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C-chart, X-chart, and the Katz Family of Distributions

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In statistical process control, the primary method used to monitor the number of nonconformities is the c-chart. The conventional c-chart is based on the assumption that the occurrence of nonconformities in samples is well modeled by a Poisson distribution. When the Poisson assumption is not met, the X-chart (individuals chart) is often used as an alternative charting scheme in practice. In this article I investigate the relative merits of the c-chart compared to the X-chart for the Katz family covering equi-, under-, and over-dispersed distributions relative to the Poisson distribution. The need to use an X-chart rather than using the c-chart depends upon whether or not the ratio of the in-control variance to the in-control mean is close to unity. The X-chart, which incorporates the information on this ratio, can lead to significant improvements under certain circumstances. Both the 3-sigma c- and X-charts fail in providing reliable information on the status of the process with a small in-control process mean when a downward mean shift occurs. In these cases, charts based on probability limits are much more appropriate.

KEY WORDS: Average run length; Maximum likelihood; Method of moments; Poisson distribution; Robustness; Statistical process control

Introduction

Statistical process control (SPC) techniques for monitoring the number of nonconformities are widely used in industry for process monitoring (see Woodall (1997) for a comprehensive review of attribute control charts). Various statistical control charts, exemplified by the c-chart, have been developed to evaluate the stability of processes characterized by such count data. These techniques are usually based on the underlying assumption that the Poisson distribution provides an appropriate model. Most available software packages that support c-chart analysis also rely on the Poisson assumption.

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The use of the Poisson distribution requires assumptions that may be overly simplistic in some applications. For example, two basic and simple generalizations of the Poisson distribution may lead to the negative binomial distribution. One model assumes that nonconformities occur in clusters. If the number of clusters has a Poisson distribution and the number of nonconformities per cluster follows a logarithmic series distribution (Johnson, Kotz and Kemp (1992)), then the observed total number of nonconformities has a negative binomial distribution (Friedman (1993)). Alternatively, the other model is based on the assumption that the parameter of the Poisson distribution itself varies when the process is in-control. If this parameter has a gamma distribution, the negative binomial marginal distribution results (Haight (1967)).

When the Poisson assumption is not met, the X-chart (individuals chart) may be used as an alternative charting procedure (Wadsworth, Stephens and Godfrey (1986), Heimann (1996), and Wheeler (1995)). Both the c- and the X-charts are widely used techniques, due in part to some common appealing aspects, such as being easy to set up, implement and interpret.

In this article we investigate the relative merits of the 3-sigma c-chart compared to the X-chart for count data generated from a variety of distributions of the Katz family. The primary objective of the article is to quantify the conditions attributing to the failure of the 3-sigma c- and the X-charts and to study the conditions under which the use of these two charts is appropriate. The implications of the results for the use of charts based on probability limits are discussed. Issues on the model specification and parameter estimation of the Katz family of distributions are also addressed.

The Katz Family

Distributions of count data over the non-negative integers can be uniquely represented by the recursive probability

$$P_{j+1} = f(j, \Theta) P_j, \quad j = 0, 1, 2, ..., \tag{1}$$

with Θ is a vector of parameters. Katz (1963) considered a system with two parameters $\Theta = (\theta_1, \theta_2)$ for which

$$f(j, \Theta) = \frac{\theta_1 + \theta_2 j}{1 + j}, \quad j = 0, 1, 2, ..., \tag{2}$$

where $\theta_1 > 0$ and $\theta_2 < 1$. It is understood that if $\theta_1 + \theta_2 j < 0$ then $P_{j+i} = 0$ for all $i > 0$.

As a commonly used recursive probability system, the Katz family has a simple probability structure in which distributions are of the Poisson, Bernoulli, or Pascal type. It can be shown that conditions $\theta_2 < 0$, $0 < \theta_2 < 1$, and $\theta_2 = 0$ give rise to the binomial distribution ($B(N, P)$), the negative binomial distribution ($NB(k, p)$) and the Poisson distribution ($P(\lambda)$), respectively, with parameters $N = -\theta_1/\theta_2, P = \theta_2/(\theta_2 - 1)$, $k = \theta_1/\theta_2, p = \theta_2$, and $\lambda = \theta_1$ (Johnson, Kotz and Kemp (1992), and Gurland (1983)). Aside from these three distributions, the Katz family contains only mild generalizations obtained from binomial distributions by permitting the two parameters to take any real values within specified ranges. The area of the (θ_1, θ_2) plane occupied by the Poisson, binomial and negative binomial distributions can be found in, for example, Johnson and Kotz (1969, pp. 42). Note that distributions in the Katz family are

Table 1: The Katz Family of Distributions

Parameters	Distribution	Ratio of Variance to Mean	Skewness	Kurtosis
$\theta_1 > 0$ and $\theta_2 = 0$	Poisson	1	> 0	> 3
$\theta_1 > 0$ and $\theta_2 < 0$	Generalized Binomial	< 1	≥ 0 iff $r \geq 1/2$	≥ 3 iff $r \geq \frac{1}{3-\sqrt{3}}$ or $r \leq \frac{1}{3+\sqrt{3}}$
$\theta_1 > 0$ and $0 < \theta_2 < 1$	Negative Binomial	> 1	> 0	> 3

related through various well-known limiting forms. For example, as $\theta_2 \to 0$, both binomial and negative binomial distributions converge to the Poisson distribution with parameter θ_1.

The probability generating function of distributions in the Katz family, $g(.)$, satisfies the equation

$$\frac{d\log g(t)}{dt} = \frac{\theta_1}{1-\theta_2 t}$$

with $g(1) = 1$. Hence,

$$g(t) = \begin{cases} \mathrm{e}^{\theta_1(t-1)} & \text{if } \theta_2 = 0 \\ [(1\text{-}\theta_2 t)/(1-\theta_2)]^{-\theta_1/\theta_2} & \text{otherwise.} \end{cases}$$

The moments about the origin are

$$\mu'_{r+1} \equiv E(X^{r+1}) = \sum_{j=0}^{r} \binom{r}{j} (\theta_1 \mu'_j + \theta_2 \mu'_{j+1}).$$

In particular, the mean is $\mu \equiv \mu'_1 = \theta_1/(1-\theta_2)$, the variance is $\sigma^2 \equiv \mu'_2 - (\mu'_1)^2 = \theta_1/(1-\theta_2)^2$, and the skewness is $E(X-\mu)^3/\sigma^3 = (1+\theta_2)\theta_1^{-1/2}$. Note that the coefficient of variation is $\sigma/\mu = \theta_1^{-1/2}$.

Despite its simple probability structure, the Katz family covers a wide spectrum of distributions with the property of being equi-, under-, or over-dispersed relative to Poisson distributions. The ratio of the variance to the mean is

$$r \equiv (1-\theta_2)^{-1}. \tag{3}$$

The Poisson distribution has $r = 1$ and is said to exhibit equi-dispersion. The binomial and negative binomial distributions are under-dispersed ($r < 1$) and over-dispersed ($r > 1$), respectively.

Table 1 lists some of the properties of the Katz family of distributions. Since violation of the condition $r = 1$ is sufficient for violation of the Poisson assumption, as far as the robustness study of control charts is concerned, the ratio r may serve as a reasonable measurement of the magnitude of the departure from the Poisson distribution.

To gain more insight into the Katz family of distributions, Figure 1 shows various shapes of distribution functions of distributions with some specific parameter values. To demonstrate the important similarities and differences between distributions with different r, four distribution functions of under- and over-dispersed distributions are presented in contrast to that of the Poisson distribution with the same mean. For illustrative purposes, the mean of all distributions is taken to be 5 and the values of r for the under- and over-dispersed distributions are chosen to be 1/2, 3/4, 5/4 and 2 in Figures 1A - 1D, respectively. In each of the four charts, the Poisson distribution serves as the reference distribution and its probability function is drawn in black while that of the under- or over-dispersed distribution is represented in the lighter bars. Except for the generalized binomial distribution with $r = 1/2$, all other distributions in Figure 1 are skewed to the right.

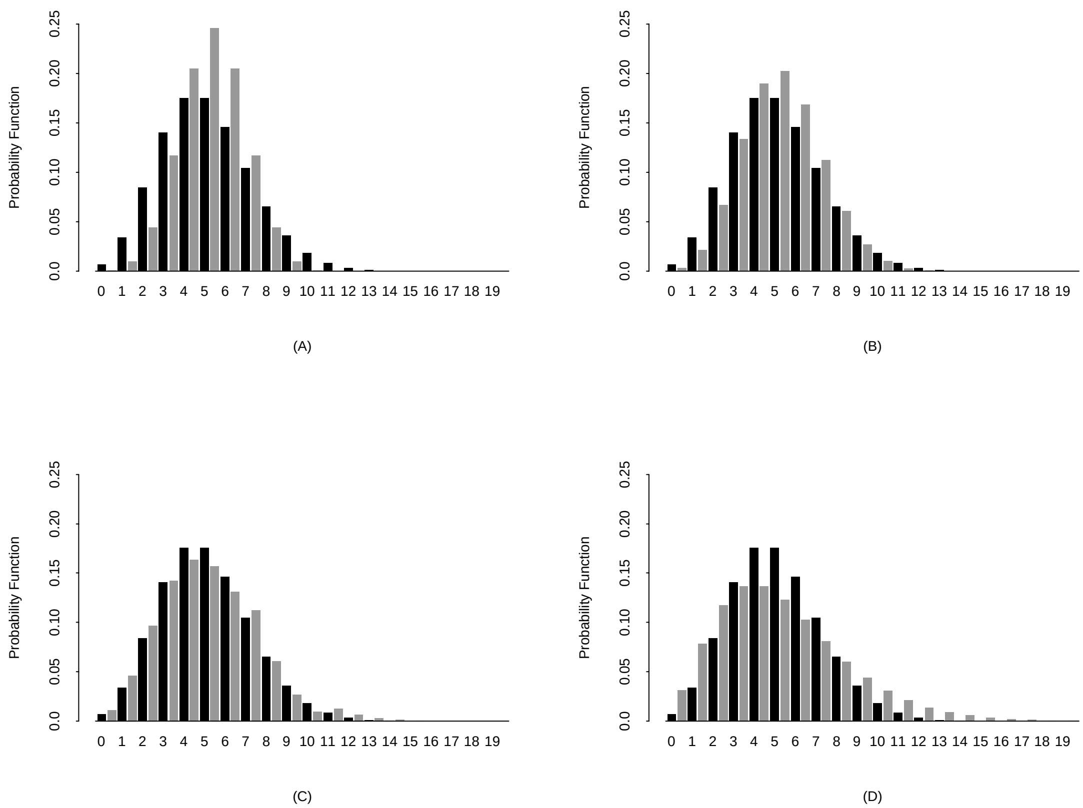


Figure 1. Under- and over-dispersed distributions (in lighter bars) in comparison to the Poisson distribution (in black) with the same mean ($\mu = 5$)
(A) $r = .5$ (B) $r = .75$ (C) $r = 1.2$ (D) $r = 2$

As noted in Katz (1963), for purposes of estimation of the parameters of the Katz family of distributions, it is more convenient to consider $\xi \equiv \theta_1(1-\theta_2)^{-1}$ and $\eta \equiv \theta_2(1-\theta_2)^{-1}$, rather than θ_1 and θ_2. Let $\{x_i\}_{i=1}^{n}$ be an in-control historical Phase I sample (see Woodall (2000) for a discussion on Phase I and Phase II of statistical process control). The logarithm of the likelihood function is

$$L(\xi, \eta) = (\sum x_i) \ln[\eta(1+\eta)^{-1}] - \frac{n\xi}{\eta} \ln(1+\eta) + \sum \ln \Gamma(\frac{\xi}{\eta} + x_i) - \sum \ln(x_i!) - n \ln \Gamma(\frac{\xi}{\eta}).$$

Setting $\frac{\partial L}{\partial \xi} = 0$ and $\frac{\partial L}{\partial \eta} = 0$ produce directly

$$n \ln(1+\eta) = \sum F(\frac{\xi}{\eta} + x_i - 1) - nF(\frac{\xi}{\eta} - 1)$$

and

$$n\xi = \sum x_i,$$

respectively, where $F(y-1) = \frac{d}{dy} \ln \Gamma(y)$ is the digamma function. Hence, the maximum likelihood estimators (MLEs) of ξ and η are given by

$$\hat{\xi}_{MLE} = \bar{x}, \tag{4}$$

and

$$n \ln(1+\hat{\eta}_{MLE}) = \sum F(\frac{\bar{x}}{\hat{\eta}_{MLE}} + x_i - 1) - nF(\frac{\bar{x}}{\hat{\eta}_{MLE}} - 1), \tag{5}$$

respectively. Equation (5) can be solved numerically.

Alternative estimates of the parameters can be obtained using the method of moments. Since $E(x_i) = \theta_1/(1-\theta_2)$ and $E(s^2) = \theta_1/(1-\theta_2)^2$, it can be shown that the method of moments estimators (MMEs) of ξ and η are

$$\hat{\xi}_{MME} = \bar{x} \quad \text{and} \quad \hat{\eta}_{MME} = \frac{s^2 - \bar{x}}{\bar{x}}, \tag{6}$$

respectively. Both MLEs and MMEs are consistent but estimates based on the method of moments are easy to compute and often provide estimates with similar estimation errors as the MLEs (Katz (1963)).

The C-chart versus the X-chart

In this paper we consider the Phase II performance of the c- and X-charts with the parameters assumed to be known. Since the mean (μ) and variance (σ^2) of the process are the same for the Poisson distribution (i.e., $r = 1$), the upper control limit (UCL) and lower control limit (LCL) of the c-chart are

$$\mu \pm k\sqrt{\mu}, \tag{7}$$

where k is a constant. More generally, one can define the UCL and LCL as

$$\mu \pm k\sigma \text{ or } \mu \pm k\sqrt{r\mu}. \tag{8}$$

The control limits in Equation (8) correspond to those of an X-chart. Although the X-chart is typically used for continuous measurement data and the c-chart is used for count data, the UCL and LCL of the two charts are identical in theory for Poisson data if the Poisson parameter is assumed to be known. If the parameter is unknown, then typically the average moving range is used to estimate the standard deviation for the X-chart in Phase I, while the sample average is used for estimating the variance used to determine the control limits of the c-chart. In this study, we consider control charts with 3-sigma limits (i.e., $k = 3$) only because these are the most frequently used in practice. We also discuss the use of probability-based control limits as recommended by Ryan and Schwertman (1997) and others.

Table 2: The In-control ARLs of c- and X-charts with 3-sigma Control Limits†

	c- and X-charts	c-chart				X-chart			
$\mu^{\ddagger}$	$r = 1$	$r = 0.75$	$r = 0.9$	$r = 1.25$	$r = 1.5$	$r = 0.75$	$r = 0.9$	$r = 1.25$	$r = 1.5$
5.0	183.4	1069.1	310.6	78.6	47.2	253.7	310.6	165.2	161.5
6.0	275.6	1911.7	493.0	107.8	61.1	477.5	176.0	219.7	198.4
7.0	174.9	894.0	288.4	76.9	46.4	244.0	288.4	148.1	138.4
8.0	269.0	1665.8	470.4	107.3	61.0	475.4	470.4	203.6	177.0
9.0	412.1	3091.8	764.4	149.3	80.0	278.0	302.3	279.9	226.9
10.0	285.7	1716.8	498.7	112.9	63.3	539.8	498.7	207.2	173.1
12.5	397.7	2657.8	720.5	146.6	78.3	320.9	312.5	260.4	203.3
15.0	283.8	1562.8	488.0	112.4	62.1	585.2	488.0	194.9	246.3
17.5	425.8	2728.4	768.7	155.2	81.3	430.8	365.4	262.3	196.2
20.0	339.7	1932.2	593.5	129.6	69.4	346.4	294.0	355.5	248.7
22.5	284.8	1485.6	485.7	112.4	61.2	296.3	485.7	297.0	210.7
25.0	443.1	2724.5	795.9	160.2	82.6	265.0	415.6	256.2	272.7
27.5	388.6	2240.7	685.6	144.1	75.3	496.2	368.2	358.1	240.1
30.0	349.9	1919.8	608.8	132.4	69.8	455.3	335.3	319.5	313.8
32.5	321.9	1699.1	554.0	123.7	65.7	427.8	312.0	290.5	283.6
35.0	301.4	1543.5	514.3	117.2	62.7	409.7	295.6	268.4	260.0
37.5	286.3	1432.4	485.4	112.3	60.4	398.3	284.0	378.7	342.6
40.0	275.4	1352.8	464.5	108.8	58.6	392.1	464.5	355.0	318.9
42.5	426.6	2444.5	755.7	155.0	79.3	390.0	449.6	336.4	299.8
45.0	413.0	2329.3	728.6	151.0	77.4	391.2	439.4	321.6	284.3
47.5	403.3	2246.6	709.1	148.0	76.1	395.3	433.1	310.0	271.6
50.0	396.7	2189.4	695.8	146.0	75.1	402.0	429.9	300.9	261.2

† $ARL \equiv 1/(1-\beta)$, where $1-\beta = P(X < LCL) + P(X > UCL)$.
‡ μ is the in-control process mean.

In-control ARLs of c- and X-charts

Table 2 presents the in-control average run lengths (ARLs) of the c- and X-charts with 3-sigma control limits for the Katz family of distributions with five different levels of r: the Poisson distribution ($r = 1$), two under-dispersed distributions ($r = 0.75$ and 0.90) and two over-dispersed distributions ($r = 1.25$ and 1.50). Note that for a given value of r, the parameter θ_2 is uniquely determined by Equation (3). The distribution with the desired process mean is obtained by taking different levels of θ_1. For example, if $r = 1.25$, then $\theta_2 = 0.2$ and $\theta_1 = 0.8\mu$. The range of the in-control process mean is chosen to be from 5 to 50, which is similar to that examined in the previously published studies such as that given in Ryan and Schwertman (1997).

The ARLs of the c- and X-charts are identical in the Poisson case ($r = 1$). When $r = 1$, the ARL starts at a low level when the process mean is small (say, less than 9), moves up as the process mean increases, and fluctuates eventually around the level 370 (= 1/0.0027) based on an underlying assumption of a normal distribution. Notice, however, that even for processes with relatively large means, most of the in-control ARLs differ considerably from this nominal value since the Poisson random variable is discrete.

When $r \neq 1$, the behavior of the ARLs of the c-chart depends greatly on whether the process is under- or over-dispersed. If $r > 1$, the probability of a false alarm of the c-chart can be considerably inflated due to the over-dispersed probability structure of the distribution;

as the case of $r = 1.50$ demonstrates. The ARL is between 60 and 80 for distributions with a mean greater than 9. These results are similar to those based on other control charts for over-dispersed in-control distributions (see, for example, Heimann (1996) for discussions on the p-chart, and Woodall and Thomas (1995) on the X-bar chart).

In comparison to the over-dispersion case, the case of $r < 1$ is less explored in the literature. As shown in Table 2, when $r < 1$, the probability of a false alarm of the c-chart may be significantly less than the nominal level (0.0027) and as a consequence, the in-control ARL could increase considerably for a under-dispersed process even with a slight departure (measured by r) from the Poisson distribution. For example, the ARL could be as high as 769 for a process with $r = 0.9$ and $\mu = 17.5$.

As can be seen from Table 2, there is a marked contrast between ARLs of the c-chart and those of the X-chart when $r \neq 1$. In general, the in-control ARLs of the X-chart are much closer to their values under the Poisson assumption $-$ a significant improvement is achieved by utilizing the information on both mean and variability of the distribution. Although the X-chart does not completely remove the distortions, the ARLs of the X-chart for both under- and over-dispersed processes are within a much tighter band around the ARL based on the Poisson distribution with the same mean in comparison to those of the c-chart.

Out-of-control ARLs for c- and X-charts

To assess the ability of c- and X-charts to detect a process mean shift, the ARLs of the two charts are calculated for processes with the 5 selected levels of r when the process mean has been shifted by a half, one, and two standard deviations of the corresponding Poisson distribution with the same in-control process mean (i.e., the same size of mean shift in absolute terms is used for the various distributions). For brevity, we will report results only for ARLs when the process mean has shifted one standard deviation (see Table 3).

Three general conclusions emerge from the results which are displayed in Table 3. First, the c-chart is not robust to the violation of the Poisson assumption. The ARLs increase for the processes with $r < 1$ and the magnitude of the increase can be substantial; as the case when the process mean has shifted upward by one standard deviation demonstrates, the ARLs can be more than 70 if $r = 0.75$. These numbers are more than twice the ARL for the Poisson process with the same mean. Although the ARLs for processes with $r > 1$ are small, the c-chart is of less practical usefulness due to its extremely low level of in-control ARLs (Table 2).

Second, the X-chart is identical to the c-chart for processes with $r = 1$ but the two charts show a remarkable difference in ARLs for processes with $r \neq 1$. The X-chart is quite robust to the departure from the Poisson assumption in either the under- or over-dispersed case, especially when an upward shift occurs, with the robustness improving as the process mean increases. If the mean shift is upward by one standard deviation, then the ARLs for most of under- and over-dispersed processes examined are approximately within the same range outlined by the saw-tooth pattern of the ARL from the Poisson distribution.

Third, both c- and X-charts with 3-sigma control limits are not effective in detecting downward changes, in particular when the process mean is relatively small. With 3-sigma limits used

Table 3: The ARLs of c- and X-charts with 3-sigma Control Limits when the Process Mean has Shifted by One Standard Deviation†

Panel A: Upward mean shift

	c- and X-charts	c-chart				X-chart			
$\mu^{\ddagger}$	$r = 1$	$r = 0.75$	$r = 0.9$	$r = 1.25$	$r = 1.5$	$r = 0.75$	$r = 0.9$	$r = 1.25$	$r = 1.5$
5.0	15.4	34.0	18.9	13.5	10.9	14.7	18.9	23.1	27.8
6.0	20.2	46.9	25.8	16.2	14.7	20.7	13.5	27.3	37.0
7.0	15.1	28.7	18.4	12.2	10.0	14.1	18.4	19.5	22.6
8.0	20.1	39.7	25.0	14.8	13.3	19.5	25.0	23.4	30.0
9.0	26.7	54.5	33.7	17.9	13.8	14.2	18.8	28.0	29.9
10.0	21.2	45.4	27.1	16.1	12.6	23.3	27.1	24.7	26.4
12.5	26.7	55.4	34.3	18.2	13.9	16.8	20.5	27.0	27.7
15.0	22.0	44.8	28.1	16.1	13.7	25.5	28.1	23.1	37.0
17.5	28.9	67.4	38.3	20.5	15.4	22.8	24.2	29.3	28.5
20.0	25.5	57.3	32.9	18.7	15.4	21.0	21.5	37.0	38.0
22.5	23.2	50.5	29.2	17.4	13.4	19.7	29.2	33.0	30.7
25.0	30.9	64.4	39.4	20.2	15.2	16.8	26.4	27.4	34.1
27.5	28.9	67.0	37.2	20.7	15.5	27.2	25.4	37.9	33.9
30.0	27.3	61.2	35.5	19.6	15.8	25.9	24.6	35.0	44.5
32.5	26.2	56.8	32.8	18.7	14.2	25.0	23.2	32.7	37.9
35.0	25.5	53.3	31.9	18.0	14.6	24.2	22.8	30.8	38.1
37.5	24.9	50.5	31.2	17.5	13.4	23.6	22.5	38.5	42.6
40.0	24.5	48.3	30.6	17.0	13.9	23.2	30.6	36.5	43.1
42.5	32.4	68.3	41.7	21.1	15.7	22.8	30.2	34.8	38.0
45.0	32.0	72.9	40.9	21.9	16.2	24.7	29.9	36.0	38.7
47.5	31.8	69.9	41.8	21.4	16.8	24.4	30.7	34.6	39.5
50.0	31.7	67.3	41.3	20.9	15.6	24.2	30.6	33.4	35.5

Panel B: Downward mean shift

	c- and X-charts	c-chart				X-chart			
$\mu^{\ddagger}$	$r = 1$	$r = 0.75$	$r = 0.9$	$r = 1.25$	$r = 1.5$	$r = 0.75$	$r = 0.9$	$r = 1.25$	$r = 1.5$
5.0	30244.9	*	244236.8	2863.8	1261.9	4194304.0	244236.8	8218.7	6901.3
6.0	46069.1	268435500.0	333795.8	4375.8	1137.9	6242685.0	57030.8	11905.8	5459.2
7.0	19459.5	13463850.0	102086.6	2737.1	1137.9	133.0	102086.6	6964.0	5185.7
8.0	33247.7	33778120.0	193342.8	4585.9	1209.8	315.3	193342.8	11346.0	5071.1
9.0	56827.7	25028040.0	285517.2	5219.5	1356.6	110.7	552.0	12349.8	5327.0
10.0	905.1	2361.7	1281.2	374.3	171.7	236.2	1281.2	8967.9	5561.8
12.5	770.9	1862.7	1079.4	293.4	137.8	73.5	198.8	2078.3	7162.5
15.0	221.9	514.0	304.3	104.5	63.3	137.4	304.3	354.4	3341.9
17.5	333.4	899.1	476.6	153.3	78.1	87.3	152.0	468.2	676.8
20.0	182.5	466.5	250.8	92.6	56.4	64.8	97.2	644.8	892.3
22.5	122.2	297.6	161.3	66.0	37.9	52.6	161.3	335.0	370.1
25.0	199.2	543.6	282.8	100.4	60.4	45.2	123.6	212.4	564.0
27.5	149.7	387.8	202.2	78.1	44.2	78.9	96.2	319.8	308.7
30.0	120.7	298.2	164.1	64.4	40.9	68.9	83.0	228.9	483.6
32.5	102.3	241.8	132.4	55.3	33.0	61.9	70.6	176.1	303.3
35.0	89.9	203.9	116.5	48.9	32.2	56.8	64.7	142.5	251.0
37.5	81.2	177.2	105.1	44.2	27.4	53.0	60.3	211.6	322.9
40.0	75.0	157.6	96.7	40.6	27.4	50.1	96.7	176.4	279.9
42.5	116.9	264.7	155.8	58.0	34.8	47.9	90.3	151.4	209.1
45.0	108.8	268.3	143.8	58.2	34.9	51.1	85.4	145.8	192.8
47.5	102.6	242.9	140.2	54.2	35.2	49.5	84.8	129.9	180.8
50.0	97.9	222.8	132.3	51.0	31.3	48.1	81.6	117.5	146.5

† $ARL \equiv 1/(1-\beta)$, where $1-\beta = P(X < LCL) + P(X > UCL)$.
‡ μ is the in-control process mean.
* ARL not computable since the probability of obtaining an out-of-control signal is effectively zero.

in the c- and X-charts, the calculated LCLs of these two charts may become negative and, thus, there is no LCL. For example, for the Poisson process ($r = 1$), there is no LCL for the c-chart when the in-control process mean is less than 9. The chart without an LCL cannot detect process improvement without the use of supplementary runs rules. Probability limits can be used, however, with any chart to determine an LCL. See, for example, Ryan and Schwertman (1997).

In summary, these results, together with those obtained from Table 2, suggest that it would be preferable to use the X-chart whenever there is an indication of any departure from the Poisson assumption in the in-control state, especially in cases that the process mean is not small and the attribute of interest is the upward mean shift. Wheeler (1995) recommends to always use the X-chart for situations in which the c-chart is traditionally used. We would like to remark, however, that if a downward mean shift is of interest, as it usually is, the X-chart would also not be acceptable, unless the in-control process mean is fairly large.

The Need to Verify the Poisson Assumption

The results presented in the previous sections emphasize the fact that the properties of the c- and X-charts differ substantially across processes with different magnitudes of dispersion. In this section we discuss several techniques for discriminating between equi-, under-, and over-dispersed distributions relative to the Poisson distribution.

From Equation (2), we have

$$u_j \equiv \frac{(j+1)P_{j+1}}{P_j} = \theta_1 + \theta_2 j. \tag{9}$$

Note that the right-hand side of Equation (9) is a linear function of j. Hence, for the Katz family of distributions, the plot of u_j against j gives a straight line with the slope =, <, or > zero for the equi-, under-, and over-dispersed distributions, respectively.

This graphical method for testing the dispersiveness for a given data set has been suggested by, for example, Katz (1963) and Ord (1967). Although different shapes of u_j curves for different distributions appear to be useful in developing strategies for determining whether the data are equi-, under-, or over-dispersed, results based only on the u_j curve using observed frequencies can be misleading as demonstrated in Jinkinson and Slater (1981), and Hoaglin, Mosteller and Tukey (1985). In terms of applications, therefore, those graphical methods may only be used as a starting point for more rigorous and sophisticated inference procedures.

Katz (1963) considered the test based on

$$J_n \equiv \sqrt{\frac{n}{2}}\left(\frac{s^2 - \bar{x}}{\bar{x}}\right), \tag{10}$$

where $s^2 = \sum(x_i - \bar{x})^2/n$, for testing the null hypothesis of equi-dispersion ($r = 1$) against the alternative hypothesis of under- or over-dispersion ($r < 1$ or $r > 1$). He found that J_n is distributed asymptotically as the standard normal distribution under the null hypothesis of equi-dispersion. It can be shown that Katz' test based on J_n is a special case of a class of statistics based on the generalized method of moments (GMM) and it has satisfactory performance for moderate size samples (Fang (2001)).

Charts Based on Probability Limits

The incidence of a false alarm is more likely to be due to a value above the UCL than to a value below the LCL, because in general, distributions in the Katz family are skewed to the right (Table 1). Consequently, the charts with 3-sigma control limits such as the c- and X-charts can perform poorly under certain circumstances, as indicated by the results presented in the previous sections. In this section, we consider charts based on probability limits. Unlike the c-chart and the X-chart, which have control limits that depend only on the first two moments of the process of interest, the chart based on probability limits can provide a refinement on the control limits over the X-chart for equi-, under-, and over-dispersed distributions, particularly when the in-control process mean is small.

We note that for Poisson processes with small means, various data transformations were recommended by Ryan (1989), Nelson (1994), and McCool and Joyner-Motley (1998), among others. Since the approach of transforming data can often make the tail areas closer to those under normality, such normalizing transformations will ameliorate the problem when the X-chart is used for processes with small means. An obvious shortcoming of this approach, however, is that a transformed statistic is plotted, rather than the statistic of interest. The optimal limits method devised by Ryan and Schwertman (1997) is rather more satisfactory in this respect. However, like the conventional c-chart, the Ryan and Schwertman's optimal c-chart is not robust to the departure of the Poisson assumption and was not intended to be.

It is also worth mentioning that, because of the special probability structures of the Katz family of distributions, increasing the size of the inspection unit results in increasing the process mean. Hence, if it is feasible, an often easy solution to increasing the process mean to a designated level is to have a larger inspection unit. However, when it is neither easy nor desirable to increase the size of the inspection unit, charts based on probability limits should be of practical use.

The UCL and LCL of the chart based on probability limits can be determined by

$$\begin{aligned} &\min UCL \\ \text{subject to: } &P(X > UCL) \leq \alpha/2 \end{aligned}$$

and

$$\begin{aligned} &\max LCL \\ \text{subject to: } &P(X < LCL) \leq \alpha/2, \end{aligned} \tag{11}$$

where the probability of a false alarm, α, is often taken to be 0.0027. Assuming that a given set of data can be approximated by a distribution from the Katz family, the solution of the UCL and LCL in Equation (11) depends on the values of θ_1 and θ_2 in Equation (2).

Table 4: The Data Used in Example 1

The 40 random numbers simulated from the negative-binomial distribution $NB(30, 0.4)$

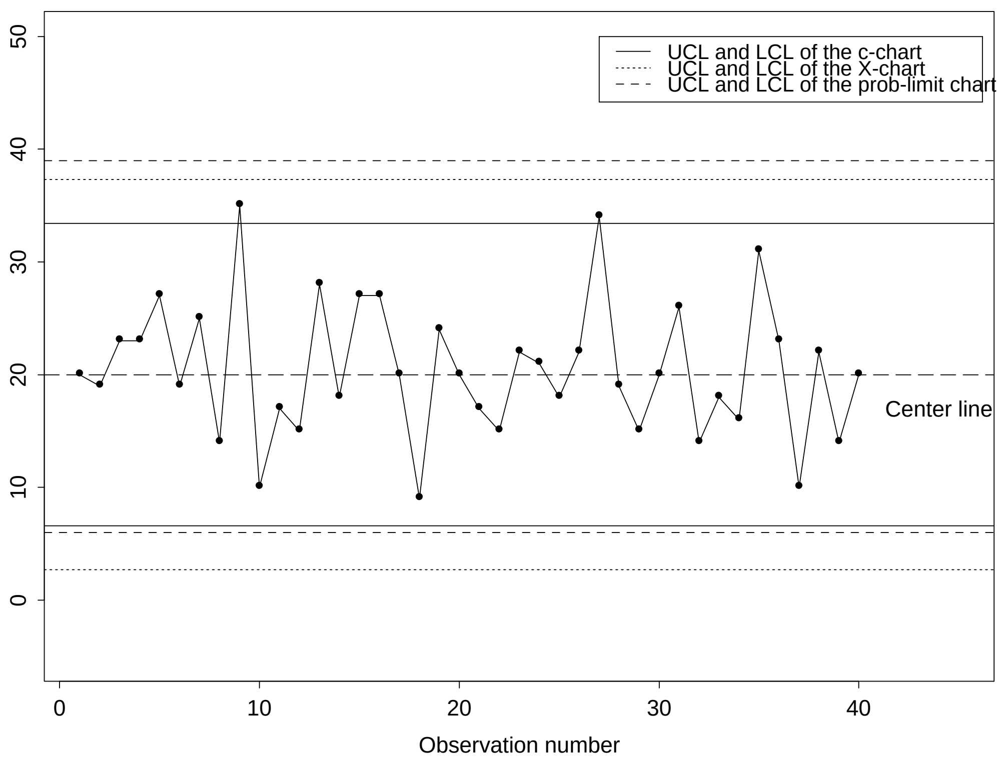
20	19	23	23	27	19	25	14	35	10	17	15	28	18	27	27	20	9	24	20
17	15	22	21	18	22	34	19	15	20	26	14	18	16	31	23	10	22	14	20

In the next section, we will utilize the method of moments to obtain parameter estimates and demonstrate the usefulness of the control chart based on probability limits. We do not consider, however, the effects of estimation error on the statistical properties of the chart.

Two Illustrating Examples

Example 1. To compare the performance of the c-chart, the X-chart, and the chart based on probability limits, we have drawn 40 samples from the negative-binomial distribution $NB(30, 0.4)$ with $\mu = 20$ and $r = 5/3$. The data are given in Table 4.

Figure 2. Control Charts for Data Used in Example 1



Since this paper focuses on the Phase II performance of the control charts with known parameters, the c-chart, X-chart and the probability-limit chart, which are shown in Figure 2, are based on $NB(30, 0.4)$. Three charts share the same centerline ($\mu = 20$) but have different upper and lower control limits. Since the data are over-dispersed, the control limits of the c-chart ($UCL = 33.42$ and $LCL = 6.58$) are too narrow; as Figure 2 demonstrates, the 9th and 27th observations are above the UCL. In contrast, improved control limits are obtained by using the X-chart and the chart based on the estimated probability limit. Although the α-risk allocated above the UCL and below the LCL differs slightly for the X-chart and the probability-limit chart based on $NB(30, 0.4)$, all of the 40 points are within the control limits of either the X-chart ($UCL = 37.32$ and $LCL = 2.68$) or the probability-limit chart ($UCL = 39$ and $LCL = 6$).

Note that the estimated mean, variance and r are 20.4, 36.3 and 1.78, respectively. Applying the Katz's test, J_n is 3.48 with p-value 0.00025, which is significant at all the usual significance levels. The MMEs of ξ and η are about 20.4 and 0.777, respectively. Hence, the estimated θ_1 and θ_2 are 11.5 and 0.437, suggesting that if the distribution for the in-control process is unknown, the negative-binomial distribution $NB(26, 0.44)$ may be a candidate based on the given sample in Table 4.

Example 2. To illustrate the fact that more appropriate control limits can be achieved by using the chart based on probability limits when the in-control process mean is small and a downward mean shift occurs, we have simulated 100 observations from two binomial distributions. The first 60 observations are from the binomial distribution $B(20, 0.3)$ with $\mu = 6.0$ and $r = 0.7$, which represents the in-control process; while the last 40 observations are from the binomial distribution $B(13, 0.3)$ with $\mu = 3.9$ and $r = 0.7$, presenting the out-of-control process with a downward mean shift. The downward mean shift, 2.1, is about one standard deviation of the in-control distribution. The data are given in Table 5.

As in Example 1, we focus on the Phase II performance of the control charts with known parameters. Hence, the c-chart, X-chart and the probability-limit chart are based on $B(20, 0.3)$, which are shown in Figure 3. The UCL for the c-chart is 13.3 and there is no LCL effectively since $(6 - 3\sqrt{6}) < 0$. Since the process is under-dispersed, the probability of a false alarm of the c-chart is much smaller than the designed 0.0027 level, and the c-chart is not sensitive to the downward mean shift (see also Panel B of Table 3). As Figure 3 indicates, all 100 observations are within the control limits of the c-chart. Although the control limits of the X-chart ($UCL = 12.2$ and no LCL effectively since $(6 - 3\sqrt{4.2}) < 0$) are tighter than those of the c-chart, 100 observations are all within the control limits of the X-chart. The probability-limit chart, which has $UCL = 11$ and $LCL = 1$, detects the process mean change.

Based on the first 60 observations (Panel A of Table 5), the estimated r is 0.58 with estimated mean 5.83 and variance 3.4. The Katz's test J_n is -2.29, which is significant at the 5 percent significance level. The MMEs of ξ and η are about 5.83 and -0.418, respectively. Hence, the estimated θ_1 and θ_2 are 10.02 and -0.718, suggesting that the binomial distribution $B(14, 0.42)$ may be a candidate for the in-control process distribution based on the given sample.

Table 5: The Data Used in Example 2

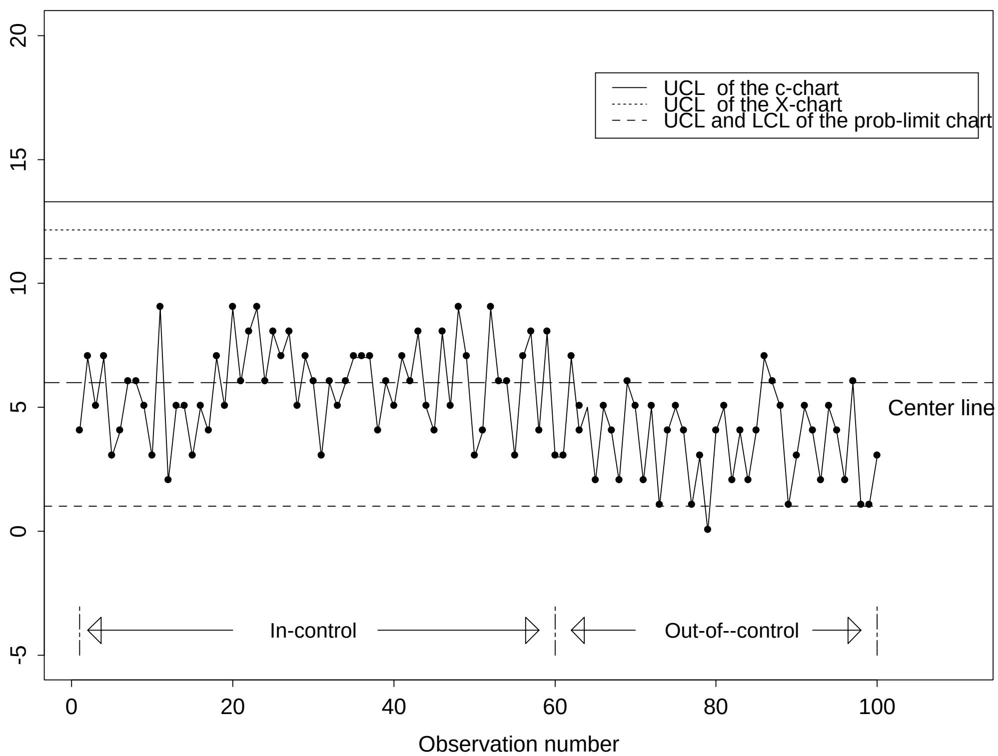
A: The 60 random numbers simulated from the binomial distribution $B(20, 0.3)$

4	7	5	7	3	4	6	6	5	3	9	2	5	5	3	5	4	7	5	9
6	8	9	6	8	7	8	5	7	6	3	6	5	6	7	7	7	4	6	5
7	6	8	5	4	8	5	9	7	3	4	9	6	6	3	7	8	4	8	3

B: The 40 random numbers simulated from the binomial distribution $B(13, 0.3)$

3	7	4	5	2	5	4	2	6	5	2	5	1	4	5	4	1	3	0	4
5	2	4	2	4	7	6	5	1	3	5	4	2	5	4	2	6	1	1	3

Figure 3. Control Charts for Data Used in Example 2



Concluding Remarks

The c-chart has proven to be useful for monitoring count data in a wide range of applications. Using the Katz family of distributions, we demonstrate that the condition $r < 1$ tends to reduce the sensitivity of the c-chart because the control limits become too wide and therefore, the detection of out-of-control processes becomes slower. On the other hand, the c-chart control limits are too narrow for the distributions with $r > 1$. Consequently, the probability of false alarms of the c-chart increases. We show that the X-chart, which incorporates (implicitly) the information on r, is superior to the c-chart for under- and over-dispersed distributions, especially when the process mean is relatively large and when the mean shift is upward. We also show that when the in-control process mean is small and a downward process mean shift occurs, neither the c- nor the X-chart is adequate. Since increasing the size of the inspection unit often results in increasing the process mean, we recommend using the X-chart based on a moderately large sized inspection unit. When it is neither easy nor desirable to increase the size of the inspection unit, charts based on probability limits can be adapted to obtain an in-control ARL value closer to a specified value with a lower control limit.

The results obtained in this article are relevant to other types of control charts for nonconformities. For example, the conventional $p-$chart is based on the binomial distribution (the case that $r < 1$); that is, the probability of occurrence of a conforming unit is constant, and successive units are independent. For processes where nonconforming units are clustered (Albin and Friedman (1989)), or where successive units are correlated (Bhat and Lal (1990)), it is likely that control limits based upon other distributions in the Katz family or in the compound Poisson class may be a better choice. In such cases, control limits with corresponding adjustments are more appropriate.

The idea of using the Katz family of distributions in the robustness study of control charts for count data can be extended to the CUSUM and EWMA charts. As illustrated by an example using negative binomial distributions in Hawkins and Olwell (1998) (Section 5.4.5), the ARLs for CUSUMs of Poisson variables are very sensitive to departures from the assumed Poisson distribution.

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