{E} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \mathcal{F}^\ell \to 0. \tag{3.3}$$

The following lemma is almost tautological.

**Lemma 3.4** *The complex (3.3) is exact (and thus a flasque resolution of $\mathcal{E}$) if and only if the complex $(A(\mathcal{A}_X), d_\lambda(X))$ is acyclic for every $X \in L \setminus \{V\}$.*

# 4 Main theorem

**Theorem 4.1** *(i) Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arbitrary $\ell$-arrangement with the intersection poset $L$. Let $\lambda = \{\lambda_1, \ldots, \lambda_n\} \in F^n$ and satisfy the following condition*

$$\sum_{X \subset H_i} \lambda_i \neq 0 \tag{4.1}$$

*for all $X \in L$ such that $\bar{\chi}(A(\mathcal{A}_X)) \neq 0$. Then $H^p(A(\mathcal{A}), d_\lambda) = 0$ for every $p < \ell$.*

*(ii) Let $\mathcal{A}$ and $L$ be as in (i) but $\mathcal{A}$ is central with $U = \bigcap_{i=1}^n H_i \neq \emptyset$. Let $\lambda$ satisfy (4.1) for $X \in L \setminus \{U\}$ but $\sum_{i=1}^n \lambda_i = 0$. Then $H^p(A(\mathcal{A}), d_\lambda) = 0$ for every $p < \ell - 1$ and $\dim H^{\ell-1}(A(\mathcal{A}), d_\lambda) = \dim H^\ell(A(\mathcal{A}), d_\lambda) = \bar{\chi}(\mathcal{A})$.*

**Proof.** First we prove (ii) assuming that (i) holds for smaller dimensions, the case of dimension 1 being trivial. As usual assume that $\alpha_1 = x_1$ and get $\bar{\mathcal{A}}$ substituting $x_1 = 1$ in $\alpha_2, \ldots, \alpha_n$. Put $\bar{\alpha}_i = \alpha_i(x_1 = 1)$ $(i = 2, \ldots, n)$ and $\bar{\lambda} = \sum_{i=2}^n \lambda_i$. The left multiplication by $\sum_{i=2}^n \lambda_i d\bar{\alpha}_i / \bar{\alpha}_i$ defines a coboundary map in $A(\bar{\mathcal{A}})$ whose cohomology we denote by $\overline{H}^*$. Since $\lambda$ satisfies (4.1) for every $X \in L \setminus \{U\}$ we have that $\bar{\lambda}$ satisfies (4.1) for every $X \in L(\bar{\mathcal{A}})$. Thus by the induction hypothesis

$$\overline{H^p} = 0, \ p \neq \ell - 1, \tag{4.2}$$

whence

$$\dim \overline{H^{\ell-1}} = |\bar{\chi}(\mathcal{A})|. \tag{4.3}$$

Since $\lambda_1 = -\sum_{i=2}^n \lambda_i$ we obtain the exact sequence (2.7) which is in the present notation

$$0 \to \overline{H^{p-1}} \to H^p(A(\mathcal{A}), d_\lambda) \to \overline{H^p} \to 0. \tag{4.4}$$

The result follows from (4.2)-(4.4).

Now we prove (i) assuming that (ii) holds for the same and smaller $\ell$. Consider the complex (3.2) of flasque sheaves on $L$ and fix $X \in L \setminus \{V\}$. Since $\mathcal{A}_X$ is a central arrangement in dimension not greater than $\ell$ we can use (ii). If (4.1) holds for $X$ then $(A(\mathcal{A}_X), d_\lambda(X))$ is acyclic by Proposition 2.1. If (4.1) does not hold for $X$ then $\bar{\chi}(A(\mathcal{A}_X)) = 0$ and (ii) implies that $(A(\mathcal{A}_X), d_\lambda(X))$ is again acyclic. In both cases Lemma 3.4 implies that (3.3) is a flasque resolution of $\mathcal{E}$. Using also Proposition 3.1 we have

$$H^p(A(\mathcal{A}), d_\lambda) = H^p(L, \mathcal{E}) \tag{4.5}$$

for every $p$.

Computation of $H^p(L, \mathcal{E})$ is a routine problem. Since $L$ is a geometric semilattice (see [10]), we have, using Lemma 3.1 from [1] and Corollary 7.3 from [10], that $H^p(L, \mathcal{E}) = 0$ for $p < \ell - 1$. This completes the proof. □

**Remark 4.2** *Due to Proposition 2.8 and Remark 2.9, the condition* (4.1) *is applicable to* $X \in L$ *with* $\dim X < \ell - 1$ *only if* $X$ *is* **Bad**. *In particular suppose that* $\mathcal{A}$ *is in general position, that is each set of not more than* $\ell$ *hyperplanes from* $\mathcal{A}$ *is independent and each bigger set has the empty intersection. Then* (4.1) *reduces to* $\lambda_i \neq 0$ *for every* $i = 1, \ldots, n$.

**Corollary 4.3** *Let* $F = \mathbb{C}$ *and* $\mathcal{L}$ *a local rank 1 system on* $M$ *of flat sections of the trivial bundle with connection* $d + \sum_{i=1}^{n} \lambda_i \omega_i$. *Suppose that for every* $X \in L$ *such that* $\mathcal{A}_X$ *does not satisfy the condition of Proposition 2.8 the number* $\sum_{X \subset H_i} \lambda_i$ *is not a nonnegative integer. Then* $H^p(M, \mathcal{L}) = 0$ *for every* $p < \ell$.

**Proof.** It follows from Corollary of [4] and Theorem 4.1. □

**Remark 4.4** *The results of [4] and [5] are proved for local systems of arbitrary finite rank. Theorem 4.1 can be generalized to this case also. More precisely, choose* $k \geq 1$ *and* $P_i \in \mathrm{End} F^k$ $(i = 1, \ldots, n)$ *such that* $P_i P_j = P_j P_i$. *Then*

$$\omega_P = \sum_{i=1}^{n} \omega_i \otimes P_i \in A_1 \otimes \mathrm{End} F^k$$

*acts on* $A \otimes F^k$ *and defines a cochain complex. All the results of the previous sections, including Theorem 4.1, hold with essentially the same proofs for the cohomology of this complex if one substitutes the condition* $\sum P_i \in \mathrm{Aut} F^k$ *for* $\sum \lambda_i \neq 0$.

# 5 Special cases

The condition of Theorem 4.1 is clearly necessary for the complex (3.3) to be exact. But it is not at all necessary for the vanishing of $H^p = H^p(A(\mathcal{A}), d_\lambda)$ for $p < \ell$. If the complex (3.3) is not exact then it defines not all 0 cohomology sheaves $\mathcal{H}^p$ and the spectral sequence with $E_2^{p,q} = H^p(L, \mathcal{H}^q)$ that converges to $H^{p+q}$. Using this spectral sequence, one can find various sufficient conditions on $\mathcal{A}$ and $\lambda$ that would imply the vanishing of $H^p$ for $p < \ell$. In this section we consider some particular cases to this effect.

First we study (by other means) the important case for applications where $\mathcal{A}$ is in general position. Recall that this means that each subset of $\mathcal{A}$ with not more than $\ell$ hyperplanes is independent and each larger subset has the empty intersection. As it was noticed in Remark 4.2, the condition of Theorem 4.1 gives $\lambda_i \neq 0$ for all $i$. This condition can be significantly relaxed.

**Proposition 5.1** *Let $\mathcal{A}$ be a general position arrangement and $\lambda \neq 0$. Then $H^p(A(\mathcal{A}), d_\lambda) = 0$ for every $p < \ell$.*

**Proof.** Recall that the algebra $A = A(\mathcal{A})$ is the factor of the exterior algebra $E = \oplus_{p=1}^{n} E_p$ over a homogeneous ideal $J = J(\mathcal{A})$. For a general position arrangement $\mathcal{A}$ we have

$$J = \oplus_{p>\ell} E_p.$$

In particular $J$ is invariant with respect to the natural action of the group $\mathrm{GL}(n, F)$ on $E$. Thus this group acts on $A$ and the action is transitive on the set $A_1 \setminus \{0\}$.

Now fix $\lambda \in F^n \setminus \{0\}$, put $\omega_\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$, and also put $\omega = \sum_{i=1}^{n} \omega_i$. There exists an isomorphism $\psi$ of $A$ sending $\omega_\lambda$ to $\omega$. Consequently $\psi$ generates an isomorphism $(A, d_\lambda) \to (A, d)$ where $d_\lambda$ and $d$ are the left multiplications by $\omega_\lambda$ and $\omega$ respectively. According to Theorem 4.1 and Remark 4.2, $H^p(A, d) = 0$ for every $p < \ell$. Hence the same is true for $(A, d_\lambda)$ which completes the proof. $\square$

The other particular case we consider is the case of an arbitrary arrangement but $\omega_\lambda = \omega_1$. Denote by $d$ the left multiplication in $A = A(\mathcal{A})$ by $\omega_1$.

**Proposition 5.2** *For any integer $p$, $H^p(A, d) = 0$ if and only if there is no $X \in L$ of dimension $\ell - p$ such that $X \cap H_1 = \emptyset$.*

**Proof.** Put $L_1 = \{X \in L | X \subset H_1\}$ and notice that $\omega_\lambda(X) = \omega_1$ if $X \in L_1$ and $\omega_\lambda(X) = 0$ otherwise. This and Proposition 2.1 imply that for every $q$ the cohomology sheaf $\mathcal{H}^q$ is 0 on $L_1$ and, on $L \setminus L_1$, coincides with the restriction of $\mathcal{F}^q$ to this set. This allows us to consider the short exact sequence

$$0 \to \mathcal{K}^q \to \mathcal{F}^q \to \mathcal{H}^q \to 0 \tag{5.1}$$

where the sheaf $\mathcal{K}^q$ is 0 on $L \setminus L_1$ and coincides with the restriction $\mathcal{F}^q_{L_1}$ of $\mathcal{F}^q$ to $L_1$. Since $\mathcal{F}^p$ is flasque, $\mathcal{F}^p_{L_1}$ is flasque also (e.g., see Proposition 1.6 of [11]). Since $L_1$ is closed $H^p(L, \mathcal{K}^q) = H^p(L_1, \mathcal{F}^q_{L_1}) = 0$ for all $p > 0$. Then the cohomology sequence of the sequence (5.1) implies that $H^p(L, \mathcal{H}^q) = 0$ for every $p > 0$ whence $E_2^{p,q} = 0$ for every $q$ and $p > 0$. This implies that $H^p(A, d) = E_2^{0,p} = \Gamma(L, \mathcal{H}^p)$.

Now suppose that there exists a non-zero $s \in \Gamma(L, \mathcal{H}^p)$. By definition of $\mathcal{F}^p$, this means that there exists $X \in L$ of dimension $\ell - p$ such that $s(X) \neq 0$. Since besides $s = 0$ on $L_1$ we have $X \cap H_1 = \emptyset$.

Conversely, suppose that there exists $X \in L$ of dimension $\ell - p$ and $X \cap H_1 = \emptyset$. Choose $a \in A_X \setminus \{0\}$. For every $Y \in L, Y \leq X$, $a \in A_p(\mathcal{A}_Y) =$

$\bigoplus_{\dim Z=\ell-p, Z\geq Y} A_Z$ since $X$ is among those $Z$. Then for every $Y \in L$ put $s(Y) = a(Y)$ if $Y \leq X$ and $s(Y) = 0$ otherwise. One easily checks that $(s(Y))_{Y\in L} \in \Gamma(L, \mathcal{H}^p)$. Since this element is non-zero the proof is complete. $\square$

In general it is hard to formulate a simple condition necessary and sufficient for the vanishing of $H^p(A, d_\lambda)$ for $p < \ell$. Let us consider some examples not covered by the above particular cases.

**Example 5.3** *Let $\mathcal{A}$ be given by the functionals $\alpha_1 = x, \alpha_2 = y, \alpha_3 = x - y, \alpha_4 = x - 1$ in the 2-dimensional affine space $\{(x, y)\}$. Then Theorem 4.1 gives the following condition on $\lambda = (\lambda_1, \ldots, \lambda_4)$ sufficient for the vanishing of $H^p(A(\mathcal{A}), d_\lambda)$ for $p = 0, 1$*

$$\lambda_1\lambda_2\lambda_3\lambda_4(\lambda_1 + \lambda_2 + \lambda_3) \neq 0.$$

*On the other hand, direct linear algebra computation gives the more relaxed necessary and sufficient condition consisting of at least one of the following inequalities*

$$\lambda_2\lambda_4 \neq 0,\ \lambda_3\lambda_4 \neq 0,\ \lambda_2(\lambda_1 + \lambda_2 + \lambda_3) \neq 0,\ \lambda_3(\lambda_1 + \lambda_2 + \lambda_3) \neq 0.$$

**Example 5.4** *Let $\mathcal{A}$ be given by the four functionals of Example 5.3 and by $\alpha_5 = y - 1$. One computes that $\dim A_0 = 1,\ \dim A_1 = 5,\ \dim A_2 = 6$. Take $\lambda = (1, 1, -2, 1, 1) \in F^5$. Then the rank of the linear map $d_\lambda : A_1 \to A_2$ is 3 whence $\dim H^1(A, d_\lambda) = 1$ and $\dim H^2(A, d_\lambda) = 3$.*

Example 5.4 shows that $H^1(A, d_\lambda)$ (for $\ell > 1$) may not vanish even if each $\lambda_i$ and the sum of all of them are not zero (cf. Theorem 6.4). Another interesting feature of this example is that $\dim E_2^{0,1} = \dim H^0(L, \mathcal{H}^1) = 2 > \dim H^1(A, d_\lambda)$ whence there exist non-zero differentials in the spectral sequence.

# 6 Arrangements in general position to infinity

In this section we consider a class of arrangements that includes all the central ones and all arrangements in general position. For this class we use a different method and give yet another sufficient condition on $\lambda$ for the vanishing of $H^p(A)$ for $p < \ell$. More importantly, under this condition we exhibit a subspace of $A_\ell$ independent of $\lambda$ that is isomorphic to $H^\ell(A)$ under the natural projection. For a real arrangement from this class, Kohno showed in Theorem 2 of [5] that the geometry of the arrangement defines a basis of $H^\ell$ independent of $\lambda$. Using [4], our results in this section can be viewed as a generalization of Kohno's.

Let $\{H_0, H_1, \ldots, H_n\}$ be a central arrangement in dimension $\ell$. Say that $H_0$ is generic with respect to $\{H_1, \ldots, H_n\}$ if any time when a set $S = \{H_0, H_{i_1}, \ldots, H_{i_k}\}$ with $k \leq \ell$ is dependent the set $S \setminus \{H_0\}$ is dependent also.

**Proposition 6.1** *Let $\mathcal{A}$ be an arrangement in the affine space $\{(x_1, \ldots, x_\ell)\}$. The following conditions are equivalent:*

*(i) For the cone $c\mathcal{A}$ of $\mathcal{A}$ in the space $\{(x_0, x_1, \ldots, x_\ell)\}$ the hyperplane $H_0$ given by $x_0 = 0$ is generic with respect to $c\mathcal{A} \setminus \{H_0\}$.*

*(ii) For any ordered subset $S$ of $\mathcal{A}$ with $|S| \leq \ell$ we have $\bigcap_{H_i \in S} H_i \neq \emptyset$.*

*(iii) Every two elements of $L$ of dimensions $r$ and $s$ respectively such that $r + s \geq \ell$ have a non-empty intersection.*

The proof is a routine exercise.

**Definition 6.2** *If an arrangement $\mathcal{A}$ satisfies the equivalent conditions of Proposition 6.1 then we say that it is in general position to infinity.*

Clearly all central arrangements and all arrangements in general position are in general position to infinity. The arrangement $x, y, x - y, x + y + 1$ in $\{(x, y)\}$ ($\mathrm{char}F \neq 2$) is another representative of this class.

For arrangements in general position to infinity we can define a boundary map in $A$ generalizing $\partial_A$ for central arrangements. We cannot use Definition 3.12 of [7] since if $\mathcal{A}$ is not central $I(\mathcal{A})$ is not invariant with respect to $\partial_E$. However regarding $A$ just as a graded linear space we have

$$A = E/I = \bar{E}/\bar{I}$$

where $\bar{E} = \oplus_{i=0}^{\ell} E_i$ and $\bar{I} = I \cap \bar{E}$.

**Lemma 6.3** *Suppose that $\mathcal{A}$ is in general position to infinity. Then $\bar{I}$ is invariant with respect to $\partial_E$.*

**Proof.** Using (ii) of Proposition 6.1 and the definition of $I(\mathcal{A})$, one sees that $\bar{I}$ is generated over $F$ by elements $e_T \partial e_S$ for ordered subsets $T$ and $S$ of $\mathcal{A}$ such that $S$ is dependent and $|T| + |S| \leq \ell + 1$. We have

$$\partial_E(e_T \partial_E e_S) = \partial_E e_T \partial_E e_S \in \bar{I}$$

which completes the proof. $\square$

Lemma 6.3 allows us to define the map $\partial_A : A \to A$ as induced by the restriction of $\partial_E$ to $\bar{E}$. If $\mathcal{A}$ is central this map coincides with $\partial_A$ from Definition 3.12 of [7]. The following properties of $\partial_A$ follow immediately from properties of $\partial_E$:

(i) $\partial_A^2 = 0$,

(ii) if $a \in A_p$ and $b \in A_q$ with $p + q \leq \ell$ then $\partial_A(ab) = (\partial_A a)b + (-1)^p a(\partial_A b)$,

(iii) if $\omega_\lambda = \sum_{i=1}^n \lambda_i \omega_i$ then $\partial_A(\omega_\lambda) = \sum_{i=1}^n \lambda_i$.

Now we can prove a generalization of Proposition 2.1.

**Theorem 6.4** *Let $\mathcal{A}$ be an arrangement in general position to infinity. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in F^n$ and $\sum_{i=1}^n \lambda_i \neq 0$. Then*

*(i) $H^p(A(\mathcal{A}), d_\lambda) = 0$ for every $p < \ell$;*

*(ii) The space $\ker \partial_A$ is isomorphic to $H^\ell = H^\ell(A(\mathcal{A}), d_\lambda)$ under the natural projection $A_\ell(\mathcal{A}) \to H^\ell$.*

**Proof.** Suppose that $a \in A_p$ with $p < \ell$. Then the properties (ii) and (iii) of $\partial_A$ imply

$$\partial_A d_\lambda(a) = (\sum_{i=1}^n \lambda_i)a - d_\lambda \partial_A(a). \tag{6.1}$$

The statement (i) follows from (6.1) immediately (using $(1/\sum \lambda_i)\partial_A$ as a homotopy).

Now let us study $H^\ell(A)$. First

$$d_\lambda(A_{\ell-1}) \cap \ker \partial_A = 0. \tag{6.2}$$

Indeed if $\partial_A d_\lambda(a) = 0$ for some $a \in A_{\ell-1}$ then by (6.1) $(\sum \lambda_i)a = d_\lambda \partial_A(a)$ whence $d_\lambda(a) = 0$. Second for arbitrary $b \in A_\ell$ put

$$b_1 = b - \frac{1}{\sum \lambda_i} d_\lambda \partial_A(b).$$

Then $b_1$ defines the same cohomology class as $b$ does and using property (i) of $\partial_A$ and (6.1) we have

$$\partial_A b_1 = \partial_A b - \frac{1}{\sum \lambda_i} \partial_A d_\lambda \partial_A(b) = 0. \tag{6.3}$$

(6.2) and (6.3) imply the statement (ii). $\square$

Under the conditions of Theorem 6.4, the space $H^\ell(A) = \ker \partial_A$ has a simple combinatorial interpretation as homology of a simplicial complex.

For any poset $P$ denote by $\Delta(P)$ the simplicial complex whose vertices are elements of $P$ and simplexes are all the strictly increasing sequences from

$P$. Let $C(P) = \oplus_{p\geq -1} C_p(P)$ be the augmented chain complex of $\Delta(P)$ with coefficients in $F$. Put $\tilde{H}_*(P;F) = H_*(C(P))$.

Now put $L^0 = L \setminus \{V\}$. There is a linear embedding

$$\theta : A(\mathcal{A})) \to C(L^0)$$

defined by $\theta(1) = 1$ and for $a_S \in A_p(\mathcal{A}) \setminus \{0\}$ with $S = (i_1, \ldots, i_p)$, $p > 0$,

$$\theta(a_S) = \sum_{\pi \in S_p} \operatorname{sign}\pi (H_{\pi i_1} \cap H_{\pi i_2} \cap \cdots \cap H_{\pi i_p}, H_{\pi i_2} \cap \cdots \cap H_{\pi i_p}, \ldots, H_{\pi i_p})$$

(cf. 3.4 in [7]). Notice that $\theta(A_p(\mathcal{A})) \subset C_{p-1}(L^0)$ for every $p$.

Let $d$ be the standard boundary map in $C(L^0)$ with respect to the orderings on simplexes induced from $L^0$.

**Lemma 6.5** *If $\mathcal{A}$ is in general position to infinity then*

$$\theta \partial_A = d\theta. \tag{6.4}$$

**Proof.** It suffices to fix $S$ as above such that $a_S \neq 0$ and check (6.4) for $a_S$. The check is straightforward. □

**Proposition 6.6** *Under the condition of Theorem 6.4*

$$H^\ell(A, d_\lambda) \cong \tilde{H}_{\ell-1}(L^0; F).$$

**Proof.** Due to Theorem 6.4 it suffices to prove that $\ker \partial_A \cong \tilde{H}_{\ell-1}(L^0;F)$. Applying (6.4) for $A_\ell$, we see that $\theta$ embeds $\ker \partial_A$ into the space of reduced $(\ell-1)$-cycles of $C(L^0)$ that coincides with $\tilde{H}_{\ell-1}(L^0;F)$. Thus it suffices to compare the dimensions of the spaces in the statement.

Let $\mu$ be the Möbius function of $L^{\mathrm{op}}$ and put $\mu(X) = \mu(V, X)$ for $X \in L$. Then Theorem 6.4 and Theorem 3.68 of [7] imply that

$$\dim H^\ell(A, d_\lambda) = |\chi(A)| = |\sum_{X \in L} \mu(X)|. \tag{6.5}$$

On the other hand, due to [10] $\tilde{H}_p(L^0;F) = 0$ for $p < \ell - 1$ whence

$$\dim \tilde{H}_{\ell-1}(L^0;F) = |\chi(\Delta(L^0))| + 1. \tag{6.6}$$

It is well known that $\chi(\Delta(L^0)) = \sum_{X \in L^0} \mu(X) = \sum_{X \in L} \mu(X) - 1$ whence (6.5) and (6.6) imply that $\dim H^\ell(A, d_\lambda) = \dim \tilde{H}_{\ell-1}(L^0;F)$. This equality completes the proof. □

**Remark 6.7** *Notice that the isomorphism $H^\ell(A, d_\lambda) \cong \tilde{H}_{\ell-1}(L^0;F)$ exists under the conditions of Theorem 4.1 also (and was used in the proof of this theorem). The difference is that under the condition of Theorem 6.4 there is an explicit representation of $H^\ell(A, d_\lambda)$ by $(\ell-1)$-cycles of $C(L^0)$.*

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