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CERTAIN INTEGRAL CURVATURE BOUNDS

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We prove that a nonvanishing weak limit of Riemannian metrics in surfaces with an integral curvature bound admits only weak cusp singularities. The result is useful toward generalizing classical uniformization theory to surfaces with boundary.

1. Introduction

Classical uniformization theorem says that in a compact Riemann surface without boundary there is a constant curvature metric in any conformal class of metrics. There have been many attempts to generalize this theory to surfaces with boundary. The main focus, started by the independent work of Troyanov [1991] and McOwen [1988], has been to study the existence or nonexistence of constant curvature metrics in surfaces with conical singularities. Much work has followed since; see, for example, [Chen and Li 1991; Chang and Yang 1988; Luo and Tian 1992].

The disadvantage of this classical approach is that the Gaussian curvature is a second-order differential operator of the metric, while the condition of prescribing conical singularities is equivalent to prescribing both the metric and its derivatives near the set of singular points at infinitesimal level. Thus, the constant curvature equation with prescribed conical singularities is an overdetermined elliptic equation. In general, one should not expect to get a clear-cut statement about the existence (or nonexistence) of solutions.

We now describe another approach. Given a compact Riemann surface Ω and a Hermitian metric g_0 on Ω, any metric g on Ω is said to be conformal to g_0 if there exists a smooth function $e^{2\varphi}$ such that $g = e^{2\varphi} g_0$ on Ω. Define the variational space $\mathcal{G}(\Omega)$ to be the set of all metrics that are conformal to g_0 and agree with g_0 on $\partial\Omega$ up to first derivatives. For each $g \in \mathcal{G}(\Omega)$, define the energy functional

$$E(g) = \int_\Omega K^2 \, dg,$$

where K is the Gaussian curvature of g. The problem is to minimize this functional with the area constraint

$$A(g) = \int_\Omega dg = \text{const.}$$

A critical point of this functional is called an extremal Hermitian metric; see [Calabi 1982]. The Euler–Lagrange equation is

$$\triangle_g K + K^2 = C,$$

where we have, in a local system,

$$-\triangle\varphi = Ke^{2\varphi} \quad \text{if } g = e^{2\varphi}|dz|^2.$$

By a theorem of Calabi, any metric solving this Euler–Lagrange equation on a surface without boundary must have constant scalar curvature. This result conforms with the classical uniformization theorem.

One of us (X. Chen) has been using the approach just described, namely the study of the variational problem of minimizing $E(g)$ in $\mathcal{G}(\Omega)$ with fixed area, with the goal of generalizing uniformization results to surfaces with boundary. Some of the ideas go back to E. Calabi (private communications).

To attack this variational problem, one first studies the weak compactness of any subset of metrics in $\mathcal{G}(\Omega)$ with finite energy and area. Generally speaking, for a sequence of metrics $\{g_n, n \in \mathbb{N}\}$ with finite energy and area, there exists a weak limit metric $\underline{g}$ such that g_n converges to $\underline{g}$ weakly in any compact subset away from a finite set of singular points. An important feature of the limit is the *bubbling phenomenon*, first observed by Sacks and Uhlenbeck in 1979, when they studied the existence theorem for harmonic maps between two spheres [Uhlenbeck 1982]. The key observation was that the noncompactness is associated with the concentration of the energy density at isolated *bubble points*. Around each such point, one can define a *weak singular angle* in the approximation sense (see Definition 2 below). If the weak singular angle is 0, it is called a weak cusp singular point. An intriguing question is that whether all points is weak cusp singular points if the weak limit metric $\underline{g} \neq 0$. In this paper, we will prove (see Section 2 for terminology and notation):

Theorem 1.1. *Let $\{g_n\}$ be a sequence of conformal metrics in domain Ω with finite area and finite energy. There exist a subsequence of $\{g_n\}$, a limit metric g_0 and a finite set of bubble points $\{p_1, p_2, \ldots, p_m\}$ such that $g_n \rightharpoonup g_0$ in $H^{2,2}_{\mathrm{loc}}(\Omega \setminus \{p_1, p_2, \ldots, p_m\})$. If $g_0 \neq 0(\varphi_0 \neq -\infty)$, then g_0 has a weak cusp singularity at each bubble point p_i. There is no ghost vertex in the bubble-tree decomposition.*

This motivates the following classification result:

Theorem 1.2 [Chen 1999]. *Let Ω be a surface without boundary and let g be an extremal Hermitian metric in $\Omega \setminus \{p_1, \ldots, p_n\}$ with finite energy and area, and having only weak cusp singularities at all singular points.*

(1) *If $\chi(\Omega) \leqslant 0$, then K is a negative constant.*

(2) *If $n \geqslant 3$ and $\chi(\Omega) = 2$ (that is, Ω is a sphere), then K is a negative constant.*

(3) *If $n = 2$ and $\chi(\Omega) = 2$, there exists no extremal Hermitian metric.*

(4) *If $n = 1$ and $\chi(\Omega) = 2$, there exists a unique extremal Hermitian metric determined by total area, and the metric must be rotationally symmetric.*

These results are critical for the generalization of uniformization theory to a surface with boundary.

The main result was stated in [Chen 1998a; 1999] without proof and was used crucially in [Chen 2001], in deriving the long-time existence of the Calabi flow in Riemannian surfaces. At present, there is strong interest in the Calabi flow for general Kähler manifolds. Clarification of this important technical step will likely be indicative of what happens in more general settings.

2. The problem from an analytic viewpoint

In a coordinate chart (D, z), any metric g can be written as

$$g = e^{2\varphi}(dx^2 + dy^2),$$

and the curvature function is

$$K = -\frac{\triangle \varphi}{e^{2\varphi}}.$$

A metric is said to have finite area C_1 and finite energy C_2 if

$$(2\text{-}1) \qquad \int_D e^{2\varphi} \, dx \, dy \leqslant C_1 \quad \text{and} \quad \int_D \frac{(\triangle \varphi)^2}{e^{2\varphi}} \, dx \, dy \leqslant C_2.$$

A sequence of metrics $\{g_n\}$, where $g_n = e^{2\varphi_n}(dx^2 + dy^2)$, is said to have finite area C_1 and finite energy C_2 if each φ_n satisfies (2-1). From now on we will always use either $\{\varphi_n\}$ or $\{g_n\}$ to denote a sequence of metrics with finite area and finite energy.

For any subdomain Ω in D, relabel the energy and area for a conformal parameter function as

$$(2\text{-}2) \qquad A_c(\varphi, \Omega) = \int_\Omega e^{2\varphi} \, dx \, dy, \quad K_c(\varphi, \Omega) = \int_\Omega \frac{(\triangle \varphi)^2}{e^{2\varphi}} \, dx \, dy.$$

A "zero metric" should have zero area and zero energy. Since the zero metric has a conformal parameter function identically equal to $-\infty$, we define $A_c(-\infty, \Omega) = K_c(-\infty, \Omega) = 0$. For notational convenience, define $\hat{H}_{\text{loc}}^{2,2}(\Omega) = H^{2,2}(\Omega) \cup \{-\infty\}$.

A sequence of functions $\{\varphi_n\} \in H^{2,2}(\Omega)$ is said to converge weakly to a function φ_0 in $\hat{H}^{2,2}_{\mathrm{loc}}(\Omega)$ if one of two mutually exclusive alternatives holds:

1 (Vanishing case): If $\varphi_0 \equiv -\infty$, then $\varphi_n \to -\infty$ uniformly in any relatively compact subdomain of Ω.

2 (Nonvanishing case): If $\varphi_0 \in H^{2,2}(\Omega)$, then $\varphi_n \rightharpoonup \varphi_0$ weakly in $H^{2,2}_{\mathrm{loc}}(\Omega)$.

Consequently, a sequence of Riemannian metrics $\{g_n\} \in H^{2,2}(\Omega)$ converges weakly to a limit metric g_0 in $H^{2,2}_{\mathrm{loc}}(\Omega)$ if and only if either

1 $g_n \to 0$ everywhere (and $g_0 \equiv 0$), or

2 $\varphi_n \rightharpoonup \varphi_0$ in $H^{2,2}_{\mathrm{loc}}(\Omega)$, where $g_n = e^{2\varphi_n} g_{\mathrm{bk}}$ and $g_0 = e^{2\varphi_0} g_{\mathrm{bk}}$, for g_{bk} a smooth background metric in Ω.

Definition 2.1. *A point p is said to be a bubble point of $\{\varphi_n\}$ if, for any $r > 0$,*

$$\liminf_{n \to \infty} \int_{D_r(p)} \frac{\triangle(\varphi_n)^2}{e^{2\varphi_n}} \, dx \, dy \geqslant \alpha > 0 \quad and \quad \liminf_{n \to \infty} \int_{D_r(p)} e^{2\varphi_n} \, dx \, dy \geqslant \beta > 0,$$

where $D_r(p)$ is a coordinate disk centered at p with radius r.

The largest possible numbers α and β are the concentration weights of the energy function and area function at point p.

For convenience we restate here the three main theorems from [Chen 1998b]. Their proofs can be found there.

Theorem 2.2. *Let $\{\varphi_n, n \in \mathbb{N}\}$ be a sequence of metrics in $H^{2,2}(D)$ with finite area C_1 and energy C_2. There exists a subsequence $\{\varphi_{n_j}, j \in \mathbb{N}\}$ of $\{\varphi_n\}$, a finite number of bubble points $\{p_1, p_2, \ldots, p_m\}$ with respect to $\{\varphi_{n_j}, j \in \mathbb{N}\}$ (where $0 \leqslant m \leqslant \sqrt{C_1 C_2/(4\pi)}$), and a metric $\varphi_0 \in \hat{H}^{2,2}_{\mathrm{loc}}(D \setminus \{p_1, p_2, \ldots, p_m\})$ such that*

$$\varphi_{n_j} \rightharpoonup \varphi_0 \quad in \ \hat{H}^{2,2}_{\mathrm{loc}}(D \setminus \{p_1, p_2, \ldots, p_m\}).$$

Theorem 2.3 (Bubbles on bubbles). *Let $\{\varphi_n\}$ be a sequence of metrics in D with finite area C_1 and finite energy C_2. Suppose that $p = 0$ is the only bubble point in D, that it has area concentration A_p and energy concentration K_p, and that there exists a metric $\varphi_0 \in \hat{H}_{2,2}(D \setminus \{p\})$ such that $\varphi_n \rightharpoonup \varphi_0$ in $\hat{H}_{2,2}(D \setminus \{p\})$. A sequence of numbers $\{\delta_n \searrow 0\}$ can be chosen to renormalize the sequence of metrics as $\phi_n(x, y) = \varphi_n(\delta_n x, \delta_n y) + \log \delta_n$, for $n \in \mathbb{N}$.*

There exists a subsequence $\{\varphi_{n_j}, j \in \mathbb{N}\}$ of $\{\varphi_n\}$, a finite number of bubble points $\{q_1, q_2, \ldots, q_m\}$ with respect to the subsequence of metrics $\{\phi_{n_j}\}$ (where $0 \leqslant m \leqslant \sqrt{A_p K_p/(4\pi^2)}$), and a metric $\phi_0 \in \hat{H}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \ldots, q_m\})$ such that

$$\phi_{n_j} \rightharpoonup \phi_0 \in \hat{H}^{2,2}_{\mathrm{loc}}(S^2 \setminus \{\infty, q_1, q_2, \ldots, q_m\}).$$

If $\phi_0 \equiv -\infty$ (vanishing case), then $m \geqslant 2$ and p is a bubble point of $\{\phi_{n_j}, j \in \mathbb{N}\}$.

Theorem 2.4 (Bubble tree). *The limit of any locally weakly convergent sequence of metrics $\{g_k, \; k \in \mathbb{N}\}$ in $\mathscr{G}(\Omega)$ encompasses the following data:*

(1) *A finite rooted tree T, possibly reduced to just the base vertex f.*

(2) *The base vertex $f \in T$ is a limit metric in Ω with a finite number of bubble points $\{p_i\}$ deleted; the edge emanating from the base vertex is $\{p_i\}$.*

(3) *Any other vertex f_s is a limit metric defined on $S^2 \setminus \{\infty, \, p_{s_i}\}$; the edges emanating from this vertices are $\{p_{s_i}\}$.*

(4) *For each pair of vertex f_{s_1} and f_{s_2} bounding a common edge in T, they are tenuously connected at the pair of respective singular points. The number of vertices whose valence $\neq 2$, is bounded from above by $\sqrt{C_1 C_2}$. The depth of the tree is also finite in a reasonable sense. Each vertex $f_I = f_{i_1 i_2 \cdots i_k}$ has the property that, if it vanishes in any point in its domain, it vanishes everywhere; in this case we call it a ghost vertex. The number of ghost vertices is finite.*

Definition 2.5. *Around each bubble point, define the weak singular angle*

$$\alpha = \lim_{r \to 0} \int_{\partial D_r(p)} k_g ds_g = \lim_{r \to 0} \lim_{n \to \infty} \int_{\partial D_r(p)} k_{g_n} ds_{g_n}$$

if the limit is well defined. Here k_g and k_{g_n} are geodesic curvatures of g and g_n.

The weak angle does exist, as we shall see in the next section, and has some interesting properties. From now on, we will always assume that $\{g_n\}$ converges, by substituting a convergent subsequence if necessary.

If the weak singular angle is zero, the bubble point p is called a *weak cusp singularity*. Our purpose is to prove:

Main Theorem 2.6. *Let $\{g_n\}$ be a sequence of conformal metrics in Ω with finite area and finite energy. There exists a subsequence of $\{g_n\}$, a limit metric g_0 and a finite set of bubble points $\{p_1, p_2, \ldots, p_m\}$ such that $g_n \rightharpoonup g_0$ in $H^{2,2}_{\mathrm{loc}}(\Omega \setminus \{p_1, p_2, \ldots, p_m\})$ (following the notations of Theorem 1.1). If $g_0 \neq 0$, then g_0 has a weak cusp singularity at each bubble point p_i, and there is no ghost vertex in the bubble-tree decomposition.*

3. Asymptotic geometry of singular points

In this section, we introduce the blowing-up process and the definition of a neck. We also prove several lemmas that are essential to our main theorem.

Definition 3.1. *For any metric φ with finite area C_1 and energy C_2 in $D \setminus \{0\}$, set*

$$\bar{\varphi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r \cos\theta, r \sin\theta) \, d\theta.$$

Lemma 3.2 [Chen 1998b]. *Suppose φ is a metric with finite area C_1 and energy C_2 in $D \setminus \{0\}$.*

(1) $\lim\limits_{r \to 0}(\varphi(r \cos\theta, r \sin\theta) + \log r) = -\infty$.

(2) $\lim\limits_{r \to 0} \bar{\varphi}_r(r) \, r$ *exists and is finite.*

(3) *There exists constants $\lambda \in (0, 1)$ and C_3, C_4 such that*

$$\frac{1}{\lambda}(\bar{\varphi}(r) + \log r) + C_3 \; \leqslant \; \varphi(r \cos\theta, r \sin\theta) + \log r \; \leqslant \; \lambda(\bar{\varphi}(r) + \log r) + C_4.$$

Similarly, one can prove:

Corollary 3.3. *Suppose ϕ is a metric in $R^2 \setminus \bar{D}_{r_0}(0)$ with finite area C_1 and energy C_2, where $\bar{D}_{r_0}(0)$ is a closed disk with radius r_0.*

(1) $\lim\limits_{r \to \infty} (\phi(r \cos\theta, r \sin\theta) + \log r) = -\infty$.

(2) $\lim\limits_{r \to \infty} \bar{\phi}_r(r) \, r$ *exists and is finite.*

(3) *There exists constants $\mu \in (0, 1)$ and C_5, C_6 such that*

$$\frac{1}{\mu}(\bar{\varphi}(r) + \log r) + C_5 \; \leqslant \; \varphi(r \cos\theta, r \sin\theta) + \log r \; \leqslant \; \mu(\bar{\varphi}(r) + \log r) + C_6.$$

Lemma 3.4 [Chen 1998b]. *Let $\{\varphi_n\}$ be a sequence of metrics with finite area C_1 and energy C_2. There exists a constant ϵ_0 such that if*

$$\max_{r \leqslant \rho} |\partial D_r|_{g_n} = \max_{r \leqslant \rho} \int_0^{2\pi} e^{\varphi_n(r \cos\theta, r \sin\theta)} r \, d\theta \; \leqslant \; \epsilon_0 \quad \text{for all } n \in \mathbb{N},$$

then $\{\varphi_n\}$ has no bubble points in D.

We can now explain the blowing-up process. To simplify the problem, let $\{g_n = e^{\varphi_n}(dx^2 + dy^2)\}$ be a sequence of metrics in D with finite area and finite energy, converging to a limit metric $g = e^{\varphi_0}(dx^2 + dy^2) \neq 0$, and having $\{0\}$ as its only bubble point. Following Lemma 3.2, we have

$$\lim_{r \to 0} \max_{0 \leqslant \theta \leqslant 2\pi} (\varphi_0(r \cos\theta, r \sin\theta) + \log r) = -\infty.$$

Then there exists $r_1 > 0$ such that

$$\max_{0 \leqslant \theta \leqslant 2\pi} (\varphi_0(r \cos\theta, r \sin\theta) + \log r) \ll 0 \quad \text{for all } r \leq r_1.$$

If n is large enough, the convergence implies that

$$\max_{0 \leqslant \theta \leqslant 2\pi} (\varphi_n(r_1 \cos\theta, r_1 \sin\theta) + \log r_1) \ll 0 \quad \text{for all } n > N;$$

equivalently, if the length of the circle $|z| = r_1$ is very small, then

$$|\partial D_{r_1}|_{g_n} = \int_0^{2\pi} e^{\varphi_n(r_1 \cos\theta, r_1 \sin\theta)} r_1 \, d\theta \leq \varepsilon \quad \text{for all } n > N.$$

According to Lemma 3.4, if ε is small enough, for each $n > N$ we can choose δ_n such that

$$|\partial D_r|_{g_n} = \int_0^{2\pi} e^{\varphi_n(r \cos\theta, r \sin\theta)} r \, d\theta < \varepsilon \quad \text{for all } r \text{ with } r_1 \geq r > \delta_n,$$

and

$$|\partial D_{\delta_n}|_{g_n} = \int_0^{2\pi} e^{\varphi_n(\delta_n \cos\theta, \delta_n \sin\theta)} \delta_n \, d\theta = \varepsilon.$$

Renormalize this sequence of metrics as

$$\phi_n(z) = \varphi(\delta_n z) + \log \delta_n \quad \text{for all } |z| < \frac{1}{\delta_n}$$

and

$$\tilde{g}_n(z) = e^{2\phi_n(z)} |dz|^2 = g_n(\delta_n z).$$

The theorems in Section 2 assert that there is a subsequence $\{\tilde{g}_{n_j}\}$ of metrics $\{\tilde{g}_n\}$, a limit metric $\tilde{g}$ and finitely many bubble points $\{q_1, q_2, \ldots, q_m\}$ such that either

$$\tilde{g} \equiv 0 (\phi_{n_j} \to -\infty) \quad \text{or} \quad \tilde{g}_{n_j} \rightharpoonup \tilde{g}$$

in $\hat{H}^{2,2}_{\text{loc}}(S^2 \setminus \{\infty, q_1, q_2, \ldots, q_m\})$.

Choose r_2 big enough that $\{q_1, q_2, \ldots, q_m\} \subset D_{r_2}$. The annulus bounded by the two circles $|z| = r_1$ and $|z| = r_2 \delta_n$ is called the neck of the blowing-up process and denoted by $\text{Neck}(r_1, r_2)$. The blowing up procedure or the renormalization procedure depends only on the filter size $\varepsilon > 0$ (once a coordinate system is fixed). Since $g_n \rightharpoonup g_0$ in $\hat{H}^{2,2}_{\text{loc}}(D \setminus \{p\})$, the surface $(g_0, D \setminus \{p\})$ and $(\tilde{g}_0, S^2 \setminus \{\infty, q_1, q_2, \ldots, q_m\})$ are called tenuously connected at p and at $z = \infty$.

Lemma 3.5. *If the limit metric g is nonzero, the weak angle α exists and is finite, and $\alpha \geqslant 0$.*

Proof. In polar coordinates (r, θ), we have

$$\alpha = \lim_{r \to 0} \int_{\partial D_r(p)} k_g \, ds_g = \frac{1}{2\pi} \lim_{r \to 0} \int_0^{2\pi} (\partial_r \varphi(r \cos\theta, r \sin\theta) r + 1) \, d\theta = \lim_{r \to 0} \bar{\varphi}'(r) + 1.$$

According to Lemma 3.2, the weak angle exists and is finite. Now we only have to prove $\alpha \geq 0$. Assume $\alpha < 0$. By the preceding equality we know that $\bar{\varphi}_r(r) r + 1 < 0$ for r small enough, say for $r < r_0$; that is,

$$\bar{\varphi}_r(r) + \frac{1}{r} < 0 \quad \text{when } r < r_0.$$

Then, for $0 < r_1 < r_2 < r_0$,

$$0 > \int_{r_1}^{r_2} \left(\bar{\varphi}_r(r) + \frac{1}{r} \right) = (\bar{\varphi}(r) + \log r)|_{r_1}^{r_2} = (\bar{\varphi}(r_2) + \log r_2) - (\bar{\varphi}(r_1) + \log r_1).$$

Letting $r_1 \to 0$, since $\bar{\varphi}(r_1) + \log r_1 \to -\infty$, it follows that

$$0 > \int_{r_1}^{r_2} \left(\bar{\varphi}_r(r) + \frac{1}{r} \right) = (\bar{\varphi}(r_2) + \log r_2) - (\bar{\varphi}(r_1) + \log r_1) \to \infty,$$

which is a contradiction. □

Definition 3.6. *Let $\{\varphi_n\}$ be a sequence of metrics in D with finite area C_1 and finite energy C_2. Suppose that $p = 0$ is the only bubble point in D in the blowing-up process just described in Lemma 3.4. We define the out angle β around p in* $\mathrm{Neck}(r_1, r_2)$ *as*

$$\beta = \lim_{r_2 \to \infty} \lim_{n \to \infty} \int_{\partial D_{r_2 \delta_n}(p)} k_{g_n} ds_n$$

if the limit exists, where k_{g_n} is the geodesic curvature.

Lemma 3.7. *In the notations in Lemma 3.4, if the (renormalized) limit metric ϕ_0 is not $-\infty$, the out angle β exists and is finite. Furthermore, $\beta \leqslant 0$.*

Proof. By the definition of β, we have

$$\beta = \lim_{r_2 \to \infty} \lim_{n \to \infty} \int_{\partial D_{r_2 \delta_n}(p)} k_{g_n} ds_n = \lim_{r_2 \to \infty} \lim_{n \to \infty} \int_{\partial D_{r_2}(p)} k_{\tilde{g}_n} d\tilde{s}_n.$$

If the limit metric $\tilde{g}_0$ does not vanish, that is, if $\phi_0 \neq -\infty$, we have

$$\beta = - \lim_{r_2 \to \infty} \int_{\partial D_{r_2}} k_{\tilde{g}_0} d\tilde{s}_0.$$

By Lemma 3.2, β is well defined and finite. Just as the argument in the proof of Lemma 3.7, we can then prove that $\beta \leqslant 0$. □

Remark 3.8. The out angle β (more precisely, its negative) can be regarded as the weak angle at $\{\infty\}$ in the blowing-up process.

When the limit metric vanishes, the weak angle can still be defined:

Lemma 3.9. *If a sequence of metrics $\{g_n = e^{\varphi_n}(dx^2 + dy^2)\}$ on D has vanishing limit metric and $\{0\}$ as its only bubble point, the weak angle at $\{0\}$ exists and finite.*

Proof. It suffices to show the existence of the limit

$$\alpha = \lim_{r \to 0} \lim_{n \to \infty} \int_{\partial D_r(p)} k_{g_n} ds_{g_n} = \lim_{r \to 0} \lim_{n \to \infty} \int_0^{2\pi} (\partial_r \varphi_n r + 1) d\theta.$$

But for any $0 < r_1 < r_2$,

$$\left| \int_0^{2\pi} (\partial_r \varphi_n (r_2 \cos \theta, r_2 \sin \theta) r_2 + 1) \, d\theta - \int_0^{2\pi} (\partial_r \varphi_n (r_1 \cos \theta, r_1 \sin \theta) r_1 + 1) \, d\theta \right|$$

$$= \int_{r_1}^{r_2} \int_0^{2\pi} \partial_r (\partial_r \varphi_n r) \, d\theta \, dr = \int_{r_1}^{r_2} \int_0^{2\pi} \triangle \varphi_n \, d\theta \, dr$$

$$\leqslant \int_{r_1}^{r_2} \int_0^{2\pi} \frac{(\triangle \varphi_n)^2}{e^{2\varphi_n}} \, d\theta \, dr \int_{r_1}^{r_2} \int_0^{2\pi} e^{2\varphi_n} \, d\theta \, dr$$

$$= K_n(r_1, r_2) A_n(r_1, r_2),$$

which tends to 0 as $n \to \infty$. Hence the limit exists and α is defined. $\qquad \square$

4. Proof of the main theorem

In the neck of the blowing-up process (page 41), the sequence of metrics has an average property around concentric circles:

Lemma 4.1 (Max-min inequality). *Let $\{g_n = e^{2\varphi_n}(dx^2 + dy^2)\}$ be a sequence of metrics with finite area and finite energy, converging weakly to a limit metric $g = e^{2\varphi}(dx^2 + dy^2)$ in $H_{\mathrm{loc}}^{2,2}(D \setminus \{0\})$ and having $\{0\}$ as its only bubble point. In* $\mathrm{Neck}(r_1, r_2)$, *there exists a constant $c \in (0, 1)$, independent of r, such that*

$$c \leqslant \left| \frac{\max_\theta (\varphi_n (r \cos \theta, r \sin \theta) + \log r)}{\min_\theta (\varphi_n (r \cos \theta, r \sin \theta) + \log r)} \right| \leqslant 1.$$

Proof. Renormalize the sequence of metrics around $p = \{0\}$ as

$$\phi_n(z) = \varphi(\delta_n z) + \log \delta_n,$$

$$\tilde{g}_n(z) = e^{2\phi_n(z)} |dz|^2 = g_n(\delta_n z).$$

Lemma 3.2 yields, for the limit metrics φ and ϕ,

$$\lambda^{-1}(\bar{\varphi}(r) + \log r) + C_3 \leq \varphi(r \cos \theta, r \sin \theta) + \log r \leq \lambda(\bar{\varphi}(r) + \log r) + C_4,$$

$$\mu^{-1}(\bar{\phi}(r) + \log r) + C_5 \leq \phi(r \cos \theta, r \sin \theta) + \log r \leq \mu(\bar{\phi}(r) + \log r) + C_6,$$

with $\lambda, \mu \in (0, 1)$. For r_1 small enough and n, r_2 big enough, we then have

$$(4\text{-}1) \qquad \lambda^{-1}(\bar{\varphi}_n(r_1) + \log r_1) + C_3 \leq \varphi_n(r_1 \cos \theta, r_1 \sin \theta) + \log r_1$$
$$\leq \lambda(\bar{\varphi}_n(r_1) + \log r_1) + C_4,$$

$$\mu^{-1}(\bar{\phi}_n(r_2) + \log r_2) + C_5 \leq \phi_n(r_2 \cos \theta, r_2 \sin \theta) + \log r_2$$
$$\leq \mu(\bar{\phi}_n(r_2) + \log r_2) + C_6,$$

where C_3, C_4, C_5, C_6 are independent of r. This last pair of inequalities leads to

$$\mu^{-1}(\bar{\varphi}_n(r_2\delta_n) + \log(r_2\delta_n)) + C_5 \leq \varphi_n(r_2\delta_n \cos\theta, r_2\delta_n \sin\theta) + \log(r_2\delta_n)$$
$$\leq \mu(\bar{\varphi}_n(r_2\delta_n) + \log(r_2\delta_n)) + C_6.$$

Combining this with (4-1), one obtains, still for r_1 small enough and n, r_2 big enough,

$$\frac{\lambda^2}{2} \leq \left| \frac{\max_\theta (\varphi_n(r_1\cos\theta, r_1\sin\theta) + \log r_1)}{\min_\theta (\varphi_n(r_1\cos\theta, r_1\sin\theta) + \log r_1)} \right| \leq 1$$

and

$$\frac{\mu^2}{2} \leq \left| \frac{\max_\theta (\varphi_n(r_2\delta_n\cos\theta, r_2\delta_n\sin\theta) + \log(r_2\delta_n))}{\min_\theta (\varphi_n(r_2\delta_n\cos\theta, r_2\delta_n\sin\theta) + \log(r_2\delta_n))} \right| \leq 1.$$

Thus the max-min inequality holds at the two boundary circles.

Set $u = -\log r = -\log\sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. The domain $D \setminus \{0\}$ becomes an infinite annulus $\{(u, \theta) : 0 < u \leq \infty, \ -\pi \leq \theta \leq \pi\}$ via this transformation. Let

$$\xi(u, \theta) = \varphi(e^{-u}\cos\theta, e^{-u}\sin\theta) - u \ , \ \xi_n(u, \theta) = \varphi_n(e^{-u}\cos\theta, e^{-u}\sin\theta) - u$$

and define $\triangle_{u,\theta} = \partial_u^2 + \partial_\theta^2$. For any small $r_1 = e^{-u_0} > 0$, define $\tilde{\varphi}(v, \theta) = \xi(v + u_0, \theta)$ and $\tilde{\varphi}_n(v, \theta) = \xi_n(v + u_0, \theta)$. Then

$$-\triangle_{v,\theta}\tilde{\varphi} = K(v + u_0, \theta)e^{2\tilde{\varphi}} \qquad \text{for all } (v, \theta) \in \tilde{D},$$
$$-\triangle_{v,\theta}\tilde{\varphi}_n = K_n(v + u_0, \theta)e^{2\tilde{\varphi}_n} \quad \text{for all } (v, \theta) \in \tilde{D},$$

where $\tilde{D} = [-1, 1] \times S^1$. There exists a constant C such that $\tilde{\varphi}, \tilde{\varphi}_n \leq C$ for n big enough; thus the right-hand sides of both equalities are uniformly bounded in $L^2(\tilde{D})$.

Define ω, ω_n by

$$\begin{cases} -\triangle\omega = K(v + u_0, \theta)e^{2\tilde{\varphi}}, \\ \omega|_{\partial\tilde{D}} = 0, \end{cases} \qquad \begin{cases} -\triangle\omega_n = K_n(v + u_0, \theta)e^{2\tilde{\varphi}_n}, \\ \omega_n|_{\partial\tilde{D}} = 0. \end{cases}$$

Then $\|\omega\|_{L^\infty}, \|\omega_n\|_{L^\infty}$ are uniformly bounded from above; the bound is actually independent of u_0, since L^2 norm of $\tilde{\varphi}(v, \theta)$ and $\tilde{\varphi}_n(v, \theta)$ in $\tilde{D}$ converge uniformly to 0 as $n \to \infty$ and $u \to \infty$.

The function $h_n = \tilde{\varphi}_n - \omega_n$ is harmonic. Consider the two concentric circles $v_1 = 0$ $(|z| = r_1)$, $v_2 = \log r_1 - \log r_2\delta_n$ $(|z| = r_2\delta_n)$. For any circle $v_3 = \log r_1 - \log r_3$ $(|z| = r_3)$ between the two, set

$$M_n(v) = \max_v h_n(v, \theta), \quad m_n(v) = \min_v h_n(v, \theta).$$

Apply Hadamard's Three-Circle Theorem to obtain

$$M_n(v_3) \leqslant$$
$$\frac{M_n(v_1)\big(\log(v_2+u_0) - \log(v_3+u_0)\big) + M_n(v_2)\big(\log(v_3+u_0) - \log(v_1+u_0)\big)}{\log(v_2+u_0) - \log(v_3+u_0)}$$

and

$$m_n(v_3) \geqslant$$
$$\frac{m_n(v_1)\big(\log(v_2+u_0) - \log(v_3+u_0)\big) + m_n(v_2)\big(\log(v_3+u_0) - \log(v_1+u_0)\big)}{\log(v_2+u_0) - \log(v_3+u_0)}.$$

Since the max-min inequality holds at the two boundary circles, we obtain

$$M_n(v_1) = \max_\theta h_n(0,\theta) = \max_\theta(\tilde{\varphi}_n(0,\theta) - \omega_n(0,\theta)) \leqslant \max_\theta \tilde{\varphi}_n(0,\theta) + C,$$
$$m_n(v_1) = \min_\theta h_n(0,\theta) = \min_\theta(\tilde{\varphi}_n(0,\theta) - \omega_n(0,\theta)) \geqslant \min_\theta \tilde{\varphi}_n(0,\theta) - C,$$

where C is the uniform bound of $\|\omega_n\|_{L^\infty}$. By the definition of $\tilde{\varphi}_n(0,\theta)$, we have

$$\max_\theta \tilde{\varphi}_n(0,\theta) = \max_\theta(\varphi_n(r_1\cos\theta, r_1\sin\theta) + \log r_1)$$
$$\leqslant \tfrac{1}{2}\lambda^2 \min_\theta(\varphi_n(r_1\cos\theta, r_1\sin\theta) + \log r_1) = \tfrac{1}{2}\lambda^2 \min_\theta \tilde{\varphi}_n(0,\theta).$$

This implies

$$M_n(v_1) \leqslant \tfrac{1}{2}\lambda^2 m_n(v_1) + 2C.$$

Similarly, we obtain

$$M_n(v_2) \leqslant \tfrac{1}{2}\mu^2 m_n(v_2) + 2C.$$

For any $v_3 \in (v_1, v_2)$, the inequality at the top of this page implies

$$M_n(v_3) \leqslant \tfrac{1}{2}\min(\lambda^2, \mu^2)m_n(v_3) + 4C;$$

thus

$$\max_\theta \tilde{\varphi}_n(v_3, \theta) \leqslant \tfrac{1}{2}\min(\lambda^2, \mu^2) \min_\theta \tilde{\varphi}_n(v_3, \theta) + 6C,$$

and so

$$\max_\theta(\varphi_n(r_3, \theta) + \log r_3) \leqslant \tfrac{1}{2}\min(\lambda^2, \mu^2) \min_\theta(\varphi_n(r_3, \theta) + \log r_3) + 6C.$$

Because $\varphi + \log r \to -\infty$ when $r \to 0$, and C is independent of r, we see that if r_1 small enough and n big enough, then

$$\min_\theta(\varphi_n(r_3, \theta) + \log r_3) \leqslant \max_\theta(\varphi_n(r_3, \theta) + \log r_3) \leqslant c \min_\theta(\varphi_n(r_3, \theta) + \log r_3)$$

for any $r_3 \in (r_2\delta_n, r_1)$, where $c \in (0, 1)$ is a constant. This immediately implies that the max-min inequality holds in Neck(r_1, r_2). $\qquad\square$

Proof of [Main Theorem 2.6](). Case 1: The renormalized limit metric ϕ does not vanish. According to the definition of the weak angle and the out angle, for r_1 small enough and r_2, n large enough we have

$$\left| \int_{|z|=r_1} k_{g_n} ds_n - \alpha \right| < \varepsilon, \quad \left| \int_{|z|=r_2\delta_n} k_{g_n} ds_n - \beta \right| < \varepsilon.$$

By Lemmas [3.7]() and [3.9](), we have $\alpha \geqslant 0$ and $\beta \leqslant 0$. If $\alpha > 0$, there exists a circle $|z| = r_{3,n}$ such that

$$\int_{|z|=r_{3,n}} k_{g_n} ds_n = \frac{\alpha}{2},$$

and we can assume that $r_{3,n}$ is maximal satisfying this condition. Define

$$\bar{\varphi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r \cos\theta, r \sin\theta) \, d\theta,$$

$$\bar{\varphi}_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(r \cos\theta, r \sin\theta) \, d\theta.$$

Then

$$
\begin{aligned}
(4\text{-}2) \quad \iint_{r_{3,n} \leqslant |z| \leqslant r_1} \frac{\alpha}{2} e^{2\bar{\varphi}_n} r \, dr \, d\theta &\leqslant \frac{1}{2} \int_0^{2\pi} \int_{r_{3,n}}^{r_1} e^{2\bar{\varphi}_n} r \int_{|z|=r} k_{g_n} ds_n \, dr \, d\theta \\
&= \pi \int_{r_{3,n}}^{r_1} e^{2\bar{\varphi}_n} r (\partial_r \bar{\varphi}_n r + 1) dr \\
&= \pi \int_{r_{3,n}}^{r_1} e^{2(\bar{\varphi}_n + \log r)} \left(\partial_r \bar{\varphi}_n + \frac{1}{r} \right) dr \\
&= \frac{\pi}{2} e^{2(\bar{\varphi}_n + \log r)} \Big|_{r_{3,n}}^{r_1}.
\end{aligned}
$$

Using the max-min inequality ([Lemma 4.1]()), one easily gets

$$(4\text{-}3) \quad e^{2(\bar{\varphi}_n + \log r)}|_{r_1} \leqslant \left(\frac{1}{2\pi} \int_0^{2\pi} e^{(\varphi_n + \log r)|_{r_1}} \, d\theta \right)^{1/c},$$

where $c \in (0, 1)$ is the constant in the max-min inequality. Since

$$\int_0^{2\pi} e^{(\varphi_n + \log r)|_{r_1}} \, d\theta < \epsilon$$

(from the blowing up process), inequalities [(4-2)]() and [(4-3)]() then imply

$$\iint_{r_{3,n} \leqslant |z| \leqslant r_1} \frac{\alpha}{2} e^{2\bar{\varphi}_n} r \, dr \, d\theta \to 0.$$

If $\alpha > 0$, then

$$\iint_{r_{3,n} \leqslant |z| \leqslant r_1} e^{2\bar{\varphi}_n} r \, dr \, d\theta \to 0$$

as $r_1 \to 0$, $r_2 \to \infty$ and $n \to \infty$. Using the max-min inequality again, we obtain

$$\iint_{r_{3,n} \leqslant |z| \leqslant r_1} e^{2\varphi_n} r \, dr \, d\theta \to 0.$$

On the other hand, for the annulus bounded by two concentric circle $|z| = r_1$ and $|z| = r_{3,n}$, we apply the Gauss–Bonnet Theorem to get

$$(4\text{-}4) \qquad \int_{|z|=r_1} k_{g_n} ds_n - \int_{|z|=r_{3,n}} k_{g_n} ds_n + \int_{r_{3,n} \leqslant |z| \leqslant r_1} K_n \, dg_n = 0,$$

while

$$\left(\int_{r_{3,n} \leqslant |z| \leqslant r_1} K_n \, dg_n \right)^2 \leqslant \int_{r_{3,n} \leqslant |z| \leqslant r_1} K_n^2 dg_n \int_{r_{3,n} \leqslant |z| \leqslant r_1} dg_n$$

and

$$\int_{r_{3,n} \leqslant |z| \leqslant r_1} dg_n = \iint_{r_{3,n} \leqslant |z| \leqslant r_1} e^{2\varphi_n} r \, dr \, d\theta \to 0.$$

This means that $\int_{r_{3,n} \leqslant |z| \leqslant r_1} K_n \, dg_n \to 0$. Taking the limit in (4-4), we get $\alpha - \frac{1}{2}\alpha = 0$, a contradiction. Thus we have proved that $\alpha = 0$ in this case.

Case 2: The renormalized limit metric ϕ_0 vanishes. If the out angle β is nonpositive, we apply the Gauss–Bonnet theorem directly:

$$\int_{|z|=r_1} k_{g_n} ds_n - \int_{|z|=r_2 \delta_n} k_{g_n} ds_n + \int_{r_2 \delta_n \leqslant |z| \leqslant r_1} K_n \, dg_n = 0.$$

Since the limit metric vanishes,

$$\int_{r_2 \delta_n \leqslant |z| \leqslant r_1} K_n \, dg_n \to 0.$$

Taking the limit in the last equality, we have $\alpha - \beta = 0$, so $\alpha = \beta = 0$.

If $\beta > 0$, we consider the bubble tree decomposition. The renormalized limit metric ϕ has two bubble points at least, by Theorem 1.2. Assume the bubble points are $\{q_1, q_2, \ldots, q_m\}$, with $m \geqslant 2$, and the weak angle at q_i is α_i. Let $h_n = \tilde{g}_n$ be the renormalized metrics, and take disks $D_{r^{(i)}}(q_i)$ around q_i with radius $r^{(i)}$ such that

$$\left| \alpha_i - \int_{\partial D_{r^{(i)}}(q_i)} k_{h_n} ds_n \right| < \varepsilon$$

for n big enough, and a fixed small ε. Set

$$\Omega = D_{r_2}(p) \setminus \bigcup_{1 \leqslant i \leqslant m} D_{r^{(i)}}(q_i)$$

(before renormalization, the disk is $D_{r_2 \delta_n}(p)$). Now apply to the Gauss–Bonnet Theorem to Ω:

$$(4\text{-}5) \qquad \int_{|z|=r_2} k_{h_n} ds_n - \sum_{i=1}^{m} \int_{|z|=r^{(i)}} k_{h_n} ds_n + \int_{\Omega} K_n dh_n = -(m-1).$$

Because $\{h_n\} \to 0$, $A_n(\Omega) = \int_{\Omega} dh_n \to 0$ when $n \to \infty$, we have

$$\left(\int_{\Omega} K_n dh_n \right)^2 \leqslant \int_{\Omega} K_n^2 dh_n \int_{\Omega} dh_n \to 0.$$

Taking limits in (4-5) we get $\sum_{i=1}^{m} \alpha_i = m - 1 + \beta$. Thus there must exist some i with $1 \leqslant i \leqslant m$ such that $\alpha_i > 0$. We can assume $\alpha_1 > 0$, and consider a pair of angles $\{\alpha_1, \beta_1\}$ (where β_1 is the out angle at the point q_1) and the blowing-up process around $\{q_1\}$. If $\beta_1 \leqslant 0$, then by the argument above we have $\alpha_1 = 0$, a contradiction. If $\beta_1 > 0$, we argue as in the case $\beta > 0$: we get a new pair of angles $\{\alpha_{11}, \beta_{11}\}$ at the bubble point q_{11} in the blowing-up process around q_1, such that $\alpha_{11} > 0$. If $\beta_{11} \leqslant 0$, there is a contradiction. If $\beta_{11} > 0$, then we apply the blowing-up once more. Since $\beta > 0$, the limit metric of renormalization is zero and the limit metric is a ghost vertex in the bubble tree. Thus, if $\alpha > 0$, we get an infinite series of ghost vertices in the bubble tree. But we know that there are only finitely ghost vertices in the bubble tree, so we reach a contradiction. This proves that α is zero in this case too, as desired. $\square$

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