Generalized translation operator and approximation in several variables

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Abstract

Generalized translation operators for orthogonal expansions with respect to families of weight functions on the unit ball and on the standard simplex are studied. They are used to define convolution structures and modulus of smoothness for these regions, which are in turn used to characterize the best approximation by polynomials in the weighted L^p spaces. In one variable, this becomes the generalized translation operator for the Gegenbauer polynomial expansions.

1 Introduction

Let w_λ denote the weight function $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ on $[-1,1]$. Let b_λ be the normalization constant of w_λ, $b_\lambda^{-1} = \int_{-1}^1 w_\lambda(s)ds$. The orthogonal polynomials with respect to w_λ are the Gegenbauer polynomials C_n^λ. The generalized translation operator with respect to w_λ is defined by

$$T_s f(t) = b_{\lambda-1/2} \int_{-1}^1 f\left(st + u\sqrt{1-s^2}\sqrt{1-t^2}\right)(1-u^2)^{\lambda-1} du. \tag{1.1}$$

It plays the role of translation for the trigonometric series and can be used to define a convolution structure $f \star g$ for $f, g \in L^1(w_\lambda, [-1,1])$,

$$(f \star g)(t) = b_\lambda \int_{-1}^1 f(s) T_t g(s) w_\lambda(s) ds, \tag{1.2}$$

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as introduced by Gelfand [17] and Bochner [9]. The convolution and the generalized translation operator have been used to study Fourier orthogonal expansions in Gegenbauer polynomials (see, for example, [3,5,6,11,22,23,28]). Using the product formula of the Gegenbauer polynomials, the generalized translation operator can also be defined by the equation

$$T_s C_n^\lambda(t) = \frac{C_n^\lambda(s)}{C_n^\lambda(1)} C_n^\lambda(t), \qquad n \ge 0. \tag{1.3}$$

The purpose of this paper is to study the generalized translation operator for weight functions defined on the unit ball $B^d = \{x : \|x\| \le 1\} \subset \mathbb{R}^d$ and on the standard simplex

$$T^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, 1 - |x| \ge 0\}, \qquad |x| = x_1 + \ldots + x_d,$$

in $\mathbb{R}^d$ and use them to study weighted approximation and orthogonal expansions in several variables. To define the weight functions, we start from the reflection invariant weight function considered by Dunkl [13].

For a nonzero vector $v \in \mathbb{R}^d$, let σ_v denote the reflection with respect to the hyperplane perpendicular to v; that is, $x\sigma_v := x - 2(\langle x, v\rangle/\|v\|^2)v$, $x \in \mathbb{R}^d$, where $\langle x, y\rangle$ denote the usual Euclidean inner product and $\|x\|$ denote the usual Euclidean norm $\|x\|^2 = \langle x, x\rangle$. The weight function h_κ is defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d, \tag{1.4}$$

in which R_+ is a fixed positive root system of $\mathbb{R}^d$, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, and κ is a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in the reflection group G generated by the reflections $\{\sigma_v : v \in R_+\}$. Then h_κ is invariant under the reflection group G, a subgroup of the orthogonal group. The simplest example is given by the case $G = \mathbb{Z}_2^d$ for which h_κ is just the product weight function

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \qquad \kappa_i \ge 0. \tag{1.5}$$

Other examples include weight functions invariant under the symmetric group and the hyperoctahedral group,

$$\prod_{1 \le i < j \le d} |x_i - x_j|^\kappa \qquad \text{and} \qquad \prod_{i=1}^d |x_i|^{\kappa_0} \prod_{1 \le i < j \le d} |x_i^2 - x_j^2|^\kappa,$$

respectively.

The weight functions on the unit ball B^d that we shall consider are of the form

$$W^B_{\kappa,\mu}(x) = h^2_\kappa(x)(1-\|x\|^2)^{\mu-1/2}, \qquad x \in B^d, \tag{1.6}$$

where $\mu \ge 0$ and h_κ is a reflection invariant weight function as in (1.4), and the weight functions on the simplex that we shall consider are of the form

$$W^T_{\kappa,\mu}(x) = h^2_\kappa(\sqrt{x_1},\ldots,\sqrt{x_d})(1-|x|)^{\mu-1/2}/\sqrt{x_1\cdots x_d}, \tag{1.7}$$

where $\mu \ge 0$ and h_κ is a reflection invariant weight function as in (1.4), and we assume that h_κ is even in each of its variables (for example, weight functions invariant under $\mathbb{Z}_2^d$ and the hyperoctahedral group on $\mathbb{R}^d$). These include the classical weight functions on these domains, which are

$$W^B_\mu(x) = (1-\|x\|^2)^{\mu-1/2}, \qquad x \in B^d, \tag{1.8}$$

on the unit ball (taking $h_\kappa(x) = 1$) and

$$W^T_\kappa(x) = x_1^{\kappa_1-1/2}\cdots x_d^{\kappa_d-1/2}(1-|x|)^{\kappa_{d+1}-1/2}, \qquad x \in T^d, \tag{1.9}$$

on the simplex (taking $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa}_i$ and $\kappa_{d+1} = \mu$). For $d = 1$, $W^T_\kappa(x)$ is the Jacobi weight function on the interval $[0,1]$.

The orthogonal structures for $W^B_{\kappa,\mu}$ on the ball and for $W^T_{\kappa,\mu}$ on the simplex are closely related to the orthogonal structure of h-harmonics on the unit sphere $S^d = \{x : \|x\| = 1\}$ of $\mathbb{R}^{d+1}$. Our study of the generalized translation operators relies on that of the weighted spherical means, studied in [36,37], which are the generalizations of the ordinary spherical means

$$T_\theta f(x) = \frac{1}{\sigma_d(\sin\theta)^d}\int_{\langle x,y\rangle=\cos\theta} f(y)d\omega(y), \tag{1.10}$$

where $\sigma_d = \int_{S^d} d\omega = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ is the surface area of S^d. The ordinary spherical means have been used as main tool for approximation on the sphere; see, for example, [6,21,19,22,26]. The weighted spherical means are defined implicitly via an integral relation. In [37] the weighted means were used to define a modulus of smoothness, which was shown to be equivalent to a K-functional and used to characterize the weighted L^p best approximation by polynomials. The similar K-functional was also defined and used to characterize the weighted best approximation on B^d and on T^d, but the modulus of smoothness was not defined since the analog of the spherical means for B^d and T^d seemed to be artificial. It has been realized only recently in [38] that the analog of the spherical means for B^d and T^d, as generalized translation operators on these regions, are of interest. Since the main purpose of [38] is to define weighted maximal functions and use them to prove results on almost everywhere convergence, the generalized translation operators themselves were not studied there. We complete this circle of ideas in the present paper.

One of our results gives an explicit integral formula for the generalized translation operator with respect to the classical weight function W_μ^B in (1.8) on the unit ball (see (3.9), which extends the formula (1.1) to several variables. No integral formula is known for any other weight functions.

The paper is organized as follows. In the next section we recall the background and results for h-harmonics and the weighted spherical means. The results on the unit ball are presented in Section 3 and the results on the simplex appear in Section 4.

2 Weighted spherical means and weighted approximation on S^{d-1}

Let h_κ be the reflection invariant weight function defined in (1.4). We denote by a_κ the normalization constant of h_κ, $a_\kappa^{-1} = \int_{S^{d-1}} h_\kappa^2(y) d\omega$, and denote by $L^p(h_\kappa^2)$, $1 \le p \le \infty$, the space of functions defined on S^{d-1} with the finite norm

$$\|f\|_{\kappa,p} := \left(a_\kappa \int_{S^{d-1}} |f(y)|^p h_\kappa^2(y) d\omega(y) \right)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(S^d)$, the space of continuous functions on S^{d-1} with the usual uniform norm $\|f\|_\infty$, since we are mainly interested in problems in approximation theory. The case $\kappa \equiv 0$ corresponds to the usual (unweighted) L^p space on S^{d-1}.

2.1 Background. The essential ingredient of the theory of h-harmonics is a family of first-order differential-difference operators, $\mathcal{D}_i$, called Dunkl operators, which generates a commutative algebra; these operators are defined by ([13])

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v \in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle v, \varepsilon_i \rangle, \qquad 1 \le i \le d,$$

where $\varepsilon_1, \ldots, \varepsilon_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$ and it plays the role similar to that of the ordinary Laplacian. Let $\mathcal{P}_n^d$ denote the subspace of homogeneous polynomials of degree n in d variables. An h-harmonic polynomial P of degree n is a homogeneous polynomial $P \in \mathcal{P}_n^d$ such that $\Delta_h P = 0$. Furthermore, let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of h-harmonic polynomials of degree n in d variables and define

$$\langle f, g \rangle_\kappa := a_\kappa \int_{S^{d-1}} f(x) g(x) h_\kappa^2(x) d\omega(x).$$

Then $\langle P, Q \rangle_\kappa = 0$ for $P \in \mathcal{H}_n^d(h_\kappa^2)$ and $Q \in \Pi_{n-1}^d$, where Π_n^d denote the space of polynomials of degree at most n in d variables. The spherical h-

harmonics are the restriction of h-harmonics to the unit sphere. It is known that $\dim \mathcal{H}_n(h_\kappa^2) = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$ with $\dim \mathcal{P}_n^d = \binom{n+d-1}{d}$.

In terms of the polar coordinates $y = ry'$, $r = \|y\|$, the h-Laplacian operator Δ_h takes the form ([35])

$$\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{2\lambda_\kappa + 1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{h,0},$$

where $\Delta_{h,0}$ is the Laplace-Beltrami operator on the sphere, and throughout this paper, we fix the value of λ_κ as

$$\lambda := \lambda_\kappa = \gamma_\kappa + \frac{d-1}{2} \qquad \text{with} \qquad \gamma_\kappa = \sum_{v \in R_+} \kappa_v. \tag{2.1}$$

Applying Δ_h to h-harmonics $Y \in \mathcal{H}_n(h_\kappa^2)$ with $Y(y) = r^n Y(y')$ shows that spherical h-harmonics are eigenfunctions of $\Delta_{h,0}$; that is,

$$\Delta_{h,0} Y(x) = -n(n + 2\lambda_\kappa) Y(x), \qquad x \in S^{d-1}, \quad Y \in \mathcal{H}_n^d(h_\kappa^2). \tag{2.2}$$

For further background materials, see [13,14] and the references in [14].

The standard Hilbert space theory shows that

$$L^2(h_\kappa^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^d(h_\kappa^2).$$

That is, with each $f \in L^2(h_\kappa^2)$ we can associate its h-harmonic expansion

$$f(x) = \sum_{n=0}^{\infty} Y_n(h_\kappa^2; f, x), \qquad x \in S^{d-1},$$

in $L^2(h_\kappa^2)$ norm. For the surface measure ($\kappa = 0$), such a series is called the Laplace series (cf. [15, Chapt. 12]). The orthogonal projection $Y_n(h_\kappa^2) : L^2(h_\kappa^2) \mapsto \mathcal{H}_n^d(h_\kappa^2)$ takes the form

$$Y_n(h_\kappa^2; f, x) := \int_{S^{d-1}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y)\, d\omega(y), \tag{2.3}$$

where the kernel $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space of h-harmonics $\mathcal{H}_n^d(h_\kappa^2)$ in $L^2(h_\kappa^2)$. The kernel $P_n(h_\kappa^2; x, y)$ enjoys a compact formula in terms of the intertwining operator between the commutative algebra generated by the partial derivatives and the one generated by Dunkl operators. This operator, V_κ, is linear and it is determined uniquely by

$$V_\kappa \mathcal{P}_n^d \subset \mathcal{P}_n^d, \qquad V_\kappa 1 = 1, \qquad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \qquad 1 \le i \le d.$$

The formula for the reproducing kernel for $\mathcal{H}_n^d(h_\kappa^2)$ is given in terms of the Gegenbauer polynomials

$$P_n(h_\kappa^2; x, y) = \frac{n+\lambda_\kappa}{\lambda_\kappa} V_\kappa[C_n^{\lambda_\kappa}(\langle \cdot, y\rangle)](x). \tag{2.4}$$

An explicit formula of V_κ is known only in the case of symmetric group S_3 for three variables and in the case of the abelian group $\mathbb{Z}_2^d$. In the latter case, V_κ is an integral operator,

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1t_1, \ldots, x_dt_d) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt, \tag{2.5}$$

where c_κ is the normalization constant determined by $V_\kappa 1 = 1$, $c_\kappa = c_{\kappa_1} \ldots c_{\kappa_d}$ and $c_r^{-1} = \int_{-1}^1 (1-t^2)^{r-1} dt$. If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int_{-1}^1 f(t)(1-t)^{\lambda-1} dt = [f(1) + f(-1)]/2.$$

One important property of the intertwining operator is that it is positive ([25]) for any reflection group,, that is, $V_\kappa p \geq 0$ if $p \geq 0$. One can also study the dual of this operator, as in [30].

2.2 Weighted spherical means, convolution and approximation. We recall the results developed in [36,37], some of which will be needed later and others are cited to show what can be expected in the cases B^d and T^d. For $f \in L^p(h_\kappa^2)$ and $g \in L^1(w_\lambda; [-1,1])$, we define a sort of convolution

$$(f \star_\kappa g)(x) := a_\kappa \int_{S^{d-1}} f(y) V_\kappa[g(\langle x, \cdot \rangle)](y) h_\kappa^2(y) d\omega. \tag{2.6}$$

For the surface measure ($h_\kappa(x) = 1$ and $V_\kappa = id$), this is called the spherical convolution in [12], and it has been used by many authors, see, for example, [7–9,19,21,22,26]. It satisfies many properties of the usual convolution in $\mathbb{R}^d$. The weighted spherical means, T_θ^κ, with respect to h_κ^2 is defined implicitly by the formula

$$b_\lambda \int_0^\pi T_\theta^\kappa f(x) g(\cos\theta)(\sin\theta)^{2\lambda} d\theta = (f \star_\kappa g)(x), \qquad 0 \leq \theta \leq \pi, \tag{2.7}$$

where g is any $L^1(w_\lambda)$ function. That $T_\theta f$ is well defined is shown in [36,37]. For $\kappa = 0$, $V_\kappa = id$, the weighted spherical means coincide with the weighted means $T_\theta f$ in (1.10). Many properties of $T_\theta f$, given in [6,22], can be extended to the weighted means $T_\theta^\kappa f$. In particular, we have

$$\|T_\theta^\kappa f\|_{\kappa,p} \leq \|f\|_{\kappa,p} \qquad \text{and} \qquad \lim_{\theta \to 0} \|T_\theta^\kappa f - f\|_{\kappa,p} = 0.$$

Consequently, the following definition of a modulus of smoothness, $\omega_r(f;t)_{\kappa,p}$, makes sense. Let $r > 0$, for $f \in L^p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(S^{d-1})$, define

$$\omega_r(f,t)_{\kappa,p} := \sup_{0\le\theta\le t} \|(I - T_\theta^\kappa)^{r/2}\|_{\kappa,p}. \tag{2.8}$$

For the unweighted case ($\kappa = 0$), such a definition was given in [26] and the case r being an even integer had appeared in several early references (see the discussion in [26]). One of the important properties of this modulus of smoothness is that it is equivalent to a K-functional.

Let $r > 0$. Recall the equation (2.2). We define $(-\Delta_{h,0})^{r/2} g$ by

$$(-\Delta_{h,0})^{r/2} g \sim \sum_{n=1}^{\infty} (n(n+2\lambda_\kappa))^{r/2} Y_n(h_\kappa^2; g)$$

for $g \in L^2(h_\kappa^2)$. Furthermore, define the function space $\mathcal{W}_r^p(h_\kappa^2)$ by

$$\mathcal{W}_r^p(h_\kappa^2) = \left\{ f \in L^p(h_\kappa^2) : (k(k+2\lambda))^{\frac{r}{2}} P_k(h_\kappa^2; f) = P_k(h_\kappa^2; g) \text{ some } g \in L^p(h_\kappa^2) \right\}.$$

The K-functional between $L^p(h_\kappa^2)$ and $\mathcal{W}_r^p(h_\kappa^2)$ is defined by

$$K_r(f;t)_{\kappa,p} := \inf \left\{ \|f - g\|_{\kappa,p} + t^r \|(-\Delta_{h,0})^{r/2}\, g\|_{\kappa,p},\ g \in \mathcal{W}_r^p(h_\kappa^2) \right\}. \tag{2.9}$$

It is equivalent to the modulus of smoothness in the following sense:

Theorem 2.1. *For $1 \le p \le \infty$, there exist two positive constants c_1 and c_2 such that for $f \in L^p(h_\kappa^2)$,*

$$c_1 \omega_r(f;t)_{\kappa,p} \le K_r(f;t)_{\kappa,p} \le c_2\, \omega_r(f;t)_{\kappa,p}.$$

The modulus of smoothness or the K-functional can be used to characterize the best approximation by polynomials. Let

$$E_n(f)_{\kappa,p} := \inf \left\{ \|f - P\|_{\kappa,p} : P \in \Pi_n^d \right\},$$

We state the direct and the inverse theorems in terms of the modulus of smoothness.

Theorem 2.2. *For $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$,*

$$E_n(f)_{\kappa,p} \le c\, \omega_r(f; n^{-1})_{\kappa,p}.$$

On the other hand,

$$\omega_r(f, n^{-1})_{\kappa,p} \le c\, n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{\kappa,p}.$$

These theorems are proved in [36,37], following closely the method developed in [26] where these theorems were essentially established in the case $\kappa = 0$. For results in the unweighted cases, see also [19]. The problem of best approximation has been studied by many authors. We refer to [6,18,21,22,24,26,31] and the references therein.

One can study summability of h-harmonic expansions (see [32,34,20]) and the convolution structure $\star_\kappa$ is useful in this direction (see [36,38]). For summability of the ordinary harmonic expansions (unweighted case), we refer to [6,7,10,18,19,21,22,26] and the references therein.

In the case of usual surface measure on S^{d-1}, there are various other moduli of smoothness that have been used to characterize the best approximation; see, for example, [27,31]. It would be nice to define one that can be given explicitly for the weighted case.

3 Generalized translation operator and Approximation on B^d

Recall the weight function $W^B_{\kappa,\mu}(x)$ defined in (1.6), in which h_κ is an reflection invariant weight function defined on $\mathbb{R}^d$. Let $a_{\kappa,\mu}$ denote the normalization constant for $W^B_{\kappa,\mu}$. Denote by $L^p(W^B_{\kappa,\mu})$, $1 \le p \le \infty$, the space of measurable functions defined on B^d with the finite norm

$$\|f\|_{W^B_{\kappa,\mu},p} := \left(a_{\kappa,\mu} \int_{B^d} |f(x)|^p W^B_{\kappa,\mu}(x) dx \right)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(B^d)$, the space of continuous function on B^d.

3.1 Background. Let $\mathcal{V}^d_n(W^B_{\kappa,\mu})$ denote the space of orthogonal polynomials of degree n with respect to $W^B_{\kappa,\mu}$ on B^d. Elements of $\mathcal{V}^d_n(W^B_{\kappa,\mu})$ are closely related to the h-harmonics associated with the weight function

$$h_{\kappa,\mu}(y_1, \ldots, y_{d+1}) = h_\kappa(y_1, \ldots, y_d)|y_{d+1}|^\mu$$

on $\mathbb{R}^{d+1}$, where h_κ is associated with the reflection group G. The function $h_{\kappa,\mu}$ is invariant under the group $G \times \mathbb{Z}_2$. Let Y_n be such an h-harmonic polynomial of degree n and assume that Y_n is even in the $(d+1)$-th variable; that is, $Y_n(x, x_{d+1}) = Y_n(x, -x_{d+1})$. We can write

$$Y_n(y) = r^n P_n(x), \qquad y = r(x, x_{d+1}) \in \mathbb{R}^{d+1}, \quad r = \|y\|, \quad (x, x_{d+1}) \in S^d, \tag{3.1}$$

in polar coordinates. Then P_n is an element of $\mathcal{V}^d_n(W^B_{\kappa,\mu})$ and this relation is an one-to-one correspondence ([34]). Under the changing variables $y \mapsto r(x, x_{d+1})$,

$h_{\kappa,\mu}$ becomes $W^B_{\kappa,\mu}$ and the elementary formula

$$\int_{S^d} P(y)d\omega = \int_{B^d}\left[P(x,\sqrt{1-\|x\|^2}\,) + P(x,-\sqrt{1-\|x\|^2}\,)\right]\frac{dx}{\sqrt{1-\|x\|^2}} \tag{3.2}$$

shows the relation between their normalization constants.

Let $\Delta_h^{\kappa,\mu}$ denote the h-Laplacian associated with $h_{\kappa,\mu}$ and $\Delta_{h,0}^{\kappa,\mu}$ denote the corresponding spherical h-Laplacian. When $\Delta_h^{\kappa,\mu}$ is applied to functions on $\mathbb{R}^{d+1}$ that are even in the $(d+1)$-th variable, the spherical h-Laplacian can be written in polar coordinates $y = r(x, x_{d+1})$ as ([35]):

$$\Delta_{h,0}^{\kappa,\mu} = \Delta_h - \langle x, \nabla\rangle^2 - 2(\lambda_\kappa + \mu)\langle x, \nabla\rangle,$$

in which the operators Δ_h and $\nabla = (\partial_1, \ldots, \partial_d)$ are all acting on x variables and Δ_h is the h-Laplacian associated with h_κ on $\mathbb{R}^d$. Define

$$D^B_{\kappa,\mu} := \Delta_h - \langle x, \nabla\rangle^2 - 2(\lambda_\kappa + \mu)\langle x, \nabla\rangle,$$

as an operator acting on functions defined on B^d. It follows that the elements of $\mathcal{V}^d_n(W_{\kappa,\mu})$ are eigenfunctions of $D^B_{\kappa,\mu}$:

$$D^B_{\kappa,\mu} P = -n(n + 2\lambda_\kappa + 2\mu)P, \qquad P \in \mathcal{V}^d_n(W^B_{\kappa,\mu}). \tag{3.3}$$

For the classical weight function $W^B_\mu(x) = (1-\|x\|^2)^{\mu-1/2}$, the operator $D^B_{\kappa,\mu}$ becomes a pure differential operator which is a classical result going back to Hermite (see [4] and [15, Chapt. 12]).

For $f \in L^2(W^B_{\kappa,\mu})$, its orthogonal expansion is given by

$$L^2(W^B_{\kappa,\mu}) = \bigoplus_{n=0}^{\infty} \mathcal{V}^d_n(W^B_{\kappa,\mu}) : \qquad f = \sum_{n=0}^{\infty} \operatorname{proj}_n^{\kappa,\mu} f,$$

where $\operatorname{proj}_n^{\kappa,\mu} : L^2(W^B_{\kappa,\mu}) \mapsto \mathcal{V}^d_n(W^B_{\kappa,\mu})$ is the projection operator, which can be written as an integral

$$\operatorname{proj}_n^{\kappa,\mu} f(x) = a_{\kappa,\mu} \int_{B^d} f(y) P_n(W^B_{\kappa,\mu}; x, y) W^B_{\kappa,\mu}(y)dy, \tag{3.4}$$

where $P_n(W^B_{\kappa,\mu}; x, y)$ is the reproducing kernel of $\mathcal{V}^d_n(W^B_{\kappa,\mu})$. The intertwining operator associated with $h_{\kappa,\mu}$, denoted by $V_{\kappa,\mu}$, is given in terms of the intertwining operator V_κ associated to h_κ and the operator $V^{\mathbb{Z}_2}_\mu$ associated to $h_\mu(x) = |x_{d+1}|^\mu$, $x \in \mathbb{R}^{d+1}$, which is given explicitly by (2.5) (setting $d = 1$ and $\kappa_1 = \mu$ there); that is,

$$V_{\kappa,\mu} f(x, x_{d+1}) = c_\mu \int_{-1}^{1} V_\kappa[f(\cdot, x_{d+1}t)](x)(1+t)(1-t^2)^{\mu-1}dt,$$

where $x \in \mathbb{R}^d$. Since polynomials in $\mathcal{V}_n(W^B_{\kappa,\mu})$ correspond to h-harmonics that are even in the last coordinates, we introduce a modified operator

$$\begin{aligned} V^B_{\kappa,\mu} f(x, x_{d+1}) &:= [V_{\kappa,\mu} f(x, x_{d+1}) + V_{\kappa,\mu} f(x, -x_{d+1})] \\ &= c_\mu \int_{-1}^{1} V_\kappa[f(\cdot, x_{d+1}t)](x)(1-t^2)^{\mu-1} dt, \end{aligned} \tag{3.5}$$

acting on functions defined on $\mathbb{R}^{d+1}$.

3.2 Generalized translation operator. For the weight function $W^B_{\kappa,\mu}$ on B^d, we define a convolution, denoted by $\star^B_{\kappa,\mu}$, as follows: For $f \in L^1(W^B_{\kappa,\mu})$ and $g \in L^1(w_{\lambda_\kappa+\mu}, [-1,1])$,

$$(f \star^B_{\kappa,\mu} g)(x) = a_{\kappa,\mu} \int_{B^d} f(y) V^B_{\kappa,\mu} g(\langle X, \cdot \rangle)(Y) W^B_{\kappa,\mu}(y) dy,$$

where $X = (x, \sqrt{1-\|x\|^2})$ and $Y = (y, \sqrt{1-\|y\|^2})$. The properties of this convolution can be derived from the corresponding convolution on the sphere. Let $f \star_{\kappa,\mu} g$ denote the convolution defined in (2.6) with respect to $h_{\kappa,\mu}$. In fact, (3.2) immediately implies the following proposition.

Proposition 3.1. *For $f \in L^1(W^B_{\kappa,\mu})$ and $g \in L^1(w_{\lambda_\kappa+\mu}, [-1,1])$,*

$$(f \star^B_{\kappa,\mu} g)(x) = (F \star_{\kappa,\mu} g)(x, \sqrt{1-\|x\|^2}), \qquad \text{where} \qquad F(x, x_{d+1}) := f(x).$$

We now define the generalized translation operator on B^d implicitly via the convolution operator.

Definition 3.2. *For $f \in L^1(W^B_{\kappa,\mu})$, the generalized translation operator $T_\theta(W^B_{\kappa,\mu}; f)$ is defined implicitly by*

$$b_{\lambda+\mu} \int_0^\pi T_\theta(W^B_{\kappa,\mu}; f, x) g(\cos\theta)(\sin\theta)^{2\lambda+2\mu} d\theta = (f \star^B_{\kappa,\mu} g)(x), \tag{3.6}$$

where $\lambda = \lambda_\kappa$, for every $g \in L^1(w_{\lambda+\mu}, [-1,1])$.

The generalized translation operator $T_\theta(W^B_{\kappa,\mu})$ is related to the weighted spherical means associated with the weight function $h_{\kappa,\mu}$ on S^d. For $F \in L^1(h^2_{\kappa,\mu})$, denote the weighted spherical means by $T^{\kappa,\mu}_\theta F$ as defined in (2.7). From the definitions of $T_\theta(W^B_{\kappa,\mu})$ and $T^{\kappa,\mu}_\theta$, Proposition 3.1 shows that the following relation holds:

Proposition 3.3. *For each $x \in B^d$, $T_\theta(W^B_{\kappa,\mu}; f, x)$ is a uniquely determined L^∞ function in θ. Furthermore, define $F(x, x_{d+1}) = f(x)$; then*

$$T_\theta(W^B_{\kappa,\mu}; f, x) = T^{\kappa,\mu}_\theta F(x, x_{d+1}), \qquad x \in B^d, \quad x_{d+1} = \sqrt{1-\|x\|^2}. \tag{3.7}$$

We could of course define the generalized translation operator by the formula (3.7). The convolution $\star^B_{\kappa,\mu}$, however, will be used in the following section. These relations allow us to derive the following properties of the generalized translation operator.

Proposition 3.4. *The generalized translation operator $T_\theta(W^B_{\kappa,\mu}; f)$ satisfies the following properties:*

(1) Let $f_0(x) = 1$, then $T_\theta(W^B_{\kappa,\mu}; f_0, x) = 1$;
(2) For $f \in L^1(W^B_{\kappa,\mu})$,

$$\operatorname{proj}^{\kappa,\mu}_n T_\theta(W^B_{\kappa,\mu}; f) = \frac{C^{\lambda_\kappa+\mu}_n(\cos\theta)}{C^{\lambda_\kappa+\mu}_n(1)} \operatorname{proj}^{\kappa,\mu}_n f;$$

(3) $T_\theta(W^B_{\kappa,\mu}) : \Pi^d_n \mapsto \Pi^d_n$ and

$$T_\theta(W^B_{\kappa,\mu}; f) \sim \sum_{n=0}^{\infty} \frac{C^{\lambda_\kappa+\mu}_n(\cos\theta)}{C^{\lambda_\kappa+\mu}_n(1)} \operatorname{proj}^{\kappa,\mu}_n f;$$

(4) For $0 \le \theta \le \pi$,

$$T_\theta(W^B_{\kappa,\mu}; f) - f = \int_0^\theta (\sin s)^{-2\lambda_\kappa-2\mu} ds \int_0^s T_t(W^B_{\kappa,\mu}; D^B_{\kappa,\mu} f)(\sin t)^{2\lambda_\kappa+2\mu} dt;$$

(5) For $f \in L^p(W^B_{\kappa,\mu})$, $1 \le p < \infty$, or $f \in C(B^d)$,

$$\|T_\theta(W^B_{\kappa,\mu}; f)\|_{W^B_{\kappa,\mu},p} \le \|f\|_{W^B_{\kappa,\mu},p} \quad \text{and} \quad \lim_{\theta\to 0} \|T_\theta(W^B_{\kappa,\mu}; f) - f\|_{W^B_{\kappa,\mu},p} = 0.$$

Proof. All these properties follow from the integral relation (3.2), the relation

$$P_n(W^B_{\kappa,\mu}; x, y) = \frac{1}{2}\Big[Y_n(h^2_{\kappa,\mu}; (x, \sqrt{1-\|x\|^2}), (y, \sqrt{1-\|y\|^2})) \\ + Y_n(h^2_{\kappa,\mu}; (x, \sqrt{1-\|x\|^2}), (y, -\sqrt{1-\|y\|^2}))\Big], \tag{3.8}$$

the connection between $T_\theta(W^B_{\kappa,\mu}; f)$ and $T^{\kappa,\mu}_\theta F$ in Proposition 3.4, and the corresponding relations for $T^{\kappa,\mu}_\theta F$ in [37]. Recall the projection operator $Y_n(h^2_{\kappa,\mu}; F)$ of h-harmonics defined in (2.3). The relations (3.2) and (3.8) show that

$$\operatorname{proj}^{\kappa,\mu}_n f(x) = Y_n(h^2_{\kappa,\mu}; F, X), \qquad X = (x, \sqrt{1-\|x\|^2}).$$

Hence, it follows from Proposition 3.4 and property (2) of Proposition 2.4 in [37] that

$$\operatorname{proj}^{\kappa,\mu}_n T_\theta(W^B_{\kappa,\mu}; f, x) = \operatorname{proj}^{\kappa,\mu}_n T^{\kappa,\mu}_\theta F(X) = Y_n(h^2_{\kappa,\mu}; T^{\kappa,\mu}_\theta F, X) \\ = \frac{C^{\lambda+\mu}_n(\cos\theta)}{C^{\lambda+\mu}_n(1)} Y_n(h^2_{\kappa,\mu}; F, X) = \frac{C^{\lambda+\mu}_n(\cos\theta)}{C^{\lambda+\mu}_n(1)} \operatorname{proj}^{\kappa,\mu}_n f(x).$$

This proves (2). The property (3) follows from (2). Since $F(x, x_{d+1}) = f(x)$ is evidently even in x_{d+1}, the definition shows that $\Delta_0^{\kappa,\mu} F(x, x_{d+1}) = D^B_{\kappa,\mu} f(x)$. Consequently, by the definition of $T_\theta^{\kappa,\mu}$ and (3.7), we conclude that

$$T_\theta^{\kappa,\mu} \Delta_0^{\kappa,\mu} F(x, x_{d+1}) = T_\theta(W^B_{\kappa,\mu}; D^B_{\kappa,\mu} f, x),$$

from which the property (4) follows from the corresponding property of $T_\theta^{\kappa,\mu}$ (Proposition 2.4 in [37]). Finally, to prove the property (5), we note that it follows from the definition of $V_{\kappa,\mu}$ at (3.5) that

$$V_{\kappa,\mu}\left[g(\langle (x, -x_{d+1}), \cdot \rangle)\right](y, y_{d+1}) = V_{\kappa,\mu}\left[g(\langle (x, x_{d+1}), \cdot \rangle)\right](y, -y_{d+1})$$

Hence, the definition of $T_\theta^{\kappa,\mu}$ shows that $T_\theta^{\kappa,\mu} F(x, x_{d+1}) = T_\theta^{\kappa,\mu} F(x, -x_{d+1})$. Consequently, by (3.2),

$$\begin{aligned} \int_{B^d} \left| T_\theta(W^B_{\kappa,\mu}; f, x) \right|^p W^p_{\kappa,\mu}(x) dx &= \int_{B^d} \left| T_\theta^{\kappa,\mu} F(x, \sqrt{1 - \|x\|^2}) \right|^p W^p_{\kappa,\mu}(x) dx \\ &= \int_{S^d} \left| T_\theta^{\kappa,\mu} F(y) \right|^p h^2_{\kappa,\mu}(y) dy. \end{aligned}$$

Let $\|\cdot\|_{\kappa,\mu,p}$ denote the $L^p(h^2_{\kappa,\mu})$ norm on S^d. For $f(x) = F(X)$, we have $\|f\|_{W^B_{\kappa,\mu},p} = \|F\|_{\kappa,\mu,p}$. Hence, the property (5) follows from the corresponding property of $T_\theta^{\kappa,\mu}$ (Proposition 2.4 in [37]). □

Recall the definition of the generalized translation operator for the Gegenbauer expansions at (1.1). The reason that $T_\theta(W^B_{\kappa,\mu})$ is called the generalized translation operator lies in the property (2), since for $d = 1$ and $\kappa = 0$ the property (2) agrees with (1.3).

Once the generalized translation operator is defined, we see that (3.6) expresses the convolution of $f \star^B_{\kappa,\mu} g$ as an integral of one variable. For $g \in L^1(w_\lambda, [-1, 1])$, its Gegenbauer expansion can be written as

$$g(t) \sim \sum_{n=0}^{\infty} \widehat{g}_n^\lambda \frac{n+\lambda}{\lambda} C_n^\lambda(t), \qquad \text{where} \qquad \widehat{g}_n^\lambda = b_\lambda \int_{-1}^{1} g(s) \frac{C_n^\lambda(s)}{C_n^\lambda(1)} w_\lambda(s) ds,$$

since the $L^2(w_\lambda, [-1, 1])$ norm of C_n^λ is equal to $C_n^\lambda(1)\lambda/(n + \lambda)$. Hence, it follows from the property (2) of Proposition 3.4 that

$$\operatorname{proj}_n^{\kappa,\mu}(f \star^B_{\kappa,\mu} g) = \widehat{g}_n^{\lambda+\mu} \operatorname{proj}_n^{\kappa,\mu} f,$$

which is the analog of the familiar property $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ of the ordinary convolution. The convolution $\star^B_{\kappa,\mu}$ also satisfies several other properties of the ordinary convolution. For example, it satisfies Young's inequality:

Proposition 3.5. *For $f \in L^q(W^B_{\kappa,\mu})$ and $g \in L^r(w_{\lambda+\mu};[-1,1])$,*

$$\|f \star^B_{\kappa,\mu} g\|_{W^B_{\kappa,\mu},p} \le \|f\|_{W^B_{\kappa,\mu},q} \|g\|_{w_{\lambda+\mu},r},$$

where $p,q,r \ge 1$, $p^{-1} = r^{-1}+q^{-1}-1$ and $\|\cdot\|_{w_{\lambda+\mu},r}$ denotes the $L^r(w_{\lambda+\mu},[-1,1])$ norm.

This follows from Proposition 3.1 and Young's inequality for $\star_{\kappa,\mu}$ in [37].

3.3 Generalized translation operator for the classical weight W^B_μ. Recall the integral formula (1.1) of the generalized translation operator for the weight function w_λ. In the case of the classical weight function W^B_μ in (1.8), it is possible to give an integral formula for the generalized translation operator in the same spirit.

To see how such a formula may look like, we turn to a further relation between functions on B^d and those on S^{d+m}, where m is a positive integer. For $f(x)$ on B^d, define $F(x,x') = f(x)$ on $\mathbb{R}^{d+m}$. Then

$$\int_{S^{d+m}} F(y)d\omega(y) = \int_{B^d} f(x)(1-\|x\|^2)^{(m-1)/2}dx.$$

As shown in [34], this relation preserves orthogonal structure. This suggests a relation between $T_\theta(W^B_\mu;f)$ on B^d and the ordinary spherical means $T_\theta f$ on S^{d+m}, similar to the one in Proposition 3.3 (which is the case $m=0$). On the other hand, the spherical means is given by the formula (1.10). Hence, it is possible to derive a formula for $T_\theta(W^B_\mu;f)$ from that of spherical means. It is this heuristic argument that suggests the following formula.

Let I denote the $d\times d$ identity matrix and define the symmetric matrix

$$A(x) = (1-\|x\|^2)I + x^T x, \qquad x = (x_1,\dots,x_d),$$

where x^T is the transpose of x (x^T is a column vector). In the following, we take $u \in \mathbb{R}^d$ as a row vector. For $u \in \mathbb{R}^d$, the inequality $1 - uA(x)u^T \ge 0$ defines an ellipsoid in $\mathbb{R}^d$ (see below).

Theorem 3.6. *For W^B_μ in (1.8) on B^d, the generalized translation operator is given by*

$$T_\theta(W^B_\mu;f,x) = A_\mu\left(\sqrt{1-\|x\|^2}\right)^{d-1} \times \int_{uA(x)u^T\le 1} f\left(\cos\theta x + \sin\theta\sqrt{1-\|x\|^2}\,u\right)(1-uA(x)u^T)^{\mu-1}du, \tag{3.9}$$

where $A_\mu = 1/\int_{B^d}(1-\|x\|^2)^{\mu-1}dx$ is the normalization constant for $W^B_{\mu-1/2}$.

Proof. Although the explicit formula of V_μ^B is known for W_μ^B, it does not seem to be easy to verify the defining formula (3.6) directly. Instead, we will verify the property (2) of Proposition 3.4. In other words, let $T_\theta^* f$ denote the right hand side of (3.9); we show that $T_\theta^* f = f$ for all $f \in \mathcal{V}_n^d(W_\mu^B)$. It is known that one basis of $\mathcal{V}_n^d(W_\mu^B)$ consists of functions of the form $C_n^{\mu+(d-1)/2}(\langle x, y\rangle)$, $y \in S^d$ ([33]). Hence, it is sufficient to show that

$$T_\theta^* C_n^{\mu+(d-1)/2}(\langle x, y\rangle) = \frac{C_n^{\mu+(d-1)/2}(\cos\theta)}{C_n^{\mu+(d-1)/2}(1)} C_n^{\mu+(d-1)/2}(\langle x, y\rangle), \qquad y \in S^d.$$

The matrix $A(x)$ has eigenvalues 1 and $\sqrt{1-\|x\|^2}$ (repeated $d-1$ times) and it is symmetric. Hence, there is a unitary matrix $U(x)$ such that

$$A(x) = U(x)\Lambda(x)U(x)^T, \qquad \Lambda(x) = \operatorname{diag}\left\{1, \sqrt{1-\|x\|^2}, \ldots, \sqrt{1-\|x\|^2}\right\}.$$

The columns of $U(x)$ are the eigenvectors of $A(x)$. In particular, the first column of $U(x)$ is $x/\|x\|$ and the other columns of $U(x)$ form an orthonormal basis of the null space of $x^T x$; that is, the other columns are mutually orthonormal and are also orthogonal to x. Changing variables $u \mapsto uU(x) := v$, the quadratic form becomes

$$uA(x)u^T = v\Lambda(x)v^T = v_1^2 + \sqrt{1-\|x\|^2}(v_2^2 + \ldots + v_d^2),$$

which suggests one more change of variables $v \mapsto \sqrt{1-\|x\|^2}\, vD^{-1}(x) := s$ with

$$D(x) = \operatorname{diag}\left\{\sqrt{1-\|x\|^2}, 1, \ldots 1\right\},$$

so that the quadratic form becomes $uA(x)u^T = ss^T$. Hence, the integral domain $uA(x)u^T \le 1$ becomes B^d in s variables. Since $U(x)$ is unitary, we have $du = dv = ds/(1-\|x\|^2)^{(d-1)/2}$. Consequently, we have

$$\begin{aligned} &T_\theta^* C_n^{\mu+(d-1)/2}(\langle x, y\rangle) \\ &\quad = a_\kappa \int_{B^d} C_n^{\mu+(d-1)/2}\left(\cos\theta\langle x, y\rangle + \sin\theta\langle s, yU(x)D(x)\rangle\right)(1-\|s\|^2)^{\mu-1} ds, \end{aligned}$$

where we have used the fact that $\langle sD(x)U^T(x), y\rangle = \langle s, yU(x)D(x)\rangle$. Since the first column of $U(x)$ is $x/\|x\|$ and U is unitary, the vector $yU(x)D(x)$ has norm

$$\begin{aligned} \|yU(x)D(x)\|^2 &= yU(x)D^2(x)U^T(x)y \\ &= yy^T - yU(I - D(x)^2)U^T y^T = 1 - \langle x, y\rangle^2, \end{aligned}$$

as $\|y\| = 1$ and $I - D^2 = \operatorname{diag}\{\|x\|^2, 0, \ldots, 0\}$. Hence, using the formula

$$A_\mu \int_{B^d} f(\langle x, y\rangle)(1-\|x\|^2)^{\mu-1} dx = b_{\mu+(d-3)/2} \int_{-1}^1 f(t\|y\|)(1-t^2)^{\mu+(d-3)/2} dt,$$

which can be easily verified as the left hand side is invariant under the rotation, we conclude that

$$T_\theta^* C_n^{\mu+(d-1)/2}(\langle x,y\rangle) = b_{\mu+(d-3)/2}$$
$$\times \int_{-1}^{1} C_n^{\mu+(d-1)/2}\left(\cos\theta\langle x,y\rangle + \sin\theta\sqrt{1-\langle x,y\rangle^2}\, t\right)(1-t^2)^{\mu+(d-3)/2}dt.$$

Using the product formula for the Gegenbauer polynomials finishes the proof. □

For $\mu = 0$, the integral formula of $T_\theta(W_\mu; f)$ holds under the limit $\mu \to 0$ and the integral domain becomes $uA(x)u^T = 1$. This case has been studied in [16]. See also [1] in which a generalized translation operator is defined for the weight function $x_{d+1}^\mu d\omega$ on $S_+^d = \{x \in S^d : x_{d+1} \ge 0\}$, which is related to $T_\theta(W_\mu; f)$, but no integral formula as above is given there.

If $d = 1$, then $A(x) = 1$ and $1 - u^T A(x)u = 1 - u^2$. Hence, the formula for $T_\theta(W_\mu^B; f, x)$ when $d = 1$ becomes

$$T_\theta(W_\mu^B; f, x) = A_\mu \int_{|u|\le 1} f\left(\cos\theta x + \sin\theta\sqrt{1-x^2}\, u\right)(1-u^2)^{\mu-1}du,$$

which agrees with the formula of $T_{\cos\theta} f(x)$ in (1.1).

Changing variables $\sqrt{1-\|x\|^2}u = sD(x)U^t(x)$, the proof of the theorem, gives an alternative expression for $T_\theta(W_\mu^B; f)$.

Corollary 3.7. *Let $U(x)$ be a unitary matrix whose first column is $x/\|x\|$ and $D(x) = \mathrm{diag}\{\sqrt{1-\|x\|^2}, 1, \ldots, 1\}$. For W_μ^B in (1.8),*

$$T_\theta(W_\mu^B; f, x) = A_\mu \int_{B^d} f\left(\cos\theta x + \sin\theta\, sD(x)U^T(x)\right)(1-\|s\|^2)^{\mu-1}ds,$$

In the above formula we take x and s as row vectors in $\mathbb{R}^d$. Recall that the first column of $U(x)$ is $x/\|x\|$ and the other columns of $U(x)$ are orthonormal vectors that are orthogonal to x (that is, $\langle x,\xi\rangle = 0$). Hence, we can write the formula for $T_\theta(W_\mu^B; f)$ as an explicit integral over B^d. For example, in the case of $d = 2$,

$$U(x) = \frac{1}{\|x\|}\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \quad \text{and} \quad D(x) = \begin{pmatrix} \sqrt{1-\|x\|^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case it is more convenient to use the polar coordinates $x = r(\cos\phi, \sin\phi)$, where $r \ge 0$ and $0 \le \phi \le 2\pi$.

Corollary 3.8. *For $d = 2$, $x = rx'$ with $x' = (x'_1, x'_2) = (\cos\phi, \sin\phi)$,*

$$T_\theta(W_\mu^B; f, x) = \\ A_\mu \int_{B^2} f\left(r\cos\theta x' + \sqrt{1-r^2}\,\sin\theta s_1 x' + \sin\theta s_2(-x'_2, x'_1)\right)(1-\|s\|^2)^{\mu-1} ds.$$

For $d > 2$ the formula of $U(x)$ can be messy. For example, in the case d is odd, we do not have a simple formula. On the other hand, there are simple expressions of $U(x)$ for $d = 4, 8, \ldots$. As examples, we give the formula of $U(x)$ for $d = 3$ and $d = 4$ below.

$$\begin{pmatrix} \frac{x_1}{\|x\|} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} & \frac{x_3}{\|x\|} \\ \frac{x_2}{\|x\|} & \frac{-x_1}{\sqrt{x_1^2+x_2^2}} & -\frac{x_2x_3}{x_1\|x\|} \\ \frac{x_3}{\|x\|} & 0 & \frac{x_2^2+x_3^2}{x_1\|x\|} \end{pmatrix} \quad \text{and} \quad \frac{1}{\|x\|}\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{pmatrix}.$$

Using these one can write down the formula of $T_\theta(W_\mu; f)$ as an explicit integral over B^d.

An interesting problem is to find an integral expression for $T_\theta(W_{\kappa,\mu}^B; f)$ with respect to other weight functions. One should perhaps start with the case that h_κ is given by the product weight function (1.5).

3.4 Modulus of smoothness, K-functional and best approximation. Property (5) of Proposition 3.4 shows that the following definition of the modulus of smoothness makes sense:

Definition 3.9. *Let $r > 0$. Define*

$$(I - T_\theta^\kappa)^{r/2} f \sim \sum_{n=0}^{\infty} \left(1 - C_n^{\lambda_\kappa+\mu}(\cos\theta)/C_n^{\lambda_\kappa+\mu}(1)\right)^{r/2} \operatorname{proj}_n^{\kappa,\mu} f.$$

For $f \in L^p(W_{\kappa,\mu}^B)$, $1 \le p < \infty$, or $f \in C(B^d)$, define

$$\omega(f;t)_{W_{\kappa,\mu}^B,p} := \sup_{\theta\le t} \left\| \left(I - T_\theta(W_{\kappa,\mu}^B)\right)^{r/2} f \right\|_{W_{\kappa,\mu}^B,p}.$$

Because of Proposition 3.4, this modulus of smoothness is related to the modulus $\omega_r(f;t)_{\kappa,\mu,p}$, defined in (2.8) but associated with $h_{\kappa,\mu}$. In fact, we have

$$\omega(f;t)_{W_{\kappa,\mu}^B,p} = \omega_r(F;t)_{\kappa,\mu,p}, \qquad F(x, x_{d+1}) = f(x). \tag{3.10}$$

Consequently, properties of $\omega_r(f;t)_{W_{\kappa,\mu}^B,p}$ can be easily obtained from those of $\omega_r(f;t)_{\kappa,\mu,p}$ (see Proposition 3.6 of [37]).

Proposition 3.10. *The modulus of smoothness $\omega_r(f;t)_{W^B_{\kappa,\mu},p}$ satisfies:*

(1) $\omega_r(f;t)_{W^B_{\kappa,\mu},p} \to 0$ *if* $t \to 0$;
(2) $\omega_r(f;t)_{W^B_{\kappa,\mu},p}$ *is monotone nondecreasing on* $(0,\pi)$;
(3) $\omega_r(f+g,t)_{W^B_{\kappa,\mu},p} \le \omega_r(f;t)_{W^B_{\kappa,\mu},p} + \omega_r(g,t)_{W^B_{\kappa,\mu},p}$;
(4) For $0 < s < r$,

$$\omega_r(f;t)_{W^B_{\kappa,\mu},p} \le 2^{[(r-s+1)/2]}\omega_s(f;t)_{W^B_{\kappa,\mu},p};$$

(5) If $(-D^B_{\kappa,\mu})^k f \in L^p(W^B_{\kappa,\mu})$, $k \in \mathbb{N}$, *then for* $r > 2k$

$$\omega_r(f;t)_{W^B_{\kappa,\mu},p} \le c\, t^{2k}\omega_{r-2k}((-D^B_{\kappa,\mu})^k f;t)_{W^B_{\kappa,mu},p}.$$

To justify the definition of this modulus of smoothness, we show that it is equivalent to a K-functional, defined using the differential-difference operator associated with $W^B_{\kappa,\mu}$ (see (3.3)). Let

$$\mathcal{W}^p_r(W^B_{\kappa,\mu}) := \left\{ f \in L^p(W^B_{\kappa,\mu}) : (-D^B_{\kappa,\mu})^{r/2} f \in L^p(W^B_{\kappa,\mu}) \right\},$$

where the fractional power of $D^B_{\kappa,\mu}$ on f is defined by

$$(-D^B_{\kappa,\mu})^{r/2} f \sim \sum_{n=0}^{\infty} (n(n+2\lambda_\kappa+2\mu))^{r/2} \operatorname{proj}^{\kappa,\mu}_n f, \quad f \in L^p(W^B_{\kappa,\mu}).$$

The K-functional between $L^p(W^B_{\kappa,\mu})$ and $\mathcal{W}^p_r(W^B_{\kappa,\mu})$ is defined by

$$K_r(f;t)_{W^B_{\kappa,\mu},p} := \inf \left\{ \|f-g\|_{W^B_{\kappa,\mu},p} + t^r \|(-D^B_{\kappa,\mu})^{r/2} g\|_{W^B_{\kappa,\mu},p} \right\},$$

where the infimum is taken over all $g \in \mathcal{W}^p_r(W^B_{\kappa,\mu})$.

Theorem 3.11. *For* $f \in L^p(W^B_{\kappa,\mu})$, $1 \le p \le \infty$,

$$c_1 \omega_r(f;t)_{W^B_{\kappa,\mu},p} \le K_r(f;t)_{W^B_{\kappa,\mu},p} \le c_2 \omega_r(f;t)_{W^B_{\kappa,\mu},p}.$$

Proof. Again let $F(x,x_{d+1}) = f(x)$. Denote by $K_r(F;t)_{\kappa,\mu,p}$ the K-functional defined in (2.9) but with respect to the weight function $h^2_{\kappa,\mu}$. Because of (3.10) and the equivalence between $K_r(F;t)_{\kappa,\mu,p}$ and $\omega_r(F;t)_{\kappa,\mu,r}$ (see Theorem 2.1), we only need to show that $K_r(f;t)_{W^B_{\kappa,\mu},p} = K_r(F;t)_{\kappa,\mu,p}$.

It follows from (3.2) that $\|\Delta^{\kappa,\mu}_{h,0} F\|_{\kappa,\mu,p} = \|D^B_{\kappa,\mu} f\|_{W^B_{\kappa,\mu},p}$. Hence,

$$K_r(f;t)_{W^B_{\kappa,\mu},p} = \inf \left\{ \|F - g_e\|_{\kappa,\mu,p} + t^r \|\Delta^{\kappa,\mu}_{h,0} g_e\|_{\kappa,\mu,p} \right\} := K^*_r(F;t)_{\kappa,\mu,p},$$

where the infimum is taken over all $g_e(x,x_{d+1}) \in \mathcal{W}^p_r(h^2_{\kappa,\mu})$ that are even in x_{d+1}. Evidently, $K^*_r(F;t)_{\kappa,\mu,p} \ge K_r(F;t)_{\kappa,\mu,p}$. To complete the proof we show that $K_r^* = K_r$. For any $\varepsilon > 0$, fix a $g \in \mathcal{W}_r^p(W_{\kappa,\mu}^p)$ such that

$$K_r(F;t)_{\kappa,\mu,p} \ge \|F - g\|_{\kappa,\mu,p} + t^r\| - \Delta_{h,0}^{\kappa,\mu} g\|_{\kappa,\mu,p} - \varepsilon.$$

Since $h_{\kappa,\mu}$ corresponds to $G \times \mathbb{Z}_2$, the spherical Laplacian $\Delta_{h,0}^{\kappa,\mu}$ commutes with the sign change in x_{d+1}. Consequently, setting $g_e(x, x_{d+1}) = [g(x, x_{d+1}) + g(x, -x_{d+1})]/2$, so that g_e is even in x_{d+1}, it follows that $\|\Delta_{h,0}^{\kappa,\mu} g_e\|_{\kappa,\mu,p} \le \|\Delta_{h,0}^{\kappa,\mu} g\|_{\kappa,\mu,p}$. This and the fact that $\|F - g_e\|_{\kappa,\mu,p} \le \|F - g\|_{\kappa,\mu,p}$, as F is even in x_{d+1}, show that $K_r^*(F;t)_{\kappa,\mu,p} \le K_r(F;t)_{\kappa,\mu,p} + \varepsilon$. As $\varepsilon > 0$ is arbitrary, the proof follows. □

One immediate consequence of the above equivalence is the following property of the modulus of smoothness, which does not follow trivially from the definition of $\omega_r(f;t)_{W_{\kappa,\mu}^B,p}$ but it is clear for the K-functional.

Corollary 3.12. *For $f \in L^p(W_{\kappa,\mu}^B)$, $1 \le p \le \infty$,*

$$\omega_r(f, \delta t)_{W_{\kappa,\mu}^B,p} \le c \max\{1, \delta^r\} \omega_r(f,t)_{W_{\kappa,\mu}^B,p}.$$

The direct and the inverse theorems for the best approximation by polynomials in $L^p(W_{\kappa,\mu}^B)$ is characterized in [37] by the K-functional. The equivalence in Theorem 3.11 allows us to state the characterization in terms of the modulus of smoothness.

Theorem 3.13. *For $f \in L^p(W_{\kappa,\mu}^B)$, $1 \le p \le \infty$,*

$$E_n(f)_{W_{\kappa,\mu}^B,p} \le c\, \omega_r(f; n^{-1})_{W_{\kappa,\mu}^B,p}.$$

On the other hand,

$$\omega_r(f; n^{-1})_{W_{\kappa,\mu}^B,p} \le c\, n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{W_{\kappa,\mu}^B,p}.$$

Finally, let us mention that, by (3.4), (3.8) and (2.4),

$$\operatorname{proj}_{\kappa,\mu} f = f \star_{\kappa,\mu}^B p_n, \qquad \text{where} \qquad p_n(t) = \frac{n + \lambda + \mu}{\lambda + \mu} C_n^{\lambda+\mu}(t).$$

Hence, all summation methods of orthogonal expansions with respect to $W_{\kappa,\mu}^B$ can be written in the form of $f \star_{\kappa,\mu}^B g_r$, where g_r is the same summation method applies to the Gegenbauer series evaluated at point $t = 1$. Since $T_\theta(W_{\kappa,\mu}^B; f)$, thus $\omega_r(f;t)_{W_{\kappa,\mu}^B,p}$, is defined in terms of $f \star_{\kappa,\mu}^B g$, the modulus of smoothness is a convenient tool for studying the summability of orthogonal expansions on B^d. For various results on the summability of orthogonal expansions with respect to $W_{\kappa,\mu}^B$, see [14,20,34,37,38] and the references therein.

4 Generalized translation operator and Approximation on T^d

Recall the weight function $W^T_{\kappa,\mu}$ defined in (1.7), in which h_κ is a weight function invariant under a reflection group G_0 and even in each of its variables. That is, h_κ is invariant under the semi-product of a reflection group G_0 and the abilian group $\mathbb{Z}_2^d$.

The definition of $L^p(W^T_{\kappa,\mu})$, $1 \le p \le \infty$, is similar to the case of $W^B_{\kappa,\mu}$. The notions such as the space of orthogonal polynomials $\mathcal{V}^d_n(W^T_{\kappa,\mu})$ and the reproducing kernel $P_n(W^T_{\kappa,\mu}; x, y)$ are also defined similarly as in the case of $W^B_{\kappa,\mu}$.

4.1 Background. Elements of $\mathcal{V}^d_n(W^T_{\kappa,\mu})$ are closely related to the orthogonal polynomials in $\mathcal{V}^d_{2n}(W^B_{\kappa,\mu})$. Let us denote by ψ the mapping

$$\psi : (x_1, \ldots, x_d) \in B^d \mapsto (x_1^2, \ldots, x_d^2) \in T^d$$

and define $(f \circ \psi)(x_1, \ldots, x_d) = f(x_1^2, \ldots, x_d^2)$. The elementary integral

$$\int_{B^d} f(x_1^2, \ldots, x_d^2) dx = \int_{T^d} f(x_1, \ldots, x_d) \frac{dx}{\sqrt{x_1 \cdots x_d}}. \tag{4.1}$$

shows that $\|f\|_{W^T_{\kappa,\mu},p} = \|f \circ \psi\|_{W^B_{\kappa,\mu},p}$. The mapping $R \mapsto P$ given by

$$P_{2n}(x) = (R_n \circ \psi)(x) \qquad x \in B^d \tag{4.2}$$

is a one-to-one mapping between $R_n \in \mathcal{V}^d_n(W^T_{\kappa,\mu})$ and $P_{2n} \in \mathcal{V}^d_{2n}(W^B_{\kappa,\mu}; \mathbb{Z}_2^d)$, the subspace of polynomials in $\mathcal{V}^d_{2n}(W^B_{\kappa,\mu})$ that are even in each of its variables (invariant under $\mathbb{Z}_2^d$). In particular, applying $D_{W^B_{\kappa,\mu}}$ on P_{2n} leads to a second order differential-difference operator acting on R_n. We denote this operator by $D^T_{\kappa,\mu}$. Then ([35])

$$D^T_{\kappa,\mu} R = -n(n + \lambda_\kappa + \mu) R, \qquad R \in \mathcal{V}^d_n(W^T_{\kappa,\mu}), \tag{4.3}$$

For the weight function (1.9), the operator is a second order differential operator, which takes the form

$$D^T_{\kappa,\mu} = \sum_{i=1}^d x_i(1 - x_i) \frac{\partial^2 P}{\partial x_i^2} - 2 \sum_{1 \le i < j \le d} x_i x_j \frac{\partial^2 P}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(\left(\kappa_i + \frac{1}{2} \right) - \lambda x_i \right) \frac{\partial P}{\partial x_i}$$

(recall $\mu = \kappa_{d+1}$ in this case). This is classical, already known in [4] at least for $d = 2$ (see also [15, Chapt. 12]).

In the following, we also denote by $\mathrm{proj}_n^{\kappa,\mu} : L^2(W^T_{\kappa,\mu}) \mapsto \mathcal{V}^d_n(W^T_{\kappa,\mu})$ the orthogonal projection operator. For $f \in L^2(W^T_{\kappa,\mu})$, it can be written as an integral

$$\mathrm{proj}_n^{\kappa,\mu} f(x) = a_{\kappa,\mu} \int_{T^d} f(y) P_n(W^T_{\kappa,\mu}; x, y) W^T_{\kappa,\mu}(y) dy,$$

where $P_n(W^T_{\kappa,\mu}; x, y)$ is the reproducing kernel of $\mathcal{V}^d_n(W^T_{\kappa,\mu})$. The relation (4.1) implies, in particular, that ([35])

$$P_n(W^T_{\kappa,\mu}; x, y) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} P_{2n}\left(W^B_{\kappa,\mu}; x^{1/2}, \varepsilon y^{1/2}\right), \tag{4.4}$$

where $x^{1/2} = (\sqrt{x_1}, \ldots, \sqrt{x_d})$ and $\varepsilon u = (\varepsilon_1 u_1, \ldots \varepsilon_d u_d)$. We define a useful operator, $V^T_{\kappa,\mu}$, acting on functions of $d+1$ variables,

$$V^T_{\kappa,\mu} F(x, x_{d+1}) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} V^B_{\kappa,\mu} F(\varepsilon x, x_{d+1}). \tag{4.5}$$

The definition of $V^T_{\kappa,\mu}$ is justified by the following fact: Let $p_n^{(\alpha,\beta)}(t)$ denote the orthonormal Jacobi polynomial of degree n associated to the weight function $w_{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta$ on $t \in [-1, 1]$. Using the relation

$$\frac{2n+\lambda}{\lambda} C^\lambda_{2n}(t) = p_n^{(\lambda-1/2,-1/2)}(1) p_n^{(\lambda-1/2,-1/2)}(2t^2 - 1), \tag{4.6}$$

we can write the reproducing kernel $P_n(W^T_{\kappa,\mu}; x, y)$ as

$$P_n(W^T_{\kappa,\mu}; x, y) = p_n^{(\lambda_\kappa+\mu-\frac{1}{2},-\frac{1}{2})}(1) V^T_{\kappa,\mu}\left[p_n^{(\lambda_\kappa+\mu-\frac{1}{2},-\frac{1}{2})}(2\langle\cdot, Y^{1/2}\rangle^2 - 1)\right](X^{1/2}).$$

where $X^{1/2} = \left(\sqrt{x_1}, \ldots, \sqrt{x_d}, \sqrt{1-|x|}\right)$.

4.2 Generalized translation operator. The operator $V^T_{\kappa,\mu}$ is used to define a convolution operator on T^d:

Definition 4.1. *For $f \in L^1(W^T_{\kappa,\mu})$ and $g \in L^1(w_{\lambda_\kappa+\mu}; [-1, 1])$, we define*

$$(f \star^T_{\kappa,\mu} g)(x) = a_{\kappa,\mu} \int_{T^d} f(y) V^T_{\kappa,\mu}\left[g\left(2\langle X^{1/2}, \cdot\rangle^2 - 1\right)\right](Y^{1/2}) W^T_{\kappa,\mu}(y) dy.$$

Recall that $|x| = x_1 + \ldots + x_d$. Using (4.1), it is not hard to show that $f \star^T_{\kappa,\mu} g$ is related to the convolution structure $f \star^B_{\kappa,\mu} g$ on B^d ([38]):

Proposition 4.2. *For $f \in L^1(W^T_{\kappa,\mu})$ and $g \in L^1(w_{\lambda+\mu}; [-1, 1])$,*

$$\left((f \star^T_{\kappa,\mu} g) \circ \psi\right)(x) = \left((f \circ \psi) \star^B_{\kappa,\mu} g(2\{\cdot\}^2 - 1)\right)(x).$$

The generalized translation operator associated with $W^T_{\kappa,\mu}$ is again defined implicitly in terms of the convolution structure.

Definition 4.3. *For* $f \in L^1(W^T_{\kappa,\mu})$*, the generalized translation operator* $T_\theta(W^T_{\kappa,\mu}; f)$ *is defined implicitly by*

$$b_{\lambda+\mu} \int_0^\pi T_\theta(W^T_{\kappa,\mu}; f, x) g(\cos 2\theta)(\sin\theta)^{2\lambda+2\mu} d\theta = (f \star^T_{\kappa,\mu} g)(x), \tag{4.7}$$

where $\lambda = \lambda_\kappa$*, for every* $g \in L^1(w_{\lambda+\mu})$*.*

The definition of $V^T_{\kappa,\mu}$ ensures that the generalized translation operator on T^d is related to the one on B^d. This also shows that the operator $T_\theta(W^T_{\kappa,\mu}; f)$ is well-defined.

Proposition 4.4. *For each* $x \in T^d$*,* $T_\theta(W^T_{\kappa,\mu}; f, x)$ *is a uniquely determined* L^∞ *function in* θ*. Furthermore,*

$$\left(T_\theta(W^T_{\kappa,\mu}; f) \circ \psi\right)(x) = T_\theta(W^B_{\kappa,\mu}; f \circ \psi, x), \qquad x \in T^d. \tag{4.8}$$

The proof follows from the definition and the elementary formula $\cos 2\theta = 2\cos^2\theta - 1$; see [38]. This relation allows us to derive properties of $T_\theta(W^T_{\kappa,\mu}; f)$.

Proposition 4.5. *The generalized translation* $T_\theta(W^T_{\kappa,\mu}; f)$ *satisfies the following properties:*

(1) Let $f_0(x) = 1$*, then* $T_\theta(W^T_{\kappa,\mu}; f_0, x) = 1$*.*
(2) For $f \in L^1(W^T_{\kappa,\mu})$*,*

$$\operatorname{proj}^{\kappa,\mu}_n T_\theta(W^T_{\kappa,\mu}; f) = \frac{P_n^{(\lambda_\kappa+\mu-1/2,-1/2)}(\cos 2\theta)}{P_n^{(\lambda_\kappa+\mu-1/2,-1/2)}(1)} \operatorname{proj}^{\kappa,\mu}_n f.$$

(3) $T_\theta(W^T_{\kappa,\mu}) : \Pi^d_n \mapsto \Pi^d_n$ *and*

$$T_\theta(W^T_{\kappa,\mu}; f) \sim \sum_{n=0}^{\infty} \frac{P_n^{(\lambda_\kappa+\mu-1/2,-1/2)}(\cos 2\theta)}{P_n^{(\lambda_\kappa+\mu-1/2,-1/2)}(1)} \operatorname{proj}^{\kappa,\mu}_n f.$$

(4) For $0 \le \theta \le \pi$*,*

$$T_\theta(W^T_{\kappa,\mu}; f) - f = 2\int_0^\theta (\sin s)^{-2\lambda_\kappa-2\mu} ds \int_0^s T_t(W^T_{\kappa,\mu}; D^T_{\kappa,\mu} f)(\sin t)^{2\lambda_\kappa+2\mu} dt.$$

(5) For $f \in L^p(W^T_{\kappa,\mu})$*,* $1 \le p < \infty$*, or* $f \in C(T^d)$*,*

$$\|T_\theta(W^T_{\kappa,\mu}; f)\|_{W^T_{\kappa,\mu},p} \le \|f\|_{W^T_{\kappa,\mu},p} \quad \text{and} \quad \lim_{\theta\to 0} \|T_\theta(W^T_{\kappa,\mu}; f) - f\|_{W^T_{\kappa,\mu},p} = 0.$$

Proof. The property (1) is an easy consequence of (4.8) and the property (1) of Proposition 3.4. Let $f \in \mathcal{V}_n(W^T_{\kappa,\mu})$. Then $f \circ \psi \in \mathcal{V}_{2n}(W^B_{\kappa,\mu})$. Hence, by

Proposition 3.4,

$$T_\theta(W_{\kappa,\mu}^T; f, x_1^2, \ldots, x_d^2) = T_\theta(W_{\kappa,\mu}^B; f \circ \psi, x) = \frac{C_{2n}^{\lambda_\kappa+\mu}(\cos\theta)}{C_{2n}^{\lambda_\kappa+\mu}(1)}(f \circ \psi)(x).$$

This proves the properties (2) and (3) upon using the relation (4.2). Let $f = \sum c_k R_k$, $R_k \in \mathcal{V}_n(W_{\kappa,\mu}^T)$. By (4.2), $P = R \circ \psi \in \mathcal{V}_n(W_{\kappa,\mu}^B)$. Then

$$\begin{aligned} (D_{\kappa,\mu}^T f) \circ \psi &= -\sum c_k k(k + \lambda_\kappa + \mu) R_k \circ \psi \\ &= -2^{-1} \sum c_k 2k(2k + 2\lambda_\kappa + 2\mu) P_{2k} \\ &= 2^{-1} \sum c_k D_{\kappa,\mu}^B P_{2k} = 2^{-1} D_{\kappa,\mu}^B (f \circ \psi), \end{aligned} \tag{4.9}$$

from which (4) follows from the property (4) of Proposition 3.4. Finally, a change of variables $x \mapsto \psi(x)$ shows that

$$\|T_\theta(W_{\kappa,\mu}^T; f)\|_{W_{\kappa,\mu}^T, p} = \|T_\theta(W_{\kappa,\mu}^T; f) \circ \psi\|_{W_{\kappa,\mu}^B, p} = \|T_\theta(W_{\kappa,\mu}^B; f \circ \psi)\|_{W_{\kappa,\mu}^B, p},$$

which is less than or equal to $\|f \circ \psi\|_{W_{\kappa,\mu}^B, p} = \|f\|_{W_{\kappa,\mu}^T, p}$ by the property (5) of Proposition 3.4. □

Using the relation to $T_\theta(W_{\kappa,\mu}^B; f)$, we can derive from Theorem 3.6 an integral formula for $T_\theta(W_\mu^T; f)$, where $W_\mu^T(x) = (x_1 \cdots x_d)^{-1/2}(1 - |x|)^{\mu - 1/2}$. One interesting question is to find such a formula for the classical weight function W_κ^T in (1.9).

In the case of $d = 1$ and $G = \mathbb{Z}_2^d$, the weight function $W_{\kappa,\mu}^T$ becomes the Jacobi weight function $w_{\kappa,\mu}(t) = 2^{\kappa+\mu} t^\kappa (1-t)^\mu$ on $[0, 1]$ (see (1.9)), whose corresponding orthogonal polynomials are $P_n^{(\kappa,\mu)}(2t - 1)$. The orthogonal expansion of f in Jacobi polynomials is defined by

$$f(t) \sim \sum_{n=0}^{\infty} a_n(f) p_n^{(\alpha,\beta)}(t), \quad \text{where} \quad a_n(f) = c_{\alpha,\beta} \int_{-1}^{1} f(s) p_n^{(\alpha,\beta)}(s) ds$$

and $c_{\alpha,\beta}^{-1} = \int_{-1}^{1} w_{\alpha,\beta}(s) ds$. The usual generalized translation operator, $S_\theta f(t)$, for the Jacobi expansion is an operator defined by ([3])

$$S_\theta f(t) \sim \sum_{n=0}^{\infty} a_n(f) p_n^{(\alpha,\beta)}(\cos\theta) p_n^{(\alpha,\beta)}(t).$$

We should emphasis, however, that the operator $S_\theta f$ is different from the case $d = 1$ of the generalized translation operator $T_\theta(W_{\kappa,\mu}^T; f)$. Even in the case of $\alpha = \lambda + \mu - 1/2$ and $\beta = -1/2$, they are different as can be seen from Proposition 4.5. The convolution structure of the Jacobi expansions defined via S_θ has a natural extension to the product Jacobi weight functions on the unit cube $[-1, 1]^d$. The convolution structure defined above works for $W_{\kappa,\mu}^T$ on the simplex.

4.3 Modulus of smoothness, K-functional and best approximation. We can also define a modulus of smoothness on T^d using the generalized translation operator; that is, for $r > 0$, define

$$\omega_r(f;t)_{W^T_{\kappa,\mu},p} := \sup_{\theta \le t} \|(T_\theta(W^T_{\kappa,\mu}) - I)^{r/2} f\|_{W^T_{\kappa,\mu},p}.$$

Evidently it is related to the modulus of smoothness on the unit ball B^d. The following relation follows immediately from Proposition 4.4 and (4.1).

Proposition 4.6. *For $f \in L^1(W^T_{\kappa,\mu})$,*

$$\omega_r(f;t)_{W^T_{\kappa,\mu},p} = \omega_r(f \circ \psi;t)_{W^B_{\kappa,\mu},p}.$$

Properties of $\omega_r(f;t)_{W^T_{\kappa,\mu},p}$ can be derived from the corresponding ones of $\omega_r(f \circ \psi;t)_{W^B_{\kappa,\mu},p}$ in Proposition 3.10. We will not write these properties down. The modulus of smoothness $\omega_r(f;t)_{W^T_{\kappa,\mu},p}$ is also equivalent to the K-functional $K_r(f;t)_{W^T_{\kappa,\mu},p}$ defined in [37]. The definition is exactly the same as the one for $W^B_{\kappa,\mu}$,

$$K_r(f;t)_{W^T_{\kappa,\mu},p} := \inf \left\{ \|f - g\|_{W^T_{\kappa,\mu},p} + t^r \|(-D^T_{\kappa,\mu})^{r/2} g\|_{W^T_{\kappa,\mu},p} \right\},$$

where the infimum is taken over all $g \in \mathcal{W}^p_r(W^T_{\kappa,\mu})$. The space $\mathcal{W}^p_r(W^T_{\kappa,\mu})$ is defined as its counterpart on B^d.

Theorem 4.7. *For $f \in L^p(W^T_{\kappa,\mu})$, $1 \le p \le \infty$,*

$$c_1 \omega_r(f;t)_{W^T_{\kappa,\mu},p} \le K_r(f;t)_{W^T_{\kappa,\mu},p} \le c_2 \omega_r(f;t)_{W^T_{\kappa,\mu},p}.$$

Proof. Because of Proposition 4.6 and Proposition 3.11, it suffices to show that

$$K_r(f;t)_{W^T_{\kappa,\mu},p} = K_r(f \circ \psi; 2t)_{W^B_{\kappa,\mu},p}.$$

By (4.9) and (4.1)

$$\begin{aligned} K_r(f;t)_{W^T_{\kappa,\mu},p} &= \inf_g \left\{ \|f \circ \psi - g \circ \psi\|_{W^B_{\kappa,\mu},p} + 2^r t^r \|(-D^B_{\kappa,\mu})^{r/2} (g \circ \psi)\|_{W^B_{\kappa,\mu},p} \right\} \\ &= \inf_{g_0} \left\{ \|f \circ \psi - g_0\|_{W^B_{\kappa,\mu},p} + 2^r t^r \|(-D^B_{\kappa,\mu})^{r/2} g_0\|_{W^B_{\kappa,\mu},p} \right\} \\ &:= K^*_r(f \circ \psi; t)_{W^B_{\kappa,\mu},p}, \end{aligned}$$

where the infimum is taken over all g_0 such that $g_0 = g \circ \psi \in \mathcal{W}^p_r(W^B_{\kappa,\mu})$. The definition clearly shows that

$$K_r(f;t)_{W^T_{\kappa,\mu},p} = K^*_r(f,t)_{W^B_{\kappa,\mu},p} \ge K_r(f \circ \psi; 2t)_{W^B_{\kappa,\mu},p}.$$

We prove that the reverse inequality holds. For any $\delta > 0$, fix a $g \in \mathcal{W}_r^p(W_{\kappa,\mu}^B)$ such that

$$K_r(f;2t)_{W_{\kappa,\mu}^T,p} \geq \|f-g\|_{W_{\kappa,\mu}^B,p} + 2^r t^r \|(-D_{\kappa,\mu}^B)^{r/2} g\|_{W_{\kappa,\mu}^B,p} - \delta.$$

Let $g_0(x) = 2^{-d}\sum_{\varepsilon\in\mathbb{Z}_2^d} R(\varepsilon)g(x)$, where $R(\varepsilon)g(x) := g(\varepsilon x)$ for $\varepsilon \in \mathbb{Z}_2^d$. Then g_0 is even in each of its variables. We claim that $R(\varepsilon)D_{\kappa,\mu}^B = D_{\kappa,\mu}^B R(\varepsilon)$. Indeed, since h_κ is even for each of its variables, it is invariant under $\mathbb{Z}_2^d$, so that $R(\varepsilon)\Delta_h = \Delta_h R(\varepsilon)$ and, furthermore,

$$\langle x, \nabla\rangle R(\varepsilon)g(x) = \sum x_i \varepsilon_i \partial_i g(\varepsilon x) = R(\varepsilon)\langle x,\nabla\rangle g(x),$$

the claimed equality follows from the definition of $D_{\kappa,\mu}^B$. It follows that

$$\|(-D_{\kappa,\mu}^B)^{r/2} g_0\|_{W_{\kappa,\mu}^B,p} \leq 2^{-d}\sum \|(-D_{\kappa,\mu}^B)^{r/2} R(\varepsilon) g\|_{W_{\kappa,\mu}^B,p} \leq \|(-D_{\kappa,\mu}^B)^{r/2} g\|_{W_{\kappa,\mu}^B,p}.$$

Clearly, we also have

$$\|f\circ\psi - g_0\|_{W_{\kappa,\mu}^B,p} \leq 2^{-d}\sum \|f\circ\psi - R(\varepsilon)g\|_{W_{\kappa,\mu}^B,p} = \|f\circ\psi - g\|_{W_{\kappa,\mu}^B,p}.$$

Consequently, since g_0 is even in each of its variables and $g_0 \in \mathcal{W}_r^p(W_{\kappa,\mu}^B)$, it follows that

$$\begin{aligned} K_r^*(f;t)_{W_{\kappa,\mu}^T,p} &\leq \|f-g_0\|_{W_{\kappa,\mu}^B,p} + 2^r t^r \|(-D_{\kappa,\mu}^B)^{r/2} g_0\|_{W_{\kappa,\mu}^B,p} \\ &\leq \|f-g\|_{W_{\kappa,\mu}^B,p} + 2^r t^r \|(-D_{\kappa,\mu}^B)^{r/2} g\|_{W_{\kappa,\mu}^B,p} \\ &\leq K_r(f;2t)_{W_{\kappa,\mu}^T,p} + \delta. \end{aligned}$$

Since δ is arbitrary, this completes the proof. □

Again, the above equivalence allows us to state the following important property of the modulus of smoothness.

Corollary 4.8. *For $f \in L^p(W_{\kappa,\mu}^T)$, $1 \leq p \leq \infty$,*

$$\omega_r(f,\delta t)_{W_{\kappa,\mu}^T,p} \leq c\,\max\{1,\delta^r\}\omega_r(f,t)_{W_{\kappa,\mu}^T,p}.$$

Furthermore, we can state the direct and the inverse theorems for the best approximation by polynomials in $L^p(W_{\kappa,\mu}^T)$, given in terms of the K-functional in [37], in terms of the modulus of smoothness.

Theorem 4.9. *For $f \in L^p(W_{\kappa,\mu}^T)$, $1 \leq p \leq \infty$,*

$$E_n(f)_{W_{\kappa,\mu}^T,p} \leq c\,\omega_r(f;n^{-1})_{W_{\kappa,\mu}^T,p}.$$

On the other hand,

$$\omega_r(f;n^{-1})_{W^T_{\kappa,\mu},p} \le c\,n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{W^T_{\kappa,\mu},p}.$$

Let us point out that in the case of $d = 1$, $\omega_r(f;t)_{W^T_{\kappa,\mu},p}$ is a modulus of smoothness for the Jacobi weight function $w_{\kappa,\mu}$ on $[-1,1]$. However, it is different from the modulus of smoothness defined in the literature using the generalized translation operator S_θ (see, for example, [3,5] for $r = 1$). Using the convolution structure, we can write

$$\mathrm{proj}_{\kappa,\mu} f = f \star^T_{\kappa,\mu} q_n, \qquad q_n(t) = p_n^{(\lambda+\mu-1/2,-1/2)}(1) p_n^{(\lambda+\mu-1/2,-1/2)}(t)$$

Hence, all summation methods of orthogonal expansions with respect to $W^B_{\kappa,\mu}$ can be written in the form of $f \star^B_{\kappa,\mu} g_r$, where g_r is the same summation method applies to the Jacobi series with $(\alpha,\beta) = (\lambda+\mu-1/2, -1/2)$. Consequently, the modulus of smoothness can be used to study the summability of orthogonal expansions on the simplex.

References

[1] I. A. Aliev and B. Rubin, Spherical harmonics associated to the Laplace-Bessel operator and generalized spherical convolutions, *Anal. Appl. (Singap.)* **1** (2003), 81–109.

[2] R. Askey, *Orthogonal polynomials and special functions*, Regional Conference Series in Applied Mathematics **21**, SIAM, Philadelphia, 1975.

[3] R. Askey and S. Wainger, A convolution structure for Jacobi series, *Amer. J. Math.* **91** (1969), p. 463-485.

[4] P. Appell and J. K. de Fériet. *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[5] H. Bavinck *Jacobi series and approximation*, Mathematical Centre Tracts, No. 39. Mathematisch Centrum, Amsterdam, 1972.

[6] H. Berens, P. L. Butzer and S. Pawelke, Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten, *Publ. Res. Inst. Math. Sci. Ser. A.* **4** (1968), 201-268.

[7] H. Berens and Luoqing Li, On the de la Vallée Poussin means on the sphere, *Results in Math.*, **24** (1993), 12-26.

[8] C. Berg, Corps convexes et potentiels spheriques. (French) *Mat.-Fys. Medd. Danske Vid. Selsk.* **37** (1969), 64 pp.

[9] S. Bochner, Positive zonal functions on spheres, *Proc. Nat. Acad. Sci.,*, **40** (1954), 1141-1147.

[10] A. Bonami and J-L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, *Trans. Amer. Math. Soc.* **183** (1973), 223-263.

[11] P. L. Butzer, Legendre transform methods in the solution of basic problems in algebraic approximation, *Functions, series, operators, Vol. I, II (Budapest, 1980)*, 277–301, Colloq. Math. Soc. Jnos Bolyai, **35**, North-Holland, Amsterdam, 1983.

[12] A. P. Calderon and A. Zygmund, On a problem of Mihlin, *Trans. Amer. Math. Soc.*, **78** (1955), 209-224.

[13] C. F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[14] C. F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Cambridge Univ. Press, 2001.

[15] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, McGraw-Hill, New York, 1953.

[16] M. I. Ganzburg, Polynomial approximation on the m-dimensional ball, in *Approximation Theory IX, Vol. I*, Vanderbilt Univ. Press, Nashville, TN, 1998, p. 141-148.

[17] I. M. Gelfand, Spherical functions in symmetric Riemann spaces, *Dokl. Akad. Nauk. SSSR.* **70** (1950), 5-8.

[18] A. I. Kamzolov, The best approximation on the classes of functions $W_p^\alpha(S^n)$ by polynomials in spherical harmonics, *Mat. Zametki*, **32** (1982), 285–293; English transl in *Math Notes*, **32** (1982), 622-628.

[19] Luoqing Li and Kunyang Wang, *Harmonic analysis and approximation on the unit sphere* Science Press, Beijing, 2000.

[20] Zh.-K, Li and Yuan Xu, Summability of orthogonal expansions of several variables, *J. Approx. Theory*, **122** (2003), 267-333.

[21] P. I. Lizorkin and S. M. Nikolskii, Approximation theory on the sphere, *Proc. Steklov Inst. Math.*, **172** (1987), 295-302.

[22] S. Pawelke, Über Approximationsordnung bei Kugelfunktionen und algebraischen Polynomen, *Tôhoku Math. J.*, **24** (1972), 473-486.

[23] S. Rafalson, An extremal relation of the theory of approximation of functions by algebraic polynomials, *J. Approx. Theory*, **110** (2001), 146–170.

[24] D. L. Ragozin, Constructive polynomial approximation on spheres and projective spaces, *Trans. Amer. Math. Soc.*, **162** (1971), 157-170.

[25] M. Rösler, Positivity of Dunkl's intertwining operator, *Duke Math. J.*, **98** (1999), 445–463.

[26] Kh. Rustamov, On approximation of functions on the sphere, *Russian Acad. Sci. Izv. Math.*, **43** (1994), 311-329.

[27] Kh. Rustamov, On the equivalence of different moduli of smoothness on the sphere, *Proc. Steklov Inst. Math.* **204** (1994), no. 3, 235–260.

[28] R. L. Stens and M. Wehrens, Legendre transform methods and best algebraic approximation, *Comment. Math. Prace Mat.* **21** (1980), 351–380.

[29] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol.23, Providence, 4th edition, 1975.

[30] K. Triméche, The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual, *Integral Transform. Spec. Funct.* **12** (2001), 349–374.

[31] M. Wherens, Best approximation on the unit sphere in $\mathbb{R}^k$, in *Functional Analysis and Approximation* (Oberwolfach, 1980), Birkhauser, Basel, 1981, 233-245.

[32] Yuan Xu, Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups, *Proc. Amer. Math. Soc.* **125** (1997), 2963–2973.

[33] Yuan Xu, Funk-Hecke formula for orthogonal polynomials on spheres and on balls, *Bull. London Math. Soc. 32* (2000), 447-457.

[34] Yuan Xu, Orthogonal polynomials and summability in Fourier orthogonal series on spheres and on balls, *Math. Proc. Cambridge Phil. Soc.*, **31** (2001), 139-155.

[35] Yuan Xu, Orthogonal polynomials on the ball and on the simplex for weight functions with reflection symmetries, *Constr. Approx.*, **17** (2001), 383-412.

[36] Yuan Xu, Approximation by means of h-harmonic polynomials on the unit sphere, *Adv. in Comp. Math*, to appear. `http://math.uoregon.edu/~yuan` (ApproxSph.ps.gz).

[37] Yuan Xu, Weighted approximation of functions on the unit sphere, *Const. Approx.*, to appear. `http://math.uoregon.edu/~yuan` (BestApp.ps.gz).

[38] Yuan Xu, Almost everywhere convergence of orthogonal expansions of several variables, submitted. `http://math.uoregon.edu/~yuan` (Almost.ps.gz).