

PBW-DEFORMATION THEORY AND REGULAR CENTRAL EXTENSIONS

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ABSTRACT. A deformation U , of a graded K -algebra A is said to be of PBW type if grU is A . It has been shown for Koszul and N -Koszul algebras that the deformation is PBW if and only if the relations of U satisfy a Jacobi type condition. In particular, for these algebras the determination of the PBW property is a *finite* and explicitly determined linear algebra problem. We extend these results to an arbitrary graded K -algebra, using the notion of central extensions of algebras and a homological constant attached to A which we call the *complexity* of A .

1. INTRODUCTION

We are interested in certain deformations of graded algebras A over a field K obtained by altering the defining relations of A . For reasons explained below, these deformations are called PBW-deformations. An example is the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra \mathfrak{g} , which is a deformation of the symmetric algebra $\mathbb{S}\mathfrak{g}$. In general, determining when a given deformation U of A is a PBW-deformation is a very difficult problem. However it has been shown by Polishchuk and Positselsky [13] and Braverman and Gaiitsgory [4] that for a Koszul algebra A it suffices to verify a comparatively simple condition on the defining relations of U . In the case $A = U\mathfrak{g}$ this condition

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coincides with the Jacobi identity for the Lie bracket on \mathfrak{g} , and so we call it the Jacobi condition.

Motivated by the cubic Artin-Schelter (AS) regular algebras, Berger [2] introduced the notion of N -Koszul algebras as a generalization of Koszul algebras. A graded algebra A is N -Koszul if all of its relations are in degree N and its Yoneda Ext-algebra is generated by $Ext^1(K, K)$ and $Ext^2(K, K)$. Two recent papers ([3] and [9]) have shown that a Jacobi type condition holds for N -Koszul algebras. In particular, Fløystad and Vatne [9] explicitly calculate the PBW-deformations of the cubic AS regular algebras. Remarkably, the stipulations in their list of PBW-deformations are identical to those given previously in [5] for central extensions of cubic AS regular algebras to be AS regular. Our work here was motivated by the desire to explain this phenomenon.

Throughout this paper the term K -algebra is used to denote an algebra over the field K that is finitely generated by degree 1 elements and has a finitely generated set of relations, i.e. an algebra of the form

$$A = K\langle x_1 \dots x_n \rangle / \langle r_1, \dots, r_m \rangle.$$

We may and will assume that none of the relations r_i are linear. Such an algebra inherits a filtration, $F^k(A)$, from the standard filtration on the free algebra $K\langle x_1, \dots, x_n \rangle$.

The free algebra $K\langle x_1 \dots, x_n \rangle$ also carries a standard grading. If the relations r_1, \dots, r_m are all homogeneous, then A inherits a grading from the free algebra. We denote the component of degree n by A_n and we use the phrase *graded K -algebra* to describe this situation. We will assume that r_1, \dots, r_m is a minimal set of relations for A . When A is graded it is also augmented by $A \rightarrow A/A_+ = K$.

Let A be a graded K -algebra as above, with relations $R = \{r_1, \dots, r_m\}$. Let T be the free algebra $K\langle x_1, \dots, x_n \rangle$. By a *deformation* of A we mean a K -algebra $U = T/\langle r_1 + l_1, \dots, r_m + l_m \rangle$, where l_1, \dots, l_m are (not necessarily homogenous) elements of T such that $deg(l_i) < deg(r_i)$ for all i . The filtered algebra U is not, in general, graded, but it has an associated graded algebra $gr(U) = \oplus gr(U)_k$, where $gr(U)_k = F^k(U)/F^{k-1}(U)$.

The classical Poincaré-Birkhoff-Witt Theorem says that for a Lie algebra \mathfrak{g} the associated graded ring $gr(U\mathfrak{g})$ is isomorphic to the symmetric algebra $\mathbb{S}\mathfrak{g}$. This observation inspired the following definition (see [13] or [4]).

Definition 1.1. The deformation U , of the graded K -algebra A , is said to be a PBW-deformation if grU is isomorphic to A as a graded algebra.

PBW-deformations of quadratic algebras have been studied recently in [8], [10] and [7]. PBW-deformations of N -homogeneous algebras have been studied recently in [3] and [9]. Deciding whether or not a given deformation is PBW appears to be an infinite linear algebra problem. The primary goal of this paper is to determine the conditions under which this can be reduced to a *finite* linear calculation and to ascertain precisely what that linear problem entails. We answer these questions by translating them into questions about a larger graded ring.

Let $T[z] = T \otimes_K K[z]$ be the polynomial extension of T . This is naturally graded by the Künneth formula. Define a function $h : T \rightarrow T[z]$ as follows. If $f = \sum_{i=0}^p f_i$, with $f_i \in T_i$ and $f_p \neq 0$, then $h(f) = \sum_{i=0}^p z^{p-i} f_i$. We call $h(f)$ the *homogenization* of f in $T[z]$.

Let A be a graded K -algebra and U a deformation of A , as above. The *central extension* of the graded K -algebra A associated to U is the algebra $D := T[z]/\langle h(r_1+l_1), \dots, h(r_m+l_m) \rangle$. This algebra is a graded K -algebra with one more generator than A and n more relations (the commutation relations for z .)

Definition 1.2. The central extension D of A associated to U is said to be a *regular central extension* if the element z in D is not a zero divisor. The central extension is *regular to degree p* if $Ann_D(z^n)_k = 0$ for all $k \leq p$ and all $n \geq 1$.

We relate the notion of PBW-deformation directly to the notion of regular central extension through the following theorem.

Theorem 1.3. *The deformation U of the graded algebra A is a PBW-deformation if and only if the associated central extension D is a regular central extension.*

It is very simple to show that the Jacobi condition of [4] or [13] for deformations of Koszul algebras is equivalent to the statement that z is not a zero divisor on D_2 . Likewise the Jacobi condition from [3] or [9] for N -Koszul algebras can be interpreted as the statement that z is not a zero divisor on D_N (and hence on $D_{\leq N}$). In both cases, the Jacobi condition assures that

the deformation is PBW. In light of the theorem above, this suggests that in general one may be able to determine the regularity of the element z from knowing its regularity only in some specific low degree. This is exactly the case and the appropriate degree is given by the following definition.

Definition 1.4. The *complexity* of the graded K -algebra A is

$$c(A) = \sup\{n \mid \text{Ext}^{3,n}(K, K) \neq 0\} - 1$$

if the global dimension of A is at least 3. For global dimension less than 3 we set $c(A) = 0$.

This homological invariant of the algebra A provides the precise measurement required to bound the size of the linear algebra problem attached to the PBW property. We can now state our main theorem.

Theorem 1.5. *Let A be a graded K -algebra of finite complexity $c = c(A)$. Let D be a central extension of A by the variable z . Then z is regular in D if and only if z is not a zero divisor on $D_{\leq c}$.*

Since the complexity of a Koszul algebra is 0 or 2, and the complexity of an N -Koszul algebra is 0 or N , the PBW deformation results cited earlier in these two cases become special cases of 1.5. On the other hand, a quadratic algebra can easily have complexity 2 without being Koszul, so even in the quadratic setting 1.5 is seen to strengthen the classical result. Moreover, the theorem has the advantage of dealing with algebras with relations in mixed degrees and does not in any way rely on homological purity.

Theorem 1.5 is a consequence of a more complete statement given in section 3 as Theorem 3.6. This Theorem also gives a simple criterion for regularity of z through a single matrix equation involving the minimal projective resolution of the trivial module K ,

$$\pi_D(M_3 f_2 + f_3 M_1) = 0.$$

This is probably the most compact and easily verified condition with which to check regularity. We refer the reader to section 3 for the necessary notation.

It may be convenient to have the regularity criterion of Theorem 1.5 re-expressed in terms of the original deformation U . This is accomplished by translating the hypothesis: z is not a zero divisor on $D_{\leq c}$, back into a set

of conditions on the relations of U . We refer the reader to section 4 for the precise formulation of this translation, which we call a Jacobi condition.

Section 2 contains the proof of Theorem 1.3 in a slightly stronger form. Section 3 introduces the necessary constructions and homological algebra required to prove Theorem 3.6, and as a corollary Theorem 1.5. In section 4 we give the Jacobi condition which solves the original problem and in section 5 we present a small number of interesting and motivating examples.

Finally, we want to note the relationship of this paper to another line of inquiry. The following lemma is easy to prove.

Lemma 1.6. *If $Ext_A^3(K, K)$ is algebraically generated by $Ext_A^1(K, K)$ and $Ext_A^2(K, K)$, then the complexity of A is no greater than the maximum of the degrees of the relations of A .*

In other words, having $Ext_A(K, K)$ generated in low cohomological degree is an effective way to limit the size of $c(A)$. Green et. al. [11] have shown N -Koszul algebras satisfy the hypothesis of the lemma, and it is well known that the class of Koszul algebras is precisely the class of graded algebras for which $Ext_A(K, K)$ is generated by $Ext_A^1(K, K)$. The lemma, together with theorem 1.5 leads us to make the following definition.

Definition 1.7. We say that the graded K -algebra A is \mathcal{K}_2 if $Ext_A(K, K)$ is generated as an algebra by $Ext_A^1(K, K)$ and $Ext_A^2(K, K)$.

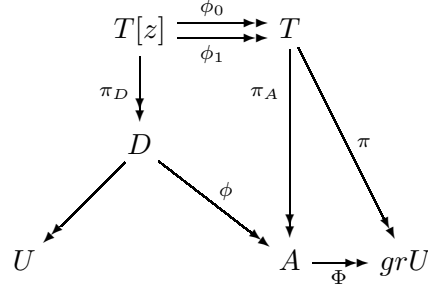
The class of \mathcal{K}_2 algebras will be studied in depth in the companion paper [6].

2. PROOF OF THEOREM 1.3

Theorem 1.3 will be an immediate consequence of the following stronger statement. Throughout we have a fixed graded K -algebra A , a deformation U and the associated central extension D .

Let $R = \{r_1, \dots, r_m\}$ be the set of relations of A and $P = \{r_1 + l_1, \dots, r_m + l_m\}$ the set of relations of U . The set of relations of D in $T[z]$ is then $h(P)$. We note: $\langle h(P) \rangle \neq h(\langle P \rangle)$ in $T[z]$. Let $\pi : T \rightarrow grU$ be the natural epimorphism. Since $\pi(R) = 0$, we may factor π through A to obtain $\Phi : A \rightarrow gr(U)$ given by $\Phi(x + \langle R \rangle) = \pi(x)$. This graded K -algebra homomorphism is always surjective.

Let $\pi_D : T[z] \rightarrow D$ be the canonical epimorphism. We consider T to be a subalgebra of $T[z]$. Let $\phi_1, \phi_0 : T[z] \rightarrow T$ be the maps which evaluate at $z = 1$ and $z = 0$ respectively. These maps and those used in the next section are summarized in the following diagram.



Theorem 2.1. *The map $\Phi : A \rightarrow grU$ is injective in degrees $1 \leq i \leq p$ if and only the central extension D is regular to degree p .*

Proof. Assume first D is not regular to degree p . This means that the set

$$S = \{x \in T[z]_{\leq p} \setminus \langle h(P) \rangle \mid z^k x \in \langle h(P) \rangle \text{ for some } k > 0\}$$

is not empty. We choose $x \in S$ of minimal degree and choose the smallest k for which $z^k x \in \langle h(P) \rangle$. We may then write $x = x' + zy$ for some $x' \in T$ and $y \in T[z]$ with $\deg(y) < \deg(x)$. Since $z^k(x' + zy) \in \langle h(P) \rangle$, $\phi_1(z^k(x' + zy)) = x' + \phi_1(y) \in \langle P \rangle$. It follows that $\pi(x')$ is 0 in $gr(U)$ and hence $x' + \langle R \rangle$ is in the kernel of Φ .

We claim that x' is not in $\langle R \rangle$, which will prove that Φ is not injective on $A_{\leq p}$. Suppose to the contrary that $x' \in \langle R \rangle$. We may choose a representation of x' as $x' = \sum_{i=1}^m \sum_j u_{i,j} r_i w_{i,j}$. Set $\tilde{x} = \sum_{i=1}^m \sum_j u_{i,j} h(r_i + l_i) w_{i,j} \in \langle h(P) \rangle$. Since $h(r_i + l_i) - r_i$ is a multiple of z (or zero) for every i , we see that $\tilde{x} - x' = zy'$ for some $y' \in T[z]$. We then have $z^k(x - \tilde{x}) = z^k(z(y - y')) \in \langle h(P) \rangle$. By the minimality of x , we conclude $y - y' = 0$. But then $x = x' + zy' = \tilde{x} \in \langle h(P) \rangle$, which is a contradiction.

For the converse, assume that D is regular to degree p . Choose a homogeneous $x \in T_{\leq p}$ such that $x + \langle R \rangle$ is in the kernel of Φ . Then there is some (not necessarily homogeneous) $y \in T$ of lower degree than x , such that $x + y$ is in $\langle P \rangle$. Let $k = \deg(x) - \deg(y) > 0$. Since $\phi_1^{-1}(\{x + y\}) \cap T[z]_{hom} = \{z^j(x + z^k h(y)) \mid j \geq 0\}$ there exists some non-negative integer j such that

$z^j(x + z^k h(y))$ is in $\langle h(P) \rangle$. Since z^j is not a zero divisor on $D_{\leq p}$, this means $x + z^k h(y)$ is in $\langle h(P) \rangle$. Since $k > 0$ we then get $\phi_0(x + z^k y) = x \in \langle R \rangle$. Hence Φ is injective, as required. \square

3. PROOF OF THEOREM 1.5

Throughout this section, let A and D be as in the previous sections, that is A is a graded K -algebra and D a central extension of $A = D/Dz$, with $z \in D_1$ central in D . Let $\phi : D \rightarrow A$ be the associated graded ring epimorphism (evaluation at $z = 0$).

We extend ϕ to the functor $\phi : \text{Mod-}D \rightarrow \text{Mod-}A$ via $M \mapsto M/zM$. We also use ϕ to denote the epimorphism $\phi : T[z] \rightarrow T$ given by evaluation at $z = 0$.

For any graded vector space $V = \bigoplus_k V_k$, let $V(j)$ be the same vector space with the shifted grading $V(j)_k = V_{j+k}$. We will be dealing with graded-free resolutions of A and D modules, with generators in many different degrees, so it is convenient to introduce the notation $A(j_1, \dots, j_p) = \bigoplus_{i=1}^p A(j_i)$.

Let (Q, δ) be a fixed minimal projective resolution of the trivial left A -module $A/A_+ = K$

$$\rightarrow Q^3 \xrightarrow{\delta_3} Q^2 \xrightarrow{\delta_2} Q^1 \xrightarrow{\delta_1} Q^0 \rightarrow K \rightarrow 0.$$

The modules Q^k are graded-free modules, but may not be finitely generated. However, the number of generators in any given degree must be finite since A is locally finite as a graded vector space. Recall that $\pi_A : T \rightarrow A$ is the canonical epimorphism. We choose a graded basis for each Q^k and we choose a matrix M_k , with homogeneous entries from T , such that right multiplication by $\pi_A(M_k)$ represents $\delta_k : Q^k \rightarrow Q^{k-1}$. Since the resolution Q is minimal, we may assume that $Q^0 = A$, $Q^1 = A(-1, \dots, -1)$ and the matrix M_1 is the column $(x_1 \ x_2 \ \dots \ x_n)^t$. Similarly we may assume the finite matrix M_2 is chosen so that $M_2 M_1 = (r_1, \dots, r_m)^t$. In general, the matrix M_n , for $n > 2$, may not be finite, but local finiteness assures that it will have at most finitely many nonzero entries in every row and at most finitely many entries of a given degree in any column.

Since the resolution Q consists of graded-free modules, we may express each Q^n as:

$$Q^n = A(m_{1,n}, m_{2,n}, \dots, m_{t_n,n}).$$

For $n > 2$, t_n may be countably infinite. We will abuse notation slightly by treating t_n as if it is finite, but the reader should have no trouble making the appropriate adjustment in the constructions that follow. Note that the shifts $m_{i,n}$ are all nonpositive, indeed $m_{i,n} \leq -n$ for all i . The matrix M_n is a t_n by t_{n-1} -matrix, whose i - j entry is a homogeneous element of T of degree $m_{i,n} - m_{j,n-1}$.

We record the following obvious lemma.

Lemma 3.1. *If the global dimension of A is at least 3, then the complexity $c(A)$ is the supremum of the degrees of the entries of the matrix M_3M_2 . In particular, $c(A)$ is finite if and only if M_3 is a finite matrix.*

We now turn to building sequences of graded-free modules for the central extension D . Let $\hat{Q}^0 = D$ and for $n \geq 1$ put

$$\hat{Q}^n = D(m_{1,n}, m_{2,n}, \dots, m_{t_n,n}) \oplus D(m_{1,n-1}, m_{2,n-1}, \dots, m_{t_{n-1},n-1})(-1).$$

We exhort the reader to remark on the extra degree shift in the second term of this definition. Let I_t denote the t by t identity matrix. For each $n > 1$, let f_n be a t_n by t_{n-2} matrix with homogeneous entries from $T[z]$ which satisfies:

$$\pi_D(M_n M_{n-1} - (-1)^{n-1} z f_n) = 0.$$

Such f_n exists because the entries of $M_n M_{n-1}$ are all in $\langle R \rangle$. Next, define matrices \hat{M}_n , with homogeneous entries in $T[z]$, by

$$\hat{M}_1 = \begin{pmatrix} M_1 \\ z \end{pmatrix}$$

and for $n \geq 2$

$$\hat{M}_n = \begin{pmatrix} M_n & f_n \\ (-1)^{n-1} z I_{t_{n-1}} & M_{n-1} \end{pmatrix}.$$

These matrices may well be infinite, but they have the same finiteness conditions on rows and columns as the matrices M_n . Finally, we define $\hat{\delta}_n : \hat{Q}^n \rightarrow \hat{Q}^{n-1}$ to be right multiplication by the matrix $\pi_D(\hat{M}_n)$ (and $\hat{\delta}_0 = 0$).

Set $\hat{Q} = \bigoplus_n \hat{Q}^n$ and $\hat{\delta} = \sum_n \hat{\delta}_n$.

Remark 3.2. There is no assurance that $(\hat{Q}, \hat{\delta})$ is a complex. Indeed, determining when it is a complex is our main concern. We do have, by construction, $\hat{\delta}_1 \hat{\delta}_2 = 0$ since $\pi_D(\hat{M}_2 \hat{M}_1) = 0$. Hence $\hat{Q}^2 \rightarrow \hat{Q}^1 \rightarrow \hat{Q}^0 \rightarrow 0$, at least, is a complex. It is not difficult to see that the homology of this complex is $H^0(\hat{Q}) = K$ and $H^1(\hat{Q}) = 0$.

For $n > 2$, $\hat{\delta}_{n-1}\hat{\delta}_n$ is given by right multiplication by the matrix $\pi_D(\hat{M}_n\hat{M}_{n-1})$, and by the centrality of z , we see that this matrix is

$$\begin{pmatrix} 0 & \pi_D(M_n f_{n-1} + f_n M_{n-2}) \\ 0 & 0 \end{pmatrix}.$$

Lemma 3.3. *Suppose that $(\hat{Q}, \hat{\delta})$ is a complex. Then the A -module complex $(\phi(\hat{Q}), \phi(\hat{\delta}))$ has the homology: $H^0(\phi(\hat{Q})) = K$, $H^1(\phi(\hat{Q})) = K(-1)$ and $H^p(\phi(\hat{Q})) = 0$ for $p > 1$.*

Proof. Since $(\hat{Q}, \hat{\delta})$ is a complex we must have $\pi_D(M_n f_{n-1} + f_n M_{n-2}) = 0$ for all $n > 2$. Applying ϕ we get $\pi_A(M_n \phi(f_{n-1}) + \phi(f_n) M_{n-2}) = 0$.

For $n > 0$, $\phi(\hat{Q}^n) = Q^n \oplus Q^{n-1}(-1)$ and $\phi(\hat{Q}^0) = Q^0$. The differential of $\phi(\hat{Q})$, for $n \geq 2$, is

$$\phi(\hat{\delta}_n) = \pi_A \begin{pmatrix} M_n & \phi(f_n) \\ 0 & M_{n-1} \end{pmatrix}.$$

To ease notation, we suppress π_A in all of the following calculations. For $n \geq 2$, let (p, p') be in the kernel of $\phi(\hat{\delta}_n)$. Then $(pM_n, p\phi(f_n) + p'M_{n-1}) = (0, 0)$. Since (Q, δ) is acyclic, we may choose $q \in Q^{n+1}$ for which $qM_{n+1} = p$. Then $(-q\phi(f_{n+1}) + p')M_{n-1} = qM_{n+1}\phi(f_n) + p'M_{n-1} = 0$ and we may choose $q' \in Q^n(-1)$ for which $q'M_n = (-q\phi(f_{n+1}) + p')$. Clearly then $\hat{\delta}_{n+1}(q, q') = (p, p')$. We have $H^n(\phi(Q)) = 0$ for all $n \geq 2$. Since $\phi(\hat{\delta}_1)$ is δ_1 on Q^1 and zero on $Q^0(-1)$, we see that $H^0(\phi(\hat{Q})) = K$.

Finally, $\ker(\phi(\hat{\delta}_1)) = \ker(\delta_1) \oplus Q^0(-1) = \text{im}(\delta_2) \oplus A(-1)$. On the other hand, $\text{im}(\phi(\hat{\delta}_2)) = \text{im}(\delta_2) \oplus (Q^2\phi(f_2) + Q^1(-1)M_1)$. By hypothesis, $Q^1M_1 = \text{im}(\delta_1) = A_+$. By degree considerations, $Q^2\phi(f_2)$ must be contained in $Q^1(-1)M_1$. Thus $H^1(\phi(\hat{Q})) = K(-1)$. \square

It is clear from the proof of the Lemma that when \hat{Q} is a complex, then $\phi(\hat{Q})$ is the mapping cone of a morphism of complexes $f : Q(-1) \rightarrow Q$. The following Lemma has the same proof as the Lemma above.

Lemma 3.4. *Suppose that $\hat{Q}^3 \rightarrow \hat{Q}^2 \rightarrow \hat{Q}^1 \rightarrow \hat{Q}^0 \rightarrow 0$ is a complex. Then the complex $\phi(\hat{Q}^3) \rightarrow \phi(\hat{Q}^2) \rightarrow \phi(\hat{Q}^1) \rightarrow \phi(\hat{Q}^0) \rightarrow 0$ has homology: $H^0 = K$, $H^1 = K(-1)$ and $H^2 = 0$.*

It seems that the following lemma should be standard, but we include its proof.

Lemma 3.5. *If the central element z is regular in D , then $(\hat{Q}, \hat{\delta})$ is a complex and a minimal projective resolution of the trivial graded D -module K .*

Proof. Assume z is regular. We have

$$\begin{aligned} z \cdot \pi_D(M_n f_{n-1} + f_n M_{n-2}) &= \pi_D(M_n z f_{n-1} + z f_n M_{n-2}) \\ &= \pi_D((-1)^n M_n M_{n-1} M_{n-2} \\ &\quad + (-1)^{n-1} M_n M_{n-1} M_{n-2}) \\ &= 0 \end{aligned}$$

Thus $\pi_D(M_n f_{n-1} + f_n M_{n-2}) = 0$, and so by Remark 3.2, $(\hat{Q}, \hat{\delta})$ is a complex.

By the definition of $\hat{\delta}_1$ we have $H^0(\hat{Q}) = K$. Consider the short exact sequence of complexes

$$0 \rightarrow z\hat{Q} \rightarrow \hat{Q} \rightarrow \phi(\hat{Q}) \rightarrow 0.$$

Since \hat{Q} is a complex of z -torsion-free modules, we have $z\hat{Q} \cong \hat{Q}(-1)$ and moreover $H(z\hat{Q}) = H(\hat{Q}(-1)) = H(\hat{Q})(-1)$. In particular, $H^0(z\hat{Q}) = K(-1)$.

If we apply this information to the long exact homology sequence induced by the short exact sequence above, recalling from Lemma 3.3 that $H^0(\phi(\hat{Q})) = K$, $H^1(\phi(\hat{Q})) = K(-1)$ and $H^n(\phi(\hat{Q})) = 0$ for $n > 1$, we obtain

$$0 \rightarrow H^1(z\hat{Q}) \rightarrow H^1(\hat{Q}) \rightarrow K(-1) \rightarrow K(-1) \rightarrow K \rightarrow K \rightarrow 0$$

and for all $n > 1$,

$$0 \rightarrow H^n(z\hat{Q}) \rightarrow H^n(\hat{Q}) \rightarrow 0.$$

Thus for all $n \geq 1$, $H^n(\hat{Q}) = H^n(z\hat{Q}) = H^n(\hat{Q})(-1)$. Since all of these modules are bounded below in internal degree, this can only happen if they are all 0. This proves that $(\hat{Q}, \hat{\delta})$ is a projective resolution of K . Since the modules in \hat{Q} are all graded and every nonzero entry of the matrices \hat{M}_n is homogeneous of degree at least 1, it follows immediately that the resolution is minimal. \square

As we will see in the following theorem, the regularity of z in D can be detected entirely by the $\hat{Q}^3 \rightarrow \hat{Q}^2 \rightarrow \hat{Q}^1$ portion of \hat{Q} .

Theorem 3.6. *Let D be a central extension of A . Then the following statements are equivalent:*

- (1) *The central element $z \in D$ is regular in D .*
- (2) *The sequence of graded D -modules $(\hat{Q}, \hat{\delta})$ is a complex.*

(3) The truncated sequence of graded D -modules $\hat{Q}^3 \rightarrow \hat{Q}^2 \rightarrow \hat{Q}^1 \rightarrow \hat{Q}^0 \rightarrow 0$ is a complex.

(4) $\pi_D(M_3 f_2 + f_3 M_1) = 0$ in D .

(5) The sequence of graded D -modules $(\hat{Q}, \hat{\delta})$ is a minimal projective resolution of K .

Proof. In light of Lemma 3.5 and Remark 3.2, we are required only to prove that (3) implies (1) and so we begin by assuming (3). Let $'\hat{Q}$ be the truncated complex $\hat{Q}^3 \rightarrow \hat{Q}^2 \rightarrow \hat{Q}^1 \rightarrow \hat{Q}^0 \rightarrow 0$.

Let $L = \text{Ann}_D(z)$. We must show that this graded two-sided ideal of D is zero. Let $\alpha : D(-1) \rightarrow Dz$ be the epimorphism $b \mapsto bz$, with kernel $L(-1)$. Since $'\hat{Q}$ consists of free D -modules, we can extend α appropriately and get a short exact sequence of complexes:

$$0 \rightarrow L \otimes_D '\hat{Q}(-1) \rightarrow '\hat{Q}(-1) \xrightarrow{\alpha} z'\hat{Q} \rightarrow 0.$$

We also have the short exact sequences of complexes

$$0 \rightarrow z'\hat{Q} \rightarrow '\hat{Q} \rightarrow \phi(''\hat{Q}) \rightarrow 0.$$

The long exact homology sequence of the second of these short exact sequences, taking into account Lemma 3.4, yields

$$0 \rightarrow H^1(z'\hat{Q}) \rightarrow H^1(''\hat{Q}) \rightarrow K(-1) \rightarrow H^0(z'\hat{Q}) \rightarrow K \rightarrow K \rightarrow 0.$$

Since $H^1(''\hat{Q}) = 0$ (by Remark 3.2), we can now conclude $H^1(z'\hat{Q}) = 0$ and $H^0(z'\hat{Q}) = K(-1)$.

The long exact homology sequence of the first short exact sequence above now yields

$$0 \rightarrow H^0(L \otimes '\hat{Q}(-1)) \rightarrow H^0(''\hat{Q}(-1)) \rightarrow K(-1) \rightarrow 0.$$

Since $H^0(''\hat{Q}(-1)) = H^0(''\hat{Q})(-1) = K(-1)$, we conclude $H^0(L \otimes '\hat{Q}(-1)) = H^0(L \otimes '\hat{Q})(-1) = 0$. But $H^0(L \otimes '\hat{Q}) = L/D_+L$. Since L is bounded below, this can only be 0 if L is 0, as required. \square

We have everything we need to prove Theorem 1.5, which follows from the restatement below.

Theorem 3.7. *Let A be a graded K -algebra of finite complexity $c = c(A)$. Let D be a central extension of A such that the extending variable z is not a zero divisor on $D_{\leq c}$. Then z is regular in D .*

Proof. From Theorem 3.6, it suffices to prove that $\pi_D(M_3f_2 + f_3M_1) = 0$. But $z \cdot \pi_D(M_3f_2 + f_3M_1) = 0$, as shown in the proof of 3.5. So it suffices to show that every entry of $M_3f_2 + f_3M_1$ has degree at most $c(A)$.

Recall that $Q^n = A(m_{1,n}, \dots, m_{t_n,n})$. We have $m_{i,1} = -1$ for all i and $\{-m_{1,2}, -m_{2,2}, \dots, -m_{t_2,2}\}$ is the set of degrees of the relations r_1, \dots, r_m ($m = t_2$). The i - j entry of M_2 thus has degree $-1 - m_{i,2}$. Similarly, the i - j entry of M_3 has degree $m_{j,2} - m_{i,3}$. It follows that the i - j entry of M_3M_2 has degree $-1 - m_{i,3}$. Since $\pi_D(M_3M_2 + f_3z) = 0$ in D , the i - j entry of f_3 must have degree $-2 - m_{i,3}$. Similarly, the i - j entry of f_2 has degree $-1 - m_{i,2}$. Putting this all together, we see that the i - j entry of $M_3f_2 + f_3M_1$ has degree $-1 - m_{i,3}$. By hypothesis, this is at most $c(A)$, as required. \square

4. JACOBI CONDITIONS

We return to the case of a deformation U of the graded algebra A . As before, let $R = \{r_1, \dots, r_m\}$ be the set of homogeneous relations of A and $P = \{r_1 + l_1, \dots, r_m + l_m\}$ be the set of nonhomogeneous relations of U . Let D be the central extension associated to this deformation.

We make an inductive definition of a nested family of subspaces of the free algebra T . Let V be the K -span of the generators x_1, \dots, x_n of T . Let $P_1 = \text{span}_K(P \cap F^1T)$. For $k > 1$, set

$$P_k = VP_{k-1} + P_{k-1}V + \text{span}_K(P \cap F^kT).$$

If $P_1 \neq 0$, we see at once that P_2 is the span of the quadratic elements of P , P_3 the span of the quadratic and cubic elements of P plus the cubic relations generated by P_2 , etc. It is clear that $P_1 \subset P_2 \subset P_3 \subset \dots$ and that $\cup_k P_k = \langle P \rangle$. It should also be clear that P_k is not, in general, equal to $\langle P \rangle \cap F^kT$. For example, if $P = \{x^2y - w, y^3 - w\}$, then $x^2w - wy^2 \in \langle P \rangle \cap F^3T$, but P_3 is just the K -span of P , which does not contain $x^2w - wy^2$.

We need one simple Lemma. Let $\phi_1 : T[z] \rightarrow T$ be the epimorphism given by evaluation at $z = 1$. Recall the homogenization function $h : T \rightarrow T[z]$. We omit the proof.

Lemma 4.1. *For all $k \geq 1$, $\phi_1(\langle h(P) \rangle_k) = P_k$.*

We can now restate Theorem 1.5 in the language of a Jacobi type condition.

Theorem 4.2. *Let A be a graded K -algebra of finite complexity $c(A)$ and let U be a deformation of A . Then U is a PBW-deformation of A if and only if $P_1 = 0$ and the following Jacobi condition is satisfied:*

$$P_{k+1} \cap F^k T \subset P_k \quad \text{for all } 1 \leq k \leq c(A).$$

Proof. Let D be the central extension of A associated to the deformation U . By combining theorems 1.5 and 1.3, we see that we are required to prove that z is not a zero divisor on $D_{\leq c(A)}$ if and only if the Jacobi condition is satisfied.

Assume the Jacobi condition and suppose also that z is a zero divisor on $D_{\leq c(A)}$. Then for some $x \in T[z]_k$, $k \leq c(A)$, with $x \notin \langle h(P) \rangle$, we have $zx \in \langle h(P) \rangle_{k+1}$. By lemma 4.1, we have $\phi_1(zx) = \phi_1(x) \in P_{k+1} \cap F^k T$. Therefore $\phi_1(x) \in P_k$. Again by 4.1, we can choose $y \in \langle h(P) \rangle_k$ with $\phi_1(y) = \phi_1(x)$. But the restriction of ϕ_1 to any homogeneous component of $T[z]$ is injective. Thus $x = y$, a contradiction.

Conversely, assume the Jacobi condition fails. Then for some $k \leq c(A)$, there is a $w \in P_{k+1} \cap F^k T$ such that $w \notin P_k$. Let $x = h(w) \in T[z]_k$. Then $\phi_1(x) = w$, so $x \notin \langle h(P) \rangle_k$, but $\phi_1(zx) = w \in P_{k+1}$, so $zx \in \langle h(P) \rangle_{k+1}$. Thus z is a zero divisor on $D_{\leq c(A)}$. \square

Remark 4.3. If A is N -Koszul then $P_i = 0$ for $i < N$ and P_N is the K -span of P . It then follows directly from Theorem 4.2 that the Jacobi condition given in [3] and [9] is equivalent to z being regular in degree N . Hence the PBW-deformation theorems in [3] and [9] for N -Koszul algebras can be recovered using the regularity theorem 1.5.

5. EXAMPLES

We conclude with a very short list of examples.

Example 5.1. Three dimensional Artin-Schelter (AS) regular algebras [1] are either quadratic or cubic. The quadratic ones are Koszul and hence have complexity two, while the cubic ones are 3-Koszul and have complexity three.

Four dimensional AS regular algebras, on the other hand, may have two, three or four generators. The 4-generated algebras are Koszul. The 3-generated algebras have two relations of degree 2 and 2 of degree 3 and satisfy a resolution of the form

$$0 \rightarrow A(-5) \rightarrow A(-4)^3 \rightarrow A(-3, -3, -2, -2) \rightarrow A(-1)^3 \rightarrow A \rightarrow K \rightarrow 0.$$

The complexity of such algebras is thus 3. Perhaps the most interesting case, from the complexity point of view, is the 2-generated AS regular algebras of dimension 4, which have one cubic relation, and one quartic relation. These algebras have been studied extensively by Lu et. al. [12]. For such an algebra there is a minimal resolution of the form

$$0 \rightarrow A(-7) \rightarrow A(-6)^2 \rightarrow A(-3, -4) \rightarrow A(-1)^2 \rightarrow A \rightarrow K.$$

Hence these algebras have complexity 5.

Infinite complexity is easily arranged.

Example 5.2. Let $A = K\langle w, x, y, z \rangle / \langle yz, zx - xz, zw \rangle$. Using the facts that $x^n z \neq 0$ in A but $yx^n z = 0$ in A , one can establish that A has global dimension 3 and a minimal projective resolution:

$$0 \rightarrow A(-3, -4, -5, \dots) \rightarrow A(-2)^3 \rightarrow A(-1)^4 \rightarrow A \rightarrow K \rightarrow 0.$$

Hence $c(A)$ is infinite.

And finally, using an essentially trivial example, we can show that even when the complexity of A is not 0, it need not be as large as the maximal degree of the relations of A .

Example 5.3. Let $A = K\langle x, y, z \rangle / \langle y^2, xyz \rangle$. Then A has relations of degrees 2 and 3, but the minimal resolution associated to A is

$$\dots A(-5) \rightarrow A(-4) \rightarrow A(-3) \rightarrow A(-2, -3) \rightarrow A(-1)^3 \rightarrow A \rightarrow K \rightarrow 0$$

and so the complexity of A is only 2.

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