Unipotent centralizers in algebraic groups

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1 Introduction

Let G be a simple algebraic group over an algebraically closed field K of characteristic $p > 0$, where p is a good prime for G, and let $u \in G$ be a unipotent element. We study the embedding of u in abelian unipotent subgroups and certain reductive subgroups of G and use this information to obtain information on $C_G(u)$. Of particular interest are $Z(C_G(u))$ and the reductive part of $C_G(u)$.

The process of embedding a unipotent element $u \in G$ in a connected, abelian, unipotent group is sometimes called *saturation*. In previous work [**?**] saturation results were achieved when $|u| = p$, where it was shown that there is a unique 1-dimensional unipotent group, the "saturation" of u, which contains u and which is contained in a *restricted* (sometimes called *good*) A_1 subgroup of G. Here an A_1 subgroup of G is restricted if all weights on the adjoint module of G are at most $2p - 2$. It was shown in [**?**] that these A_1 subgroups provide the basis for a variant of the Steinberg tensor product theorem, hence the name. It was also shown in [**?**] that $C_G(u)$ can be factored as a product of the unipotent radical and the centralizer of a restricted A_1 subgroup containing u.

In this paper we extend these results to cover unipotent elements of arbitrary order. We describe a certain 1-dimensional torus which normalizes $C_G(u)$ and then decompose $Z(C_G(u))^0$ into indecomposable summands, each invariant under the torus and such that u is contained in one of the summands, say W. If $|u| = p^r > p$, we show that the saturation of $u^{p^{r-1}}$, described above, coincides with the subgroup of elements of order at most p in W.

We next establish the existence of a pair of reductive groups J, R such that each is the centralizer of the other, u is a semiregular unipotent element of J, and R is the reductive part of $C_G(u)$. These subgroups provide insight into a number of issues surrounding unipotent elements. For instance, the reductive part of $C_G(u)$ appears as the centralizer of a reductive group and this is helpful in understanding the component group of $C_G(u)$. The restriction $L(G) \downarrow JR$ is relatively easy to understand, so that one can further restrict to find the Jordan blocks of u, fixed points, etc. Also, $Z(C_G(u))^0 \leq J$, so it may be possible to work within J to study this interesting subgroup of $C_G(u)$.

A little background material is necessary before we can state the main results. In particular we will make use of a certain type of 1-dimensional torus, which we now define.

Let T_0 be a maximal torus of $C_G(u)$ and set $D = C_G(T_0)'$, the derived group of a Levi subgroup of G. Then u is a *distinguished* unipotent element of D in the sense of Bala-Carter. This means that $C_D(u)^0$ is unipotent and lies in a uniquely determined parabolic subgroup $P = QL$ of D such that u is in the dense orbit of P on Q and such that $\dim(L) = \dim(Q/Q')$.

Let $T \leq Z(L) < D$ be a 1-dimensional torus such that T acts by weight 2 on all fundamental roots in $\Pi(D) - \Pi(L)$. We will say that T is a *u-distinguished 1-dimensional torus* if there exists a nilpotent element $e \in L(G)$ such that $C_G(u) = C_G(e)$ and T acts on $\langle e \rangle$ via weight 2. Note that this implies that T normalizes $C_G(u)$ and hence T also normalizes $Z(C_G(u))$.

The existence of nilpotent elements e as above follows from the existence of G-equivariant correspondences, called Springer maps, between the set of unipotent elements of G and the set of nilpotent elements of $L(G)$.

If T is any 1-dimensional torus, then a closed abelian unipotent group W will be called *T-homocyclic* if T acts on W without fixed points, $exp(W) = p^a$, and $W = W^{p^0} > W^p > W^{p^2} > \cdots > W^{p^a} = 1$ with successive quotients having dimension 1. Here W^{p^c} denotes the subgroup of W generated by elements w^{p^c} for $w \in W$. An inductive argument then shows that $W_{p^i} = W^{p^{a-i}}$ for $1 \leq i \leq a$, where W_{p^i} denote the group of elements of W whose order divides p^i.

The following result extends earlier work of Proud [**?**].

Theorem 1 *Let $u \in G$ be unipotent of order p^r. There exist a u-distinguished 1-dimensional torus T acting without fixed points on $Z(C_G(u))^0$ and a decomposition $Z(C_G(u))^0 = W_1 \oplus \cdots \oplus W_s$ (direct sum as abstract groups), such that each W_i is T-homocyclic, $exp(W_i) \geq exp(W_j)$ for $i \leq j$, and $u \in W_1$.*

With u and T as in Theorem 1, $W = W_1 = \langle u^T \rangle$ is a T-homocyclic group containing u. Clearly W is determined once T is given. However we shall see by example that when $|u| > p$, different choices of T may yield different homocyclic groups. Nonetheless, the next result shows that the subgroup of elements of order at most p in such a group is uniquely determined and coincides with the saturation of $u^{p^{r-1}}$.

Theorem 2 *Let $u \in G$ be unipotent of order p^r and let $W \leq Z(C_G(u))^0$ be a T-homocyclic group containing u, where T is a u-distinguished 1-dimensional torus. Then $W^{p^{r-1}} = U$, where U is the saturation of $v = u^{p^{r-1}}$.*

We now turn to the reductive part of $C_G(u)$. For this result we wish to allow for reductive groups which are not necessarily connected. So we use the term *reductive* to mean a group D for which D^0 is a connected reductive group and the component group is of order prime to p.

Theorem 3 *Let $u \in G$ be unipotent. There are reductive subgroups J and R of G such that*

i) $C_G(J) = R$ and $C_G(R) = J$.

ii) u is a semiregular element of J^0.

iii) $C_G(u) = QR$, where $Q = R_u(C_G(u))$.

In order to prove Theorem 3 it will be convenient to work with a certain reductive subgroup E of J which contains u and satisfies $C_G(E) = C_G(J) = R$. In particular, when $|u| = p$, then E can be taken as a restricted A_1 containing u. Here $R = C_G(E)$ and J can be defined to be $C_G(R)$, which we will show is reductive. If $|u| > p$ and G is of exceptional type, then the definition of E is more complicated and will be given explicitly in the tables of Lemma 5.4.

With T as in Theorem 1, there is an action of T on $C_G(u)$. The next theorem shows that for suitable choice of T the set of fixed points under this action is the reductive part of $C_G(u)$, recovering a result of Premet.

Theorem 4 *If $u \in G$ is unipotent, then there is a u-distinguished 1-dimensional torus $T < J$, with J as in Theorem 3, such that $C_G(u) \cap C_G(T) = C_G(J)$, the reductive part of $C_G(u)$.*

The group $Z = Z(C_G(u))^0$ is a group of considerable interest. At the moment, even the dimension of this subgroup remains a mystery. Combining the above results does provide some information. Theorem 1 establishes a certain decomposition of Z, while Theorem 3 implies that Z is contained in a particular reductive subgroup of G, namely $J = C_G(R)$. Also, Lemmas 2.2 and 2.4, to follow, show that $\overline{u^G \cap Z} = Z$ and $uZ^p \subset u^G$.

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2 Preliminaries

Fix notation as above and take G to be simply connected. Since p is assumed to be a good prime there is a G-equivariant correspondence, called a Springer map, between the set of unipotent elements of G and the set of nilpotent elements of $L(G)$.

When dealing with classical groups we fix a convenient Springer map. For SL we use the map $u \to u-1$ and for Sp or SO we use the map $u \to (1-u)/(1+u)$. In the latter case the map is self-inverse. In the case of orthogonal groups the classical group is not simply connected, but this map lifts to one for the simply connected group.

Fix $u \in G$ a unipotent element with $|u| = p^r$. Let e denote the nilpotent element which corresponds to u under the chosen Springer map. We remark that when working with exceptional groups we may later replace e by a conjugate so that $e \in L(W)$, with W as in Theorem 2.

We begin with two lemmas that were first established by Proud [**?**] with different proofs.

Lemma 2.1 *Set $Z = Z(C_G(u))^0$. Then*

i) $Z(C_G(u)) = Z(G) \times Z$.

ii) $Z = R_u(Z(C_G(u))$.

Proof We first show that if $s \in Z(C_G(u))$ is semisimple, then $s \in Z(G)$. Let B be a Borel subgroup containing su and $U = R_u(B)$. By assumption, $C_U(u) \leq C_U(s)$. If $C_U(s) = U$, then $s \in Z(G)$ as claimed. Supposing this is not the case let $E > C_U(s)$ be minimal among su-invariant subgroups of U which normalize $C_U(s)$. Note that minimality implies $[E, u] \leq C_U(s)$. We have $[[s, u], E] = 1$ and $[[u, E], s] \leq [C_U(s), s] = 1$. Hence, the three subgroups lemma implies $[[E, s], u] = 1$, so that $[E, s] \leq C_U(u) \leq C_U(s)$. Then s centralizes $E/[E, s]$ and $[E, s]$, so that that $E \leq C_U(s)$, a contradiction.

Next observe that $Z(C_G(u)) = C_G(C_G(u))$, so that by a result of Springer-Steinberg [SS, III, 3.15], all unipotent elements of $Z(C_G(u))$ are in Z. The result now follows from the Jordan decomposition and the fact that $Z(C_G(u))$ is abelian. ■

Lemma 2.2 *Set $Z = Z(C_G(u))^0 = R_u(Z(C_G(u))$.*

i) $Z = \overline{u^G \cap Z}$.

ii) Z has exponent $|u|$.

iii) $C_G(u) \sim_G C_G(v)$ if and only if $u \sim_G v$.

Proof Let $v \in Z$. Then $C_G(v) \geq C_G(u)$, so that $\dim(v^G) \leq \dim(u^G)$. Now Z is an irreducible variety and G has only finitely many unipotent classes. It follows that exactly one such class has dense intersection with Z and we take v to be a representative of this class. Then $u \in \overline{v^G \cap Z} \subset \overline{v^G}$. But the latter set is the union of v^G together with classes of strictly smaller dimension. It follows that $u \in v^G$, so that $u^G = v^G$ and $\overline{u^G \cap Z} = Z$ giving (i). Also, the subgroup of elements of Z of order strictly less than the exponent of Z is a closed subvariety, so (ii) follows from (i). Finally, for (iii) we must show that if $C_G(u)$ and $C_G(v)$ are conjugate, then so are u and v. For this we may assume that $C_G(u) = C_G(v)$. Then (i) implies that both $u^G \cap Z$ and $v^G \cap Z$ are dense in Z and as mentioned above there is just one class with this property. ■

At this point we review the Bala-Carter classification of unipotent elements. Let $u \in G$ be a unipotent element and let T_0 be a maximal torus of $C_G(u)$.

Then $C_G(T_0)$ is a Levi subgroup of G and $D = C_G(T_0)'$ is a reductive group with u a distinguished element. This means $C_D(u)$ does not contain a nontrivial torus. There is a corresponding distinguished parabolic subgroup $P = QL$ of D, with unipotent radical Q and Levi factor L. Here the term *distinguished* means that $\dim(L) = \dim(Q/Q')$. Further, there is a dense orbit of P on Q, called the Richardson orbit, and u lies in this dense orbit. There is a similar classification of nilpotent orbits of G on $L(G)$.

With P as above, fix a system of fundamental roots for D so that P is a standard parabolic subgroup with respect to this system. Take a 1-dimensional torus $T \leq L$ such that for α a fundamental root we assume that T acts on the root group U_α by weight 0 or 2, according to whether or not α is in the root system of L. Then $T \leq Z(L)$ and determines a labelling of the Dynkin diagram of D by 0's and 2's. As mentioned in the introduction, we call T a *u-distinguished* torus if it acts on $\langle e \rangle$ with weight 2 where $e \in L(G)$ is a nilpotent element such that $C_G(u) = C_G(e)$.

Let α be a root such that the corresponding root subgroup, U_α, is in the system of root groups and contained in Q. Then T acts on U_α via the weight $2r$, where r is the level of α, as defined in [**?**]. Namely, write $\alpha = \sum c_i\beta_i + \sum d_j\gamma_j$, where β_i and γ_j range over those fundamental roots with T-label 0 and 2, respectively. Then $r = \sum d_j$. So if u is a regular element, r is just the height of α. It is shown in [**?**] that the descending central series of Q has successive quotients isomorphic to the direct sum of the root groups of a given level.

The next lemma is essentially (4.5) of [**?**]. It will be used several times in what follows, so we include a proof for completeness.

Lemma 2.3 *Let $P = QL$ be a distinguished parabolic subgroup of G and let u and e be in the open dense orbits of P on Q and $L(Q)$, respectively. Then Q acts transitively on the cosets uQ' and $e + L(Q')$, respectively.*

Proof Given u, we have

$$\dim(L) + \dim(Q) = \dim(P) = \dim(u^P) + \dim(C_P(u)).$$

As u is assumed to be in the dense orbit of P on Q we also have

$$\dim(u^P) = \dim(Q).$$

Hence, using the fact that P is distinguished we have

$$(*) \quad \dim(C_P(u)) = \dim(L) = \dim(Q/Q').$$

Consider the map on Q given by $q \to u^q$. Translating by u^{-1} we see that

$$\dim(u^Q) = \dim\{u^{-1}u^q : q \in Q\} \leq \dim(Q').$$

On the other hand,

$$(**) \quad \dim(Q/Q') + \dim(Q') = \dim(Q) = \dim(u^Q) + \dim(C_Q(u)).$$

Hence, $\dim(C_P(u)) \geq \dim(C_Q(u)) \geq \dim(Q/Q')$ and combining this with $(*)$ we have $\dim(C_Q(u)) = \dim(Q/Q')$. Therefore, $(**)$ yields

$$\dim(u^Q) = \dim(Q').$$

Orbits of unipotent groups are closed and Q' is irreducible, so this implies that $u^{-1}u^Q = Q'$ and hence $uQ' = u^Q$, which gives the assertion for u. Essentially the same argument gives the assertion for e. ∎

Lemma 2.4 *Let $u \in G$ be a unipotent element and set $Z = R_u(Z(C_G(u)))$. Then $u \in Z$ and $uZ^p \subset u^G$.*

Proof Let T_0 be a maximal torus of $C_G(u)$. Lemma 2.1 implies $u \in Z \leq C_G(T_0)'$, the semisimple part of a Levi subgroup. Set $D = C_G(T_0)'$, so that u is a distinguished unipotent element of D and we let P be the corresponding distinguished parabolic subgroup of D. Set $Q = R_u(P)$. By Corollary 5.2.2 of [**?**] $C_D(u)^0 \leq Q$ so that $u \in Z \leq Q$ and u is in the Richardson orbit of P on Q. Then Lemma 2.3 shows that uQ' is fused under the action of Q. Also Q/Q' is of exponent p, so that $Z^p \leq Q'$ and the result follows. ∎

Lemma 2.5 *Let notation be as above with $P = QL < D$ and let $Q^{(i)}$ denote the ith term of the descending central series of Q.*

i) $Q^{(i)}$ is the product of root groups U_α for α a root of level at least i.

ii) $Q^{(i)}/Q^{(i+1)}$ is isomorphic to the direct sum of root groups for roots of level i and has the structure of a K-vector space with T inducing scalars corresponding to weight $2i$.

iii) $Q/Q^{(p)}$ has exponent p.

iv) $\exp(Q) = |u|$.

v) Let k be minimal with $Q^{(k)} = 1$. Then $p^r \geq k > p^{r-1}$, where $|u| = p^r$.

Proof (i) and (ii) follow from results in [**?**]. Each of the groups $Q^{(i)}$ is a product of root groups, so Corollary 12.3.1 of [**?**] shows that $Q/Q^{(p)}$ has exponent p, giving (iii). Suppose $|u| = p^r$, so that $\exp(Q) \geq p^r$. On the other hand the P-orbit of u is dense in Q and the set of elements of Q having order at most p^r is closed. So (iv) follows. Finally, (v) follows from Testerman [**?**]. ∎

The next lemma establishes the existence of u-distinguished tori.

Lemma 2.6 *Assume $u \in G$ is unipotent.*

i) If $e \in L(G)$ is a nilpotent element satisfying $C_G(u) = C_G(e)$, then there is a u-distinguished 1-dimensional torus T acting on $\langle e \rangle$ with weight 2.

ii) Any two u-distinguished 1-dimensional tori of G are conjugate by an element of $N_G(C_G(u))$.

Proof (i) We have $C_G(u) = C_G(e)$ and $C_G(e) = C_G(ce)$ for all $0 \neq c \in K$, so we look for a torus acting on $\langle e \rangle$. In view of the Bala-Carter classification of nilpotent elements we may assume that e is distinguished. Indeed, if T_0 is a maximal torus of $C_G(u)$, then $e \in C_{L(G)}(T_0) = L(C_G(T_0))$, so e is a distinguished nilpotent element of $L(D)$ where $D = C_G(T_0)'$.

There is a distinguished parabolic subgroup P of D such that $P = QL$, where $Q = R_u(P)$, L is a Levi subgroup, and e is in the Richardson orbit of P on $L(Q)$. There is a 1-dimensional torus T with $T \leq Z(L)$ and T acting by weight 2 on fundamental roots not in the base of $\Sigma(L)$. Let $L(Q)_i$ denote the T-weight space of $L(Q)$ corresponding to weight i. Then $L(Q) = L(Q)_2 \oplus L(Q)_4 \oplus \cdots$, and it follows from [**?**] that $L(Q') = L(Q)_4 \oplus L(Q)_6 \oplus \cdots$.

Each of the weight spaces $L(Q)_i$ is L-invariant and L has a dense orbit on $L(Q)_2 \cong L(Q)/L(Q')$. Elements of this orbit correspond to images of distinguished nilpotent elements. Hence there exists $v \in L(Q)_2$ such that $e + L(Q') = v + L(Q')$. Then Lemma 2.3 shows that e and v are conjugate under the action of Q. Hence, adjusting T, if necessary, we may take $e \in L(Q)_2$. Then T normalizes $\langle e \rangle$, as required.

(ii) Suppose $T, \bar{T}$ are u-distinguished 1-dimensional tori. Let $e, \bar{e}$ be the corresponding nilpotent elements. Lemma 2.2 and the existence of a Springer map imply that e and $\bar{e}$ are G-conjugate, so choose $g \in G$ such that $\bar{e}^g = e$. Then $C_G(e) = C_G(\bar{e})^g = C_G(u)^g = C_G(e)^g$ so that $g \in N_G(C_G(e)) = N_G(C_G(u))$. Now $\bar{T} \leq C_G(T_0^c)'$ for some $c \in C_G(u)$, so adjusting g by an element of $C_G(u) = C_G(e)$, we may assume that $\bar{T}^g \leq C_G(T_0)' = D$.

Now $T, \bar{T}^g < N_D(\langle e \rangle)$. On the other hand, u is distinguished in D, so $C_D(u)^0 = C_D(e)^0 = V$, a unipotent group. Hence $N_D(\langle e \rangle) = VT = V\bar{T}^g$, so a further conjugation by an element of V establishes the result. ■

Lemma 2.7 *Let u be unipotent in G and assume T is a u-distinguished 1-dimensional torus. Then the following conditions hold.*

i) $C_Z(T) = 1$, where $Z = Z(C_G(u))^0$.

ii) If $W = \langle u^T \rangle$ is T-homocyclic, then there is an element $e \in L(W)$ such that $C_G(u) = C_G(e)$ and such that T normalizes $\langle e \rangle$ and acts by weight 2.

iii) If $W = \langle u^T \rangle$ is T-homocyclic, then T acts by weight 2 on W/W^p.

Proof By assumption $T \leq N(C_G(u))$ and there is a maximal torus T_0 of $C_G(u)$ such that $T \leq D = C_G(T_0)'$. Moreover, u is a distinguished unipotent element of D and we let $P = QL$ denote the uniquely determined distinguished parabolic subgroup of D such that $C_D(u)^0 \leq Q$. Then $W \leq Z \leq Q$.

Since T is u-distinguished, conjugating by an element of Q, if necessary, we may assume $T \leq Z(L)$ and T acts by weight 2 on root groups for fundamental roots in $\Pi(D) - \Pi(L)$. As mentioned earlier, the descending central series of Q satisfies the property that successive quotients are each isomorphic to the direct product of root groups for roots of a given level at least 1 and these quotients each have a vector space structure with T inducing scalars. In particular, Q/Q' is isomorphic to the direct product of root groups of level 1 and T acts by weight 2. In view of these comments $C_Q(T) = 1$. In particular $C_Z(T) = 1$, proving (i).

Now, uQ' is in the dense orbit of L on Q/Q' and Q/Q' is isomorphic to the direct product of root groups for those roots of level 1. We can thus view Q/Q' as a K-vector space on which T acts by scalars. Also, $W \leq Q$ and $WQ'/Q' \cong W/W^p$ is 1-dimensional and T-invariant. It follows that WQ'/Q' is a 1-space and also (iii) holds.

We have $L(Q) = L(Q)_2 \oplus L(Q)_4 \oplus \cdots \oplus L(Q)_{2k}$ for suitable k, where T acts by weight j on $L(Q)_j$. Then T stabilizes the intersection of $L(W)$ with each of these weight spaces. Moreover, as WQ'/Q' is a 1-space, we conclude that $L(W)_2$ is a 1-space, say $L(W)_2 = \langle e \rangle$.

As Q/Q' is isomorphic to the direct sum of root groups of level 1, there is a natural L-isomorphism between Q/Q', viewed as affine space, and $L(Q)/L(Q')$. Under this isomorphism, $U_\alpha(c)Q'$ corresponds to $ce_\alpha + L(Q')$. It follows that the 1-space WQ'/Q' corresponds to $L(WQ')/L(Q')$ and hence $L(WQ')$ contains an element projecting to the dense orbit of L on $L(Q)/L(Q')$. Now, $L(WQ') = L(W)_2 \oplus L(Q')$, so e projects to an element of the dense orbit of L on $L(Q)/L(Q')$.

At this point Lemma 2.3 implies that e is in the dense orbit of P on $L(Q)$, so u and e have the same parametrization in the Bala-Carter classifications of unipotent and nilpotent elements, respectively.

Also, since $W \leq Z(C_G(u))$, we have $C_G(u) = C_G(W) \leq C_G(L(W)) \leq C_G(e)$. The Springer map implies that there exists $e_0 \in L(G)$ such that $C_G(u) = C_G(e_0)$ and it follows that e_0 is in the dense orbit of P on $L(Q)$. So $C_G(e) \geq C_G(u) = C_G(e_0) \cong C_G(e)$. Hence, $C_G(u) = C_G(e)$, proving (ii). ■

Lemma 2.8 *Let u be a unipotent element of G.*

(i) If $|u| = p$ and $u \in R$ with R a subsystem subgroup of G, then $u \in A \leq R$, where A is a restricted A_1 subgroup of G. In particular, the saturation of u is also contained in R.

(ii) Let T be a u-distinguished 1-dimensional torus. Then there is a subsystem subgroup R of G such that u is a semiregular unipotent element of R, and $T < R$ is a u-distinguished torus of R.

Proof (i) Here we are assuming $|u| = p$ and $u \in R$, a subsystem subgroup. Let T_1 be a maximal torus of $C_G(R)$. Then $T_1 \leq C_G(u)$. On the other hand, by Theorem 1.2 of [**?**], $C_G(u) = QC_G(A)$, where $Q = R_u(C_G(u))$ and A is a restricted A_1 subgroup containing u. Conjugating by an element of Q we may assume $T_1 < C_G(A)$ and hence $A < E = C_G(T_1)'$, the semisimple part of a Levi subgroup of G. If $R = E$ then there is nothing to prove. Otherwise, R is a proper subsystem subgroup of E and since p is a good prime for G, there is a semisimple element $s \in E$ such that $R \leq C_E(s) < E$. Now A is also a restricted A_1 subgroup of E, so as above we can conjugate, if necessary, to assume $A \leq C_E(s)'$. Continuing in this way we eventually obtain the assertion.

(ii) This time we must find R. Fix T. Then by definition there is a maximal torus T_0 of $C_G(u)$ such that $T < C_G(T_0)' = D$, where D is the semisimple part of a Levi subgroup. Also, $u \in D$ is a distinguished unipotent element of D. Let $P = QL$ be the corresponding distinguished parabolic subgroup of D, where $Q = R_u(P)$ and L is chosen so that $T < L$. If u is semiregular in D there is nothing to prove. Otherwise, there is a noncentral semisimple element s of D contained in $C_D(u)$. Here $R_u(C_D(u))$ has finite index in $C_D(u)$ and conjugating s by an element of this group, if necessary, we can assume T centralizes s. Therefore, T is contained in the subsystem subgroup $F = C_D(s) = C_D(s)'$ which contains u.

Now T determines a labelled diagram of F. Also, T is u-distinguished in D so the labels are among $0, 2, 4, \cdots$. We claim that the only possible labels are 0 and 2. Let F_2 be the Levi subgroup of F generated by a suitable maximal torus together with all root subgroups corresponding to fundamental roots in the system for which the T-label is 0 or 2 and their negatives. Using the fact that every positive root in the root system of F is the sum of fundamental roots, we see that F_2 contains all root subgroups of F for which T acts by weight 2. By definition there is a nilpotent element $e \in L(G)$ such that $C_G(u) = C_G(e)$ and T acts by weight 2 on $\langle e \rangle$. It follows that $e \in L(F_2)$. So if $F_2 < F$, then F_2 is a proper Levi subgroup of F and hence e is centralized by a nontrivial torus of E, contradicting the fact that $C_D(e) = C_D(u)$ and u is distinguished. This proves the claim.

Next, we argue that T is u-distinguished in F. It is clear from a T-weight consideration that $P_F = P \cap F$ is a parabolic subgroup of F. Write $P_F = Q_F L_F$, the Levi decomposition with $Q_F = Q \cap F$ and $L_F = L \cap F$. We argue that P_F is distinguished. Consider the T-weight space $L(Q_F)_2$. Then [**?**] implies $L(Q_F)_2 \cong L(Q_F)/L(Q_F')$ and L_F acts on this space with a dense orbit. In particular, $\dim(L_F) \geq \dim(L(Q_F))_2$. On the other hand, $e \in L(Q_F)_2$ and we know that $C_L(e)$ is finite. Hence $\dim(L_F) = \dim(L(Q_F))_2$, so that P_F is distinguished.

If u is semiregular in F, then we set $R = F$ and the lemma is established. Otherwise, repeat the above a finite number of steps until we reach this point.

■

3 Decomposing $Z(C_G(u))$

In this section we establish Theorem 1. Let $u \in G$ be unipotent, let T_0 be a maximal torus of $C_G(u)$, and set $D = C_G(T_0)'$, the semisimple part of a Levi subgroup. Then D is a reductive group containing $Z(C_G(u))$. Let $e \in L(G)$ correspond to u under a Springer map which is the usual one if G is of classical type. Lemma 2.6 shows there exists $T < D$, a u-distinguished 1-dimensional torus of G with e the associated nilpotent element. We have $C_G(u) = C_G(e)$, so that T normalizes $C_G(u)$, but $T \cap C_G(u)$ is finite (of order at most 2). Note that $C_{L(G)}(T_0) = L(C_G(T_0)) = L(D)$, so that $e \in L(D)$. Finally, we observe that u is a distinguished unipotent element in D. For this reason we are often able to reduce to the case where u is distinguished.

For the next result let D be as above, with u a distinguished unipotent element of D. Let $P = QL$ be the corresponding distinguished parabolic subgroup of D. Then u is in the Richardson orbit of P on Q. Let k be minimal with $Q^{(k)} = 1$. We take $T \leq Z(L)$, so that for some system of fundamental roots T acts by weight 2 on fundamental roots in $\Pi(D) - \Pi(L)$. Then T acts by scalars with weights $2, 4, \ldots, 2k-2$ on the vector spaces $Q/Q^{(2)}, Q^{(2)}/Q^{(3)}, \ldots, Q^{(k-1)}/1$. Lemma 2.5 shows that if $|u| = p^r$, then $p^r \geq k > p^{r-1}$.

The above filtration of Q can be refined to a T-invariant filtration of closed normal subgroups where successive quotients have dimension 1. Now let $J \leq Q$ be a T-invariant closed subgroup. Then intersecting J with terms of this filtration we see that J has a T-invariant filtration where the quotient of successive terms is a 1-dimensional unipotent group with T acting via a nonzero weight.

Lemma 3.1 *Let J be a connected abelian group of exponent p admitting the action of a 1-dimensional torus T and assume that J has a filtration by closed normal subgroups such that successive quotients have dimension 1 with T acting with nonzero weights.*

(i) There is a T-invariant decomposition $J = \bigoplus J(c)$ where $J(c) \cong G_a$ with T acting by weight c.

(ii) If W is a T-invariant subgroup of J, then $J = W \oplus R$ for some T-invariant subgroup R.

Proof (i) Let E be the last term in the given filtration, so that E is a 1-dimensional unipotent group with T acting without fixed points. Inductively, there is a decomposition $J/E = E_1/E \oplus \cdots \oplus E_k/E$, where each summand is a closed T-invariant 1-dimensional unipotent group. It will suffice to show that for $1 \leq j \leq k, E_j = E \oplus R_j$, where R_j is T- invariant. For then $J = E \oplus R_1 \oplus \cdots \oplus R_k$.

In view of the previous paragraph we may assume J has dimension 2. By 14.2.6 of [**?**] we have $J \cong K^2$. So the coordinate ring $K[J] = K[x, y]$ and T induces a group of locally finite linear transformations. Now T preserves the group structure of J so stabilizes the subspace M of group homomorphisms from J to K. Suppose $m \in M$. Then $m(x, y) = m(x, 0) + m(0, y)$ and from here we see that $m(x, y) = f(x) + g(y)$, where $f(x) = \sum a_i x^{p^i}$ and $g(y) = \sum b_j y^{p^j}$.

There is a decomposition $M = \bigoplus M_n$, where the sum is over integers and T acts on each M_n via weight n. We can choose r and $f \in M_r$ such that E is not contained in $\ker(f)$. Indeed, x and y are both contained in M and at least one of these restricts nontrivially to E. So we could work with a component of x or y. Then $f : J \to K$ and for $c \in K^*$, we have $c^r f = (T(c))f = f \circ T(c)$. Now restrict to E and get $c^r f(e) = f(T(c)e)$ for all $e \in E$. We are assuming that T acts nontrivially on E, so choosing e with $f(e) \neq 0$, we conclude that $E \cap \ker(f) = 0$. Also $\ker(f)$ is T-invariant, so $J = E \oplus \ker(f)$, completing the proof of (i).

(ii) Here we again proceed inductively. Intersect the terms of the given filtration of J with W to get a filtration of W and then choose $E \leq W$. Then inductively we can write $J/E = W/E \oplus R/E$. Now use (i) to decompose R/E into a direct sum of 1-dimensional T-invariant unipotent groups. Then apply the above argument to the preimage of each summand to obtain a T-invariant complement to W. ■

Lemma 3.2 *Let $u \in G$ be unipotent and T a u-distinguished 1-dimensional torus. There is a decomposition $Z(C_G(u))^0 = W_1 \oplus W_2 \oplus \cdots \oplus W_t$ (direct sum as abstract groups), where each W_i is a T-invariant homocyclic group. Moreover, if W is any T-invariant homocyclic subgroup of $Z(C_G(u))^0$ of exponent equal to that of $Z(C_G(u))^0$, then there is a T-invariant subgroup W_1 such that $Z(C_G(u))^0 = W \oplus W_1$.*

Proof Set $Z = Z(C_G(u))^0$. If T_0 is a maximal torus of $C_G(u)$, then $Z < C_G(T_0) = D$ and u is a distinguished unipotent element in D. Hence, there is a T-invariant filtration of Z such that successive quotients have dimension 1 with T acting via nonzero weights. We now proceed inductively only assuming that T is a 1-dimensional torus and A is a T-invariant connected abelian unipotent group with a filtration as described above for Z.

The lemma follows from the previous lemma if $exp(A) = p$, so assume $exp(A) = p^k$ with $k > 1$. Write $A > A^p > \cdots > A^{p^{k-1}} = V$, so that V is T-invariant and of exponent p. Inductively, there is a decomposition $A/V = A_1/V \oplus \cdots \oplus A_s/V$ with each summand T-homocyclic.

Reorder if necessary so that $exp(A_1) = p^k$ and set $U = A_1^{p^{k-1}}$, so that $U \leq V$. As A_1A^p/A^p has dimension 1, we see that U also has dimension 1 (consider the map $a_1 \to a_1^{p^{k-1}}$).

First assume that $U < V$. Then $A_1^{p^{k-1}} < A^{p^{k-1}}$, so that $A_1 < A$. Inductively we can write $A_1 = B_1 \oplus S$, a T-invariant decomposition with B_1 homocyclic of exponent p^k. Then $A_1 = B_1V$ and hence $S \cong A_1/B_1$ is of exponent p. Using induction again (this time the second assertion) we have $A/S = A_1/S \oplus B_2/S \oplus \cdots \oplus B_r/S$ with each summand T-homocyclic. Then $A = B_1(B_2 \cdots B_r)$. Moreover, $B_1 \cap (B_2 \cdots B_r) \leq S$ and $B_1 \cap S = 1$. Thus, $A = B_1 \oplus (B_2 \cdots B_r)$, a direct sum of abstract groups. Use induction to decompose $B_2 \cdots B_r$, thereby completing the proof of the first assertion.

Now assume $U = V$, so that $A^{p^{k-1}} = A_1^{p^{k-1}}$. Consider the map $\phi : A \to U$ sending $x \to x^{p^{k-1}}$. As $\dim(U) = 1$, it follows that A_1 is T-homocyclic. Let $K = ker(\phi)$, so that $A = A_1K$. Then K has codimension 1 in A and $A_1 \cap K = A_1^p$ has codimension 1 in A_1. Now $A_1 \cap K$ is T-invariant and homocyclic in K of maximal exponent, so inductively, $K = (A_1 \cap K) \oplus L$, for some T-invariant subgroup L. It follows that $A = A_1 \oplus L$, a direct product of abstract groups. Applying induction to L we have established the first assertion.

The second assertion follows along the same lines. Given W, we set $U = W^{p^{k-1}}$ and set $V = A^{p^{k-1}}$ as before. Set $A_1 = WV$. Inductively, there is a decomposition $A/V = A_1/V \oplus A_2/V \oplus \cdots \oplus A_s/V$. If $A_1 = A$, then $U = V$ and so $A = W$ is homocyclic. Suppose $A_1 < A$. We can write $V = U \oplus S$, with S invariant under T. If $U < V$, then $A_1 = W \oplus S$ and we proceed as in the fourth paragraph with W replacing B_1 to get the assertion. And if $U = V$, then $A_1 = W$ and we proceed as in the fifth paragraph, to complete the proof. ■

Lemma 3.3 *Theorem 1 holds.*

Proof As in the proof of the last lemma we can work in $C_G(T_0)'$ for T_0 a maximal torus of $C_G(u)$ and reduce to the case where u is distinguished in G. So there is a distinguished parabolic subgroup $P = QL$, such that u is in the Richardson orbit of P on Q and $C_G(u)^o \le Q$ (see 5.2.2 of [?]). In particular, $Z = Z(C_G(u))^0 \le Q$ and Lemma 2.1(i) shows that $u \in Z$.

We have $u \in Q - Q'$, so that ZQ'/Q' is nontrivial and T-invariant, where T is a u-distinguished 1-dimensional torus. Also, Q/Q' has a vector space structure with T inducing scalars. So ZQ'/Q' is a subspace of Q/Q'.

The previous lemma yields a decomposition $Z = W_1 \oplus \cdots \oplus W_s$ into T-homocyclic subgroups (direct sum as abstract groups). Then for $1 \le i \le s, W_iQ'/Q'$ is a subspace of Q/Q' of dimension at most 1. Reorder the T-homocyclic summands, $W_1, \ldots, W_s$, if necessary, so that $uQ' \in W_1 \cdots W_hQ'$ with h minimal. Minimality of h implies that the subspaces $W_1Q'/Q', \ldots, W_hQ'/Q'$ are independent. (We note that Lemma 2.9 of [?] shows that $C_G(u)Q'/Q'$ typically has dimension 1 and so usually $h = 1$, as well.) Lemma 2.3 shows that uQ' is fused in Q, so that there is an element $g \in Q$, such that $u^g \in W_1 \oplus \cdots \oplus W_h$. Set $E = W_1 \oplus \cdots \oplus W_h$. We have $E \cap Q' = E^p$.

Write $C = \langle (u^g)^T \rangle$. We claim that C is T-homocyclic. It follows from the above that CQ'/Q' is a 1-space of $W_1 \cdots W_hQ'/Q'$ and hence CE^p/E^p has dimension 1. The previous lemma implies $C = C' \oplus V'$ where C' is T-homocyclic of exponent p^r and V' is T-invariant. We assume $V' \neq 1$, since otherwise there is nothing to prove. Write $C \cap E^p = (C' \cap E^p) \oplus R$ for some T-invariant closed subgroup R. As CE^p/E^p has dimension 1, $CE^p/E^p = C'E^p/E^p$. Then $C \le C'E^p$, so that $C = C'(C \cap E^p)$ and we can take $V' = R$. In particular, $V' \le E^p$.

Write $u^g = cv$, where $c \in C'$ and $v \in V'$. Then $v \neq 1$ as otherwise $u^g \in C'$ and $C = C'$ which we are assuming false. Assume $v \in E^{p^k} - E^{p^{k+1}}$ for some $k \ge 1$ and write $v = a^{p^k}$ for some $a \in E - E^p$.

We have $C = \langle (u^g)^T \rangle$ so there are integers c_s and elements $t_s \in T$ such that $\prod_s ((u^g)^{c_s t_s}) = v$. However $C = C' \oplus V'$ and $u^g = cv$, so $\prod_s (c^{c_s t_s}) = 1$ and $\prod_s v^{c_s t_s} = v$.

Now E/E^p has a vector space structure with scalar action induced by conjugation of elements of T. Hence all 1-spaces are T-isomorphic, in particular those spanned by cE^p and aE^p. We have $\prod_s (c^{c_s t_s}) = 1$ and so $\prod_s (a^{c_s t_s})E^p = 1$. Write $\prod_s (a^{c_s t_s}) = e^p$, for some $e \in E$. Then $v = \prod_s v^{c_s t_s} = \prod_s a^{p^k c_s t_s} = (\prod_s (a^{c_s t_s}))^{p^k} = e^{p^{k+1}}$. But then $v \in E^{p^{k+1}}$, which is not the case. This contradiction proves the claim.

From the claim we have $u^g \in C$, a T-homocyclic group. Setting $C = J_1$, we can apply the previous lemma to obtain a decomposition $A = Z(C_G(u))^0 = J_1 \oplus \cdots \oplus J_s$ with each summand a T-homocyclic group. Then $u^g \in J_1 \leq Z = Z(C_G(u))^0$. It follows that $C_G(u) = C_G(u^g)$, so that $Z(C_G(u))^0 = Z(C_G(u^g))^0 = (Z(C_G(u))^0)^g$. That is $Z = Z^g$. Conjugating the above decomposition of A by g^{-1} we have $u \in J_1^{g^{-1}}$, a $T^{g^{-1}}$- homocyclic group. Now replace T by $T^{g^{-1}}$. As $g \in Q$, $T^{g^{-1}}$ is also u-distinguished and contained in $C_G(T_0)'$, so the assertion follows. ■

4 Compatibility issues

In this section we prove Theorem 2. Let $u \in G$ be a unipotent element of G of order p^r and let $v = u^{p^{r-1}}$, an element of order p. It was shown in [?] that v is contained in a unique 1-dimensional unipotent group, say U, which is contained in a restricted A_1 subgroup of G. So U can be legitimately called "the saturation" of v. In this section we show that U is contained in each T-homocyclic group, W, containing u, where T is a u-distinguished 1-dimensional torus.

Lemma 4.1 *It suffices to establish Theorem 2 for u a semiregular unipotent element of G.*

Proof Let T be a u-distinguished 1-dimensional torus of G such that $W = \langle u^T \rangle$ is T-homocyclic. Lemma 2.8 shows that there is a subsystem subgroup R containing T such that u is a semiregular unipotent element of R. Then $W = \langle u^T \rangle \leq R$, so that $W \leq Z(C_R(u))$.

Also, Lemma 2.8(i) shows that $v < U < A \leq R$, where A is a restricted A_1 subgroup of G. It follows from the definition that A is also a restricted A_1 subgroup of R. At this point the unicity of the saturation implies that U is also the saturation of v within R and so we may work entirely within R. In the next paragraph we argue that we can pass to simple factors of R, replacing G by a simple factor and u by its projection to such a factor. Here we note that v might project trivially to some simple factors of R. Indeed later (see Lemma 5.4) we shall see that for exceptional groups v projects trivially to all but one simple factor.

Let E be a simple factor of R where the projection of v is nontrivial and let subscript E denote projection to E. Then u_E is a semiregular unipotent element of E, T_E is a u_E-distinguished torus, and $W_E = \langle u_E^{T_E} \rangle$ is a T_E-homocyclic subgroup. If we show that $(W_E)^{p^{r-1}} = U_E$, the saturation of v_E, then it will follow that $W^{p^{r-1}} = U$. Indeed, this will show that U_E is contained in a restricted A_1 subgroup of E and hence U is contained in a restricted A_1 subgroup of R. But Proposition 4.3 of [?] shows that restricted A_1 subgroups of R are also restricted in G and it follows that U is the saturation of v. So for purposes of proving Theorem 2 we may now work entirely with E, completing the proof. ■

In view of the previous lemma we now assume u is semiregular (hence distinguished) in G and let $P = QL$ be the corresponding distinguished parabolic subgroup of G. Then u lies in the dense orbit of P on Q and we may assume $T < Z(L)$. The labelled diagram associated to u is the same as the labelling associated with T. Let α be a root such that the corresponding root subgroup, U_α, is invariant under T and contained in Q. Then T acts on U_α and the corresponding weight is $2r$, where r is the level of α (see the discussion prior to Lemma 2.3).

The following lemma is a consequence of results in [**?**] and will be useful for the proof of the theorem.

Lemma 4.2 *Let D be a reductive group over K, let P be a parabolic subgroup of D such that $Q = R_u(P)$ has nilpotence class strictly less than p, and let $v \in Q$. Then there is at most one 1-dimensional unipotent subgroup, V, of Q such that $v \in V = V^T$, where $T \le P$ is a 1-dimensional torus acting by weight 2 on V and having no weights on $L(Q)$ which are a multiple of $2p$.*

Proof Assume the hypotheses and that $v \in V = V^T \cong G_a$. By Proposition 5.4 of [**?**] there are commuting elements $e_0, e_1, \dots, e_n \in L(Q)$ such that for $t \in K$ we have $V(t) = exp(e_0t) \cdot exp(e_1t^p) \cdots exp(e_nt^{p^n})$. Here exp is the uniquely determined, P-equivariant, isomorphism $L(Q) \to Q$ such that the tangent map is the identity, where we view $L(Q)$ as an algebraic group via the Hausdorff formula.

Suppose T acts on V via weight r. A computation using the P-equivariance of exp and the fact that $e_0, e_1, \cdots, e_n$ commute shows that for $0 \le i \le n, e_i$ is a T-weight vector of $L(Q)$ of weight rp^i. So by hypothesis $e_i = 0$ for $i > 0$. Hence, $V(t) = exp(e_0t)$ for all $t \in K$.

Now suppose that $v \in V' = V'^{T'}$ is another such 1-dimensional group, where $T' < P$ is a 1-dimensional torus satisfying the hypothesis. Applying the above analysis we obtain $V'(t) = exp(f_0t)$ for some $f_0 \in L(Q)$. As $v \in V \cap V'$, we have $v = exp(e_0c) = exp(f_0d)$, for nonzero scalars c, d. However, exp is an isomorphism, so $\langle e_0 \rangle = \langle f_0 \rangle$ and hence $V = V'$. ■

At this point we separate the discussions of the exceptional and classical groups. In outline the proofs are similar, but the details differ.

4.1 Exceptional groups

For this subsection assume that G is a simple algebraic group of exceptional type. As above we assume $u \in G$ is a semiregular unipotent element and let T be a u-distinguished 1-dimensional torus as in the discussion prior to 4.2.

Lemma 4.3 *Let $W \leq Z(C_G(u))$ be a T-homocyclic group such that $u \in W - W^p$ and T acts by weight 2 on W/W^p. Then*

(i) If $|u| > p$, then $|u| = p^2$ and T acts by weight $2p$ on W^p.

(ii) $W^p < F < D$ where D is the subsystem subgroup determined by all root subgroups for roots of level a multiple of p (see the discussion prior to Lemma 2.3) and F is the product of all root subgroups for positive roots in the root system of D having positive T-weight.

(iii) If W' is another T-homocyclic group generated by conjugates of u and such that T acts by weight 2 on W'/W'^p, then W and W' are T-isomorphic.

Proof (i) Fix $u \in W \leq C_G(u) < Q$, where $Q = R_u(P)$ and $P = QL$ is a distinguished parabolic subgroup with $T \leq Z(L)$. Now T acts on successive quotients of the descending central series of Q inducing weight $2r$ on the quotient $Q^{(r)}/Q^{(r+1)}$.

Assume $|u| > p$. It then follows from 0.4 of [**?**] that $|u| = p^2$ and Q has nilpotence length strictly less than p^2. So $W^{p^2} = 1$ and the p-power map on W induces a surjective map $W/W^p \to W^p$.

Write $u = U(1)$ as a product of root elements, where we order so that the roots have nonincreasing levels. As W is T-invariant, the image of W/W^p in Q/Q' is a 1-space and since T induces scalars on Q/Q', the nonzero elements of this 1-space can be obtained by conjugating $U(1)Q'$ by elements of T. So conjugating $U(1)$ by elements of T it follows that we can write elements of W/W^p as images of elements of W of form $U(c) = U_1(c)U_2(c^2)U_3(c^3)\cdots$, where $U_i(c^i)$ is a product of root group elements of the form $U_\alpha(d_\alpha c^i)$, where α is a root of level i and $d_\alpha \in K$ is a scalar. Also $U_1(c) \neq 1$. Lemma 2.5(iii) implies that $Q/Q^{(p)}$ has exponent p, so forming $U(1)^p$ and conjugating by elements of T we see that elements of W^p have the form $U(c)^p = V_p(c^p)V_r(c^r)V_s(c^s)\cdots$, where $p < r < s < \cdots < p^2$ and again the terms are products of root elements for roots of the indicated levels. Now $U(c)U(d) \equiv U(c+d) \pmod{W^p}$, so as W is abelian,

$$(*) \qquad U(c)^pU(d)^p = U(c+d)^p.$$

We claim that $V_p(c^p) \neq 1$ for $c \neq 0$. Consider the sequence $Q^{(p)} > Q^{(p+1)} > Q^{(p+2)} > \cdots$ and choose $p \leq t < p^2$ minimal such that $V_t(c^t) \neq 1$. Working in $Q^{(t)}/Q^{(t+1)}$, which is isomorphic to the direct product of root groups of level t, $(*)$ yields an equation $\bar{V}_t(c^t)\bar{V}_t(d^t) = \bar{V}_t((c+d)^t)$, where bars denote images in $Q^{(t)}/Q^{(t+1)}$ of the corresponding elements. Projecting to root groups we obtain a polynomial identity which is only possible for $t = p$, establishing the claim.

An inductive argument using $(*)$ and the commutator relations implies that $r, s, \cdots$ are all multiples of p whenever the corresponding term is nontrivial. This implies that W^p is contained in the product of root subgroups corresponding to positive roots of level a multiple of p, which gives (ii). Also W^p is a 1-dimensional unipotent group and $W^p \cong W^pQ^{(p+1)}/Q^{(p+1)}$ so that T acts by weight $2p$, establishing (i).

(iii) Suppose W' is another T-homocyclic group as described in (iii). Write $W = \langle u^T \rangle$ and $W' = \langle u'^T \rangle$, where u' is a conjugate of u. The result is immediate if $|u| = p$ as both groups are isomorphic to G_a with T acting by weight 2. So assume that u and u' both have order p^2. Consider the group $R = W \oplus W'$ (external direct sum). This group admits the action of T.

Now consider the subgroup $J = \langle (uu')^T \rangle$ of R. Viewing R/R^p as a 2-dimensional vector space with T inducing scalars, we see that J projects to a 1-space. In particular, J is a proper T-invariant subgroup of R. At this point the proof of Lemma 3.1 shows $J = K \oplus L$, where K is T-homocyclic of exponent p^2 and L is T-invariant. Also, as J is proper, L has exponent p, so $L \leq R^p$.

Write $K = \langle (xy)^T \rangle$ where $x \in W, y \in W'$. If $x \in W^p$, then $K < W'R^p$ and hence $uu' \in J < W'R^p$, which is not the case. Hence, $x \in W - W^p$ and similarly $y \in W' - W'^p$.

Consider the projection map $\pi : R \to W$ and let π_K denote the restriction to K. We claim that π_K is an isomorphism, so that $W \cong K$. If we can show this, then by symmetry $W' \cong K$, which will give the result.

Now π commutes with the action of T and clearly x is in the image of π_K. It follows that π_K is surjective and since $\ker(\pi_K)$ is T-invariant, π_K is also injective. To show π_K is an isomorphism we must show that $\partial\pi_K$ is an isomorphism. It will suffice to show $\partial\pi_K$ is injective. We can regard R^p as a K-vector space and (i) shows that T acts by scalars corresponding to weight $2p$. Also $K^p < R^p$ has dimension 1 and is T-invariant, hence K^p is a subspace. As $(xy)^p \in K^p$ it follows that K^p is not contained in either W^p or W'^p. So the projection map $K^p \to W^p$ is an isomorphism and hence the image of $\partial\pi_K$ contains the T weight space of $L(W)$ corresponding to weight $2p$. If $\ker(\partial\pi_K) \neq 0$, then $\ker\partial\pi_K = L(K)_2$ and so $L(K)_2 < L(W') = \ker(\partial\pi)$. But then $L(KR^p)/L(R^p) = (L(K)_2 \oplus L(R^p))/L(R^p) \leq (L(W') + L(R^p))/L(R^p)$. So from the isomorphism $L(R/R^p) \cong L(R)/L(R^p)$ we have $L(KR^p/R^p) \leq L(W'R^p/R^p)$. However, this is impossible as KR^p/R^p is a subspace of R/R^p not contained in $W'R^p/R^p$. Hence $\ker\partial\pi_K = 0$, completing the proof. ■

The next lemma deals with semiregular unipotent classes in exceptional groups. Let $u \in G$ be a semiregular unipotent element with $|u| = p^2$. The previous result showed that $u^p \in D$, where D is certain subsystem subgroup of G such that the T-labels of fundamental roots are $2p$ or 0. Lemma 4.3(ii) shows that $u^p \in D^*$ where D^* is the product of all direct factors of D except for those where all labels are 0. In most cases u^p is a regular unipotent element of D^*, so u^p and D^* correspond to the same Dynkin diagram. The other cases are listed with an asterisk in the table of the next lemma and the Dynkin diagram of D^* for these cases is indicated.

Lemma 4.4 *Assume G is of exceptional type and u is a semiregular unipotent element. The type of u and its pth power are listed in the table below. Also presented is a subsystem subgroup D^* containing u^p (see the explanation above).*

u	u^p $(p = 5, 7, ...)$	D^*
G_2	A_1	
F_4	$A_2A_1^2, A_1^3, A_1$	
E_6	$A_2A_1^2, A_1^3, A_1$	
$E_6(a_1)$	$(A_1^3)^*, (A_1)^*$	$A_2A_1^2, A_2$
E_7	$A_3A_2A_1, A_2A_1^3, A_1^3, A_1^2, A_1$	
$E_7(a_1)$	$(A_2^2A_1)^*, (A_1^4)^*, (A_1)^*, A_1$	$A_3A_2A_1, A_2A_1^3, A_2$
$E_7(a_2)$	$(A_2A_1^2)^*, (A_1^3)^*, A_1$	$A_3A_2A_1, A_3A_1$
E_8	$-, A_4A_2A_1, A_2^2A_1^2, A_2A_1^3, A_1^4, A_1^3, A_1^2, A_1$	
$E_8(a_1)$	$-, (A_3A_2A_1)^*, (A_2A_1^2)^*, (A_1^4)^*, (A_1^2)^*, (A_1)^*, A_1$	$A_4A_2A_1, A_3A_1^2, A_2A_1^3, A_2A_1, A_2$
$E_8(a_2)$	$-, (A_2^2A_1^2)^*, (A_1^4)^*, (A_1^2)^*, (A_1)^*, A_1$	$A_4A_2A_1, A_3A_1^2, A_2^2, A_2$

Proof The unipotent element u has a labelled diagram which defines a distinguished parabolic subgroup, $P = QL$. This is just a Borel subgroup of G except for the cases $E_6(a_1), E_7(a_1), E_7(a_2), E_8(a_1)$, and $E_8(a_2)$, where the labelled diagram is $222022, 2220222, 2220202, 22202222, 22202022$, respectively. In the regular cases the labelling is given by all 2's. The type of u^p is given by Lawther [**?**].

Now the previous lemma shows that $u^p \in D$, the group generated by root subgroups for which the root has level a multiple of p. Using the labelling, it is straightforward to find all such positive roots and a corresponding base for the root system of D. We illustrate the method with an example. Assume $G = E_8$, with u a regular element and $p = 13$. Here we look for roots of height a multiple of 13. Note that the root of greatest height has height 29, so the only possibilities are roots of height 13 and 26. It is easy to see that the only root of height 26 is 23465321 and the roots of height 13 are $11233210, 12232111, 12232210, 11232211, 11222221$. From here we can see that D has type $A_2A_1^3$ with base $\{11233210, 12232111\} \cup \{12232210\} \cup \{11232211\} \cup \{11222221\}$. So in this case u^p and D are of the same type.

Exactly the same process can be used for the other types. In some cases simple factors of D have all labels 0 and we ignore these factors, letting D^* be the product of the remaining factors, as above. For later reference in the following table we present a system of simple roots for the subsystem group D^* in those cases where D^* and u^p do not have the same type. Most of the simple roots have T-label $2p$. However, in each case one or two fundamental roots of D^* are also fundamental roots for G (in the fixed system) and these roots have T-label 0. ∎

u	p	$\Pi(D^*)$
$E_6(a_1)$	5	$\{111111, \alpha_4\}, \{112210\}, \{011221\}$
	7	$\{112221, \alpha_4\}$
$E_7(a_1)$	5	$\{1111110, \alpha_4, 0111111\}, \{1011111, 0112210\}, \{1122100\}$
	7	$\{1112211\}, \{1122111\}, \{0112221\}, \{1122210, \alpha_4\}$
	11	$\{1223321, \alpha_4\}$
$E_7(a_2)$	5	$\{1122100, \alpha_6, 0112211\}, \{1111111, \alpha_4\}, \{1112210\}$
	7	$\{\alpha_4, 1122211, \alpha_6\}, \{1223210\}$
$E_8(a_1)$	7	$\{11122110\}, \{11221110, 01122111\}, \{11222100, \alpha_4, 11111111, 01122210\}$
	11	$\{12232211\}, \{11233211\}, \{12233210, \alpha_4, 11222221\}$
	13	$\{22343210\}, \{12343211\}, \{11233321\}, \{12233221, \alpha_4\}$
	17	$\{13354321\}, \{22344321, \alpha_4\}$
	19	$\{23454321, \alpha_4\}$
$E_8(a_2)$	7	$\{11221111\}, \{01122221, 12232100\}, \{\alpha_4, 11222110, \alpha_6, 11122111\}$
	11	$\{12343211\}, \{22343210\}, \{\alpha_4, 12233221, \alpha_6\}$
	13	$\{12344321, \alpha_4\}, \{22343221, \alpha_6\}$
	17	$\{23465432, \alpha_6\}$

Lemma 4.5 *Theorem 2 holds if G is of exceptional type.*

Proof We consider $v = u^p \in D^*$, where D^* is as in 4.4. Let J denote the saturation of v in G. Then 2.8(i) implies $J < D^*$ and that $v \in A \leq D^*$, where A is a restricted A_1 subgroup of G. Then $v \in J < A$, as J is contained in all restricted A_1's of G that contain v (see Theorem 1.3 of [**?**]). By definition, A is a restricted A_1 subgroup of D^* and J is the saturation of v in D^*.

We have $N_A(J) = JT'$ where T' is a 1-dimensional torus normalizing J and hence U. Now T' determines a labelling of the Dynkin diagram of D^*. If v is a regular element of D^*, then the labelling consists of all 2's. Otherwise, the labels can be $0, 1, 2$. The labels are determined by the weights on $L(D^*)$ and, since all simple components of D^* are of type A_r, these can be quickly calculated from the action of v on the natural module. For later reference we observe that it follows from these comments and the above two tables, that there do not exist nonzero weights of T' on $L(D^*)$ which are multiples of p.

First assume that v and D^* are of the same type. Then v is a regular element of D^*. Let U be the unipotent radical of the Borel subgroup of D^* containing v. Note that $V = W^p \leq C_{D^*}(v)^0 \leq U$. Note also that $U \geq C_{D^*}(v)^0 = C_{D^*}(J)^0$ implies that $J < U$.

We will apply Lemma 4.2 to show that $V = J$. From the information presented in the table of Lemma 4.4 we see that U has nilpotence class strictly less than p. So it only remains to verify the information on weights. By construction T induces a torus on D^* acting by weight $2p$ on each root element for a fundamental root. So the induced action of T on $L(D^*)$ can be factorized through a Frobenius morphism. That is, there is a 1-dimensional torus $T_1 < D^*$ inducing the same group as T but having weight 2 on fundamental root elements. It follows that all weights of T_1 on $L(U)$ are even and strictly less than $2p$. Moreover, 4.3(i) implies that T_1 has weight 2 on V. Also T' has all labels 2 in this case so the weights of T' are the same as those of T_1. At this point Lemma 4.2 gives $V = J$, as required.

A similar argument will be applied to settle the remaining cases, where v and D^* are not of the same type. In these cases $\Pi(D^*)$ contains at least one fundamental root (either α_4 or α_6) which has T-label 0. It will suffice to work with a simple factor of D^* (see the last paragraph of the proof of Lemma 4.1). Let E be a simple factor of D^* for which the base contains a fundamental root of G. Note that E is of type A_r for some r.

First assume there is just one such label and it occurs at an end node. From the previous table we see that $E = A_2$ and we consider the projection $\pi(v)$ to E. Lemma 4.3 shows that v lies in the product of root groups affording nonzero weights, so $\pi(v)$ lies within the unipotent radical of a T-invariant maximal parabolic subgroup of E corresponding to an end node. This unipotent radical is a natural module for a Levi factor of the parabolic and T induces scalars in this action. Since $\pi(V)$ is generated by the T-conjugates of $\pi(v)$, it follows that $\pi(V)$ is contained in the unipotent radical in question and is a 1-space. It follows that $\pi(V)$ is root subgroup of E. At this point, it is clear that $\pi(V)$ is the saturation of $\pi(v)$ and we have the result.

There remain seven cases to consider. Consider first $E_8(a_1)$ with $p = 7$. Let E denote the A_4 factor which has T_0-labelling 2022. Let P denote the corresponding parabolic subgroup. So P is the stabilizer of a flag $0 < V_1 < V_3 < V_4 < V_5$ of the usual module, where V_i denotes an i-space. Now $\pi(v) \in E$ is a unipotent element of type A_3. As above, $\pi(V) < Q$, the unipotent radical of P. Also $\pi(v)$ acts on V_5 as $J_4 \oplus J_1$, the sum of Jordan blocks of the indicated sizes. Choose bases $\{v_1, v_2, v_3, v_4\}, \{v_5\}$ for the Jordan blocks, where in the first case $\pi(v)$ stabilizes the spaces spanned by each initial set of the given basis. If V_1 is not spanned by v_1, then $\pi(v)$ acts on V_5/V_1 as a single Jordan block. However, Q acts trivially on V_3/V_1, so this is impossible. It follows that $V_1 = \langle v_1 \rangle$ and then $V_3 = \langle v_1, v_2, v_5 \rangle$. Similarly, $V_4 = \langle v_1, v_2, v_3, v_5 \rangle$. The saturation of $\pi(v)$ in E, say J_E, preserves the Jordan blocks of $\pi(v)$, so it follows from the above that $\pi(J) < Q$. Indeed, we can choose a restricted A_1 subgroup, say A, with $\pi(v) \in \pi(J) < A < A_3$. The maximal torus in $N_A(\pi(J))$ preserves the Jordan decomposition so is contained in P.

At this point we can apply Lemma 4.2 as in the earlier cases. It is clear from the construction that Q has class strictly less than p and we have shown that $\pi(V)$ and $\pi(J)$ both lie in Q. As before we get a torus T_0 associated with T by factoring through a Frobenius morphism and a torus T' from a restricted A_1 containing v. Both have all weights less than p on $L(Q)$, so 4.2 yields the result. This settles the case of $E_8(a_1)$ for $p = 7$. The cases $E_7(a_1)$ with $p = 5$, $E_7(a_2)$ with $p = 5$, and $E_8(a_1)$ with $p = 11$ are entirely similar.

We are left with the $E_8(a_2)$ cases with $p = 7, 11$ and the $E_7(a_2)$ case with $p = 7$. We provide details for the first of these, which is the most difficult. Here $E = A_4$ and $\pi(v)$ has type A_2A_1. In this case P is a parabolic with T_0 labelling 0202, so that P is the stabilizer of a flag of shape $0 < V_2 < V_4 < V_5$ and $Q = R_u(P)$ acts trivially on successive quotients. As before $\pi(v) \in Q$ and since Q is T-invariant we also have $\pi(V) \leq Q$. We must show that $\pi(J) \leq Q$. Now $\pi(v)$ induces $J_3 \oplus J_2$ on V_5 so $\pi(v)$ acts trivially on a unique 2-space, which must then be V_2. Also, $\pi(v)$ acts trivially on a unique 2-space of V_5/V_2, so this must be V_4/V_2. Now $\pi(J) < D$ is the saturation of $\pi(v)$ so must lie in A_2A_1 and preserve the relevant flags in each of the natural modules. It follows that $\pi(J) < Q$, as required. This completes the proof of the lemma. ∎

4.2 Classical Groups

Now we consider the case where G is of classical type. There are both complications and simplifications available here. On the one hand, unipotent elements can now have arbitrarily large order. On the other hand, considerations can all be reduced to considerations for groups of type A where matrix computation provides some insight. It will be convenient to work with the actual classical groups, so we take $G = SL(V), Sp(V)$, or $SO(V)$, with $\dim(V) = n + 1$. Recall that p is a good prime, so this means $p \neq 2$ in the symplectic and orthogonal cases.

In view of Lemma 4.1 we assume u is a semiregular unipotent element of G. Recall that T is a u-distinguished 1-dimensional torus such that $W = \langle u^T \rangle$ is T-homocyclic. Let e be a nilpotent element of $L(G)$ such that $C_G(u) = C_G(e)$ and T acts on $\langle e \rangle$ with weight 2.

For $G \neq SO(V)$ with n odd, this means that u is a regular element when viewed as an element of $SL(V)$. In the even dimensional orthogonal case we have $u \in B_kB_s$, where $n + 1 = 2(k + s + 1), k \neq s$, and u projects to a regular element of each factor.

The following remark is in order when $G = SO(V)$ and $\dim(V)$ is even. Let $u \in B_kB_s$ as above. Then $B_kB_s = C_G(\tau)$ for τ a suitable involutory automorphism of G. Indeed, τ corresponds to a diagonal involution of shape $(-1)^{2k+1}(1)^{2s+1}$ in the orthogonal action. Then τ acts on $A = Z(C_G(u))^0$ and this group can be decomposed $A = A_+ \times A_-$, where τ centralizes the first factor and inverts the second. Arguing as in Lemma 2.8 but allowing for graph automorphisms we see that we may take $T < B_kB_s$, so that T acts on each factor of A and $W = \langle u^T \rangle$ is contained in $A_+ \leq B_kB_s$.

We now aim to prove Theorem 2. We first note that in view of the above discussion we can assume that $u \in G \leq SL(V)$ and that u is a regular unipotent element of $SL(V)$. Indeed, this is immediate except for the even dimensional orthogonal groups and here $u \in B_kB_s$ and we work with each projection separately. So in this case we replace u and T by their projections to one of the factors.

Define an automorphism of G as follows. If $G = SL(V)$, set $\delta = 1$. Otherwise δ is taken as an involutory automorphism of $SL(V)$ such that $G = SL(V)_\delta$.

Lemma 4.6 *The following conditions hold.*

i) $C_{SL(V)}(T) = T_{SL(V)}$*, a* δ*-invariant maximal torus of* $SL(V)$*.*

ii) δ *permutes the* $T_{SL(V)}$*-root subgroups of* $SL(V)$*.*

iii) u *is contained in a uniquely determined Borel subgroup, say* $B = B^\delta$*, of* $SL(V)$*.*

Proof We have $u \in G \leq SL(V)$, where $\dim(V) = n + 1$. Also T acts on V with weights $n, n-2, \ldots, -(n-2), -n$. So all weight spaces are 1-dimensional and hence $C_{SL(V)}(T) = T_{SL(V)}$, a maximal torus of $SL(V)$. Also $T \leq SL(V)_\delta$, so $T_{SL(V)}$ is δ-invariant, proving (i). It follows that δ permutes the $T_{SL(V)}$-root subgroups of $SL(V)$, since these are the minimal $T_{SL(V)}$-invariant unipotent subgroups of $SL(V)$. This gives (ii). Finally, since $u \in SL(V)$ is a regular element, it lies in a uniquely determined Borel subgroup, B, of $SL(V)$ and this must also be invariant under δ. This establishes (iii) and completes the proof of the lemma. ■

Lemma 4.7 *Theorem 2 holds if* G *is of classical type.*

Proof Write $\Pi(SL(V)) = \{\alpha_1, \ldots, \alpha_n\}$ and, as above, assume u is a regular unipotent in $SL(V)$. Let B be the unique Borel subgroup of $SL(V)$ containing u. Conjugating by an element of B we may assume

$$e = e_{\alpha_1} + e_{\alpha_2} + \cdots + e_{\alpha_n}.$$

Then $Z = C_{SL(V)}(u)^0$ is the (abelian) lower triangular unipotent group where the entries are constant on each subdiagonal. Then, as indicated at the end of the proof of Lemma 2.3, we have $N_G(\langle e \rangle) = ZT$. But also $\langle e \rangle$ is normalized by a 1-dimensional torus which is diagonal with weights $n, n-2, \ldots, -(n-2), -n$. Consequently further conjugation by an element of Z allows us to assume T is this diagonal torus.

Adjusting u by an element of T, if necessary, we have

$$u = U_{\alpha_1}(1)U_{\alpha_2}(1)\cdots U_{\alpha_n}(1)y,$$

where $y \in Z$ has all 0's on the subdiagonal. Then conjugating by $T(c)$ we have

$$u^{T(c)} = U_{\alpha_1}(c^2)U_{\alpha_2}(c^2)\cdots U_{\alpha_n}(c^2)y^{T(c)}.$$

Matrix calculation shows that

$$(u^{T(c)})^{p^{r-1}} = U_{\delta_1}(c^{2p^{r-1}})U_{\delta_2}(c^{2p^{r-1}})\cdots U_{\delta_k}(c^{2p^{r-1}})z,$$

where $\delta_1 = \alpha_1 + \ldots + \alpha_{p^{r-1}}, \delta_2 = \alpha_2 + \ldots + \alpha_{p^{r-1}+1}, \ldots, \delta_k = \alpha_k + \ldots + \alpha_n$, $k = n - p^{r-1} + 1$, and z is a product of root elements for roots of height greater than p^{r-1}.

Elements of Z have the form $1 + a_1 m_1 + \alpha_2 m_2 + \cdots + a_{n-1} m_{n-1}$, where for each i, a_i is a constant and m_i is the lower triangular matrix with 0's except on the ith subdiagonal where the value is 1. One checks that m_i and m_j commute for all i, j and so it follows that

$$(1 + a_1 m_1 + \alpha_2 m_2 + \cdots + a_{n-1} m_{n-1})^{p^{r-1}} = 1 + a_1^{p^{r-1}} m_{p^{r-1}} + a_2^{p^{r-1}} m_{2p^{r-1}} + \cdots,$$

and so $Z^{p^{r-1}}$ is contained in the product of root subgroups of height a positive multiple of p^{r-1}.

Hence $u^{p^{r-1}} \in W^{p^{r-1}} \leq Z^{p^{r-1}} \leq D = \langle U_{\pm\delta_1}, \ldots, U_{\pm\delta_k} \rangle$. Write $n = sp^{r-1} + t$, with $0 \leq s < p$ (as $n < p^r$) and $0 \leq t < p^{r-1}$. We then find that $D = (A_s)^{t+1}(A_{s-1})^{p^{r-1}-t-1}$ and a base for the root system of each factor is given by certain of the roots $\delta_1, \ldots, \delta_k$. Also, from the above expression for elements of $Z^{p^{r-1}}$, we conclude that nonidentity elements of this group are regular unipotent elements of D.

Now T is centralized by δ and D is generated by those root subgroups for roots of level a multiple of p^{r-1}. It follows that $D = D^\delta$ and T induces a group of inner automorphisms on D. Also, $u^{p^{r-1}} \in D_\delta$, a product of classical groups. Let J denote the saturation of $v = u^{p^{r-1}}$ in D_δ. Then $v \in J \leq A$, a restricted A_1 of D_δ. Now for classical groups, restricted A_1's are just A_1's having restricted action on the classical module. It follows that A is also restricted in G and hence J is the saturation of v in G. Write $N_A(J) = JT'$, for T' a 1-dimensional torus. Now $J < D$ and must lie in the unique Borel subgroup, B_D, of D containing v. Also $W^{p^{r-1}} \leq C_D(v) \leq B_D$. Then B_D is the unique Borel containing J and $W^{p^{r-1}}$, so $B_D = B_D^T = B_D^{T'}$. Moreover $s < p$, so the unipotent radical of B_D has class less than p and we apply Lemma 4.2. Indeed, T induces a 1-dimensional torus, T_D, of D and after factoring the action of T_D through a Frobenius morphism, as in the proof of 4.5, we conclude that there is a unique 1-dimensional unipotent group of $R_u(B_D)$ containing v and invariant under both T and T'. Hence $W^{p^{r-1}} = J$, completing the proof. ∎

4.3 Uniqueness issues

In this section we show that the T-homocyclic group produced in in Theorem 1 is not necessarily canonical. Let $u \in G$ be unipotent. If $|u| = p$, then there does exist a canonical saturation, so assume $|u| = p^r > p$. Let T be a u-distinguished 1-dimensional torus such that $W = \langle u^T \rangle$ is a T-homocyclic group.

It follows from Lemma 2.4 that $uW^p \subset u^G$. So for $1 \neq w \in W^p$ there exist an element $g \in G$ with $uw = u^g$. Then $u \in W, W^{g^{-1}}$ and $W^{g^{-1}}$ is a $T^{g^{-1}}$-homocyclic group. Also $u^g \in W \leq Z$, so $C_G(u) = C_G(u^g)$ and hence $g \in N_G(C_G(u))$. So a necessary condition for W to be unique is that $g \in N_G(W)$, for all such g.

In some cases this condition does hold. However, we present two examples showing that this is not always the case.

Let $G = SL_4(K)$ with $p = 2$. Set

$$u = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

a regular unipotent element. A computation shows that $C = C_G(u)$ consists of matrices of form

$$\begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & a & 1 & \\ c & b & a & 1 \end{pmatrix}$$

for $a, b, c \in K$. Now choose T such that $T(c)$ is diagonal of shape $(c^3, c^1, c^{-1}, c^{-3})$. One then checks that $W = \langle u^T \rangle$ is T-homocyclic, where W consists of elements of the form

$$\begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & a & 1 & \\ a^3 + ab & b & a & 1 \end{pmatrix}$$

for $a, b \in K$. Also $W^2 = C^2$ is the 1-dimensional unipotent group consisting of matrices of form

$$\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ b & 0 & 1 & \\ 0 & b & 0 & 1 \end{pmatrix}$$

for $b \in K$. Letting U be the group of lower triangular unipotent matrices, we calculate that $N_U(W) = C$. On the other hand, fusion of elements of uW^p must be achieved by elements of $N_U(W)$, so W is not canonical in this case.

The following is a somewhat more complicated example with $p = 3$ and $u \in SL_6$. Let T consist of diagonal matrices of shape $(c^5, c^3, c^1, c^{-1}, c^{-3}, c^{-5})$ and let $W < SL_6$ consist of matrices of the form

$$\begin{pmatrix} 1 & & & & & \\ a & 1 & & & & \\ -a^2 & a & 1 & & & \\ b & -a^2 & a & 1 & & \\ a^4 + ab & b & -a^2 & a & 1 & \\ -a^2 b & a^4 + ab & b & -a^2 & a & 1 \end{pmatrix}$$

for $a, b \in K$. Once again calculation shows that $N_U(W) = C_U(W)$.

5 Theorem 3: the reductive part of $C_G(u)$

In this section we prove Theorem 3. We must produce a reductive subgroup J containing u satisfying certain properties. We will first produce a connected reductive subgroup E containing u and argue that $R = C_G(E)$ is a complement in $C_G(u)$ to $Q = R_u(C_G(u))$. The isogeny type of G is irrelevant for this purpose as u is a unipotent element. In the special case where $|u| = p$, we will take E as a restricted A_1 containing u and then set $J = C_G(R)$.

The following result settles the case where G is of classical type.

Lemma 5.1 *Theorem 3 holds if G is of classical type.*

Proof We first describe a suitable reductive subgroup E. To describe this subgroup we review some information given on pp.476-477 of [**?**].

We may assume $G = SL(V), Sp(V)$, or $SO(V)$. If W is a linear, symplectic, or orthogonal space we will use the notation $I(W) = GL(W), Sp(W)$, or $O(W)$ to denote the corresponding isometry group of W.

There is a decomposition $V = \sum_{1\leq i\leq k} V_i$, such that u acts on V_i as the sum of r_i Jordan blocks of size i and the summands are orthogonal with respect to the underlying form. For each i there is a tensor decomposition $V_i = W_i \otimes Z_{r_i}$ and a containment $I(V_i) \geq I(W_i) \circ I(Z_{r_i})$ such that $u \in G \cap \prod I(W_i)$. Certain parity conditions are necessary in the symplectic and orthogonal groups. In the symplectic case r_i is even if i is odd ($Sp(V_i) \geq SO(W_i) \circ Sp(Z_{r_i})$), whereas in the orthogonal case r_i is even if i is even ($SO(V_i) \geq Sp(W_i) \circ Sp(Z_{r_i})$).

Set $E = G \cap \prod I(W_i)$. In view of the particular Springer correspondence that we use for classical groups it follows that $e \in L(E)$. In view of the above parity conditions involving i and r_i it follows that u projects to a regular unipotent element in each of the classical groups W_i and that T is chosen so that it projects to a regular torus of each $I(W_i)$. That is, the projection of T to $I(W_i)$ acts on W_i with weights $i-1, i-3, \ldots, -(i-3), -(i-1)$.

Set $R = G \cap \prod I(Z_{r_i})$. By 3.7 of [**?**] we have $C_G(u) = RQ$, where $Q = R_u(C_G(u))$. We claim that $C_G(E) = R$. Now $C_G(E) \leq C_G(u)$, so $C_G(E) = RQ_0$ for $Q_0 = Q \cap C_G(E)$. The subspaces V_i are homogeneous components of V under the action of E, so that $C_G(E)$ and hence Q_0 acts on each of the spaces V_i. Then $Q_0 \downarrow V_i \leq C_{GL(V_i)}(I(W_i)^0) = GL(Z_{r_i})$. But $R \downarrow V_i$ contains $I(Z_{r_i})^0$ and normalizes $Q_0 \downarrow V_i$. It follows that $Q_0 \downarrow V_i = 1$ for all i and hence $Q_0 = 1$, as required. Hence, $C_G(E) = R$, which proves the claim.

At this point we define the subgroup J. We will set $J = E$ except for some special situations in orthogonal groups. Namely, suppose $G = SO(V)$ and there exist exactly two summands V_i and V_j with the property that $r_i = r_j = 1$. Here we set $V_{i,j} = V_i \oplus V_j$ and let $J = G \cap (I(V_{i,j}) \circ \prod_{k\neq i,j} I(W_k))$. In this exceptional case, the projection of u to $I(V_i \oplus V_j)$ is a semiregular unipotent element and the projection of T is the product of regular tori of $I(V_i)$ and $I(V_j)$.

We claim that $C_G(R) = J$. First consider the case where $J = E$. We argue that $C_G(R)$ must stabilize each of the spaces V_i. This is for the most part immediate from the fact that the spaces V_i are homogeneous components for V under the action of R. However, there is a subtlety here when $G = SO(V)$ and there exist subscripts i with $\dim(Z_{r_i}) = 1$. Here $I(Z_{r_i})$ is of order 2, but the corresponding involution of $O(V_i)$ does not lie in the special orthogonal group. If there are two or more such subspaces, then for every pair i, j, the product of the two corresponding involutions is contained in G, and using this we see that $C_G(R)$ must indeed stabilize each V_i, except when there are exactly two such subspaces, and this is precisely the case we have temporarily excluded. We note that this is not an issue for $G = SL(V)$. For if $\dim(Z_{r_i}) = \dim(Z_{r_j}) = 1$ for $i \neq j$, then there are scalars in R with distinct actions on the subspaces V_i and V_j. So $C_G(R)$ leaves invariant each V_i and $C_G(R) \downarrow V_i < C_{I(V_i)}(I(Z_{r_i})) = I(W_i)$. Hence, $J \leq C_G(R) \leq G \cap \prod I(W_i) = J$, proving the claim in this case.

Now consider the excluded case where there are exactly two subscripts i, j such that $r_i = r_j = 1$. Here $R \cap I(V_{i,j}) = \langle \tau \rangle$, where τ is the involution inducing -1 on $V_{i,j} = V_i \oplus V_j$ and acting trivially on all V_k for $k \neq i, j$. So in this case the homogeneous components of R on V are the subspaces $V_{i,j}, V_k$ for $k \neq i, j$ and the above argument shows that $C_G(R) = J = G \cap (I(V_{i,j}) \circ \prod_{k \neq i,j} I(W_k))$. This establishes the claim and the lemma follows. ■

We now consider exceptional groups where the analysis is more complicated. The following lemma will be used to establish the existence of a subgroup J with the required properties.

Lemma 5.2 *Let T_0 be a maximal torus of $C_G(u)$ and let E be a semisimple subgroup of $L = C_G(T_0)$ containing u. Assume*

i) E is not contained in a proper parabolic subgroup of L'.

ii) $R = C_G(E)$ is a complement in $C_G(u)$ to $R_u(C_G(u))$.

Then $J = C_G(R)$ is a reductive subgroup of G containing E and u is a semiregular unipotent element of J. Also $R = C_G(J)$.

Proof Let $J = C_G(R) = C_G(C_G(E)) \geq E$. As $T_0 \leq Z(L)$, we have $T_0 \leq C_G(E) = R$. Setting $V = R_u(J)$ we then have $V < J = C_G(R) \leq C_G(T_0) = L$. However, E normalizes V and by hypothesis, E is not contained in a proper parabolic subgroup of L'. It follows that $V = 1$, so that J is reductive.

We next show that u is a semiregular unipotent element of J. To this end, let s be a semisimple element in $C_J(u)$. Then $s \in C_G(u) \cap C_G(R)$. But $C_G(u) = QR$ where $Q = R_u(C_G(u))$, so all semisimple elements of $C_G(u) \cap N(R)$ are contained in R. But then $s \in R \leq C_G(J)$ and so $s \in Z(J)$. Finally, $R \leq C_G(J) \leq C_G(E) = R$, so $R = C_G(J)$, completing the proof of the lemma. ■

When u has order p, the next lemma shows that E can be taken as a restricted A_1 containing u. For unipotent elements of order greater than p, additional analysis is required.

Lemma 5.3 *Theorem 3 holds if* $|u| = p$.

Proof Assume $|u| = p$. Then Theorem 1.1 of [**?**] shows that u is contained in a restricted A_1 subgroup of G, say E. Moreover, Theorem 1.2 of [**?**] shows that $C_G(u) = QC_G(E)$, where $Q = R_u(C_G(u))$ and $R = C_G(E)$ is reductive. We can use Lemma 5.2 to complete the proof of Theorem 3 (setting $J = C_G(R)$) once we show that E is not contained in a proper parabolic subgroup of L'. Now Theorem 1.1(iv) of [**?**] shows that E is L-completely reducible. This means that if E is contained in a proper parabolic subgroup of L', then E is contained in a Levi subgroup of that parabolic. But if this occurs, then E would centralize a nontrivial torus of L', whereas T_0 is a maximal torus of $C_G(u)$ and $T_0 \leq Z(L)$. Thus, E cannot lie in a proper parabolic of L' and the proof is complete. ■

Lemma 5.4 *Theorem 3 holds if* G *is of exceptional type.*

Proof In view of the previous lemma we may assume $|u| > p$. By the Bala-Carter classification of unipotent elements there is a Levi subgroup L of G, such that u is a distinguished unipotent element of L'. Indeed, $L = C_G(T_0)$ where T_0 is a maximal torus of $C_G(u)$. From the possibilities for L, the order formula of Testerman, [**?**], and the fact that p is a good prime, it is straightforward to determine the precise possibilities and we will list these later in this proof.

In all cases $L' = L_0F$, with L_0 a simple factor of L' and $F = 1, A_1$ or A_2, and where the projection of u to L_0 has order greater than p. The projection of E to L_0, say E_0, will be either a simple group or the product of a simple group and a restricted A_1 subgroup of L_0. In the former case write $E = E_1E_2$, where $E_1 = E_0$ and E_2 is either trivial (if $F = 1$) or is a restricted A_1 of F containing the projection of u. In the latter case we again write $E = E_1E_2$, but here $E_0 = E_1C \leq L_0$, where C has type A_1 and we will take E_2 to be a restricted A_1 in the group CF. So in all cases, either $E = E_1$ is simple or $E = E_1E_2$, where E_1 is simple and E_2 has type A_1.

The following examples may help clarify the construction. Say $L = D_5A_2 < E_8$. If u_0 has type D_5, then $E = E_1E_2$, where $E_0 = E_1 = B_4$ and $E_2 = B_1 < A_2$. On the other hand, suppose u_0 has type $D_5(a_1)$, then $E_0 = G_2B_1(< B_3B_1 < D_5)$. Here we set $E_1 = G_2$ and E_2 a diagonal A_1 in the group $B_1B_1 < B_1A_2$.

In all cases either $E = E_1$ is simple or $E = E_1E_2$. After identifying E, our main objective is to check that $R = C_G(E)$ is a complement in $C_G(u)$ to $Q = R_u(C_G(u))$. From here Lemma 5.2 will complete the proof of Theorem 3. For this last step we need to know that E is not contained in a proper parabolic subgroup of L'. It will be clear from the construction that E_0 is not contained in a proper parabolic of L_0. And writing $L' = L_0F$ as before, the argument of the previous lemma shows that the projection of E to F (a restricted A_1 containing a distinguished unipotent element) cannot lie in a proper parabolic subgroup. In view of the above comments, our primary goal is to describe E_0 and $C_G(E)$.

The possibilities where $|u| > p$ and the groups E, J, and R are listed in the following tables and is partly based on information in the tables of [**?**]. We find E as above. The computation of R and J will be discussed later. In Table 1 - Table 5 we reserve the symbol A_1 for a group generated by opposite long root subgroups of G and use $\hat{A}_1$ for other connected groups of Lie rank 1.

Table 1. $E_8, |u| > p$

u	p	E	J	R
D_5	7	B_4	B_4	B_3
D_5A_1	7	B_4A_1	B_4A_1	$A_1\hat{A}_1$
$D_6(a_1)$	7	$B_4\hat{A}_1$	$B_4\hat{A}_1$	$A_1A_1 \cdot 2$
$E_7(a_4)$	7	$B_4\hat{A}_1$	D_6A_1	$A_1.2$
$E_6(a_1)$	7	C_4	C_4	$A_2 \cdot 2$
D_5A_2	7	$B_4\hat{A}_1$	$B_4\hat{A}_1$	$T_1 \cdot 2$
D_6	7	B_5	B_5	B_2
E_6	$7, 11$	F_4	F_4	G_2
$D_7(a_2)$	7	$B_4\hat{A}_1$	B_4B_2	$T_1 \cdot 2$
A_7	7	C_4	C_4	$\hat{A}_1$
$E_6(a_1)A_1$	7	C_4A_1	C_4A_1	$T_1 \cdot 2$
$E_7(a_3)$	7	B_5A_1	D_6A_1	$A_1 \cdot 2$
$E_8(b_6)$	7	$C_4\hat{A}_1$	$C_4\hat{A}_1$	Sym_3
$D_7(a_1)$	7	$B_5\hat{A}_1$	$B_5\hat{A}_1$	$T_1 \cdot 2$
E_6A_1	$7, 11$	F_4A_1	F_4A_1	$\hat{A}_1$
$E_7(a_2)$	$7, 11$	E_7	E_7	A_1
$E_8(a_6)$	7	B_4	B_4	Sym_3
D_7	$7, 11$	B_6	B_6	$\hat{A}_1$
$E_8(b_5)$	$7, 11$	$F_4\hat{A}_1$	$F_4\hat{A}_1$	Sym_3
$E_7(a_1)$	$7, 11, 13$	E_7	E_7	A_1
$E_8(a_5)$	$7, 11$	$B_6\hat{A}_1$	D_8	Sym_2
$E_8(b_4)$	$7, 11, 13$	E_7A_1	E_7A_1	Sym_2
E_7	$7, \ldots, 17$	E_7	E_7	A_1
$E_8(a_4)$	$7, 11, 13$	B_7	D_8	Sym_2
$E_8(a_3)$	$7, \ldots, 17$	E_7A_1	E_7A_1	Sym_2
$E_8(a_2)$	$7, \ldots, 19$	E_8	E_8	1
$E_8(a_1)$	$7, \ldots, 23$	E_8	E_8	1
E_8	$7, \ldots, 29$	E_8	E_8	1

Table 2. $E_7, |u| > p$

u	p	E	J	R
D_4	5	G_2	G_2	C_3
D_4A_1	5	G_2A_1	B_3A_1	C_2
A_5	5	C_3	C_3	G_2
$D_5(a_1)$	5	$G_2\hat{A}_1$	$B_3\hat{A}_1$	$A_1T_1 \cdot 2$
A_5'	5	C_3	C_3	$A_1\hat{A}_1$
A_5A_1	5	C_3A_1	C_3A_1	$\hat{A}_1$
$D_5(a_1)A_1$	5	$G_2\hat{A}_1$	$G_2\hat{A}_1$	$\hat{A}_1$
$D_6(a_2)$	5	$G_2\hat{A}_1$	D_6	A_1
$E_6(a_3)$	5	C_3A_1	C_3A_1	$\hat{A}_1 \cdot 2$
D_5	$5, 7$	B_4	B_4	$A_1\hat{A}_1$
$E_7(a_5)$	5	$C_3\hat{A}_1$	$C_3\hat{A}_1$	Sym_3
A_6	5	G_2	G_2	$\hat{A}_1$
D_5A_1	$5, 7$	B_4A_1	B_4A_1	$\hat{A}_1$
$D_6(a_1)$	$5, 7$	$B_4\hat{A}_1$	D_6	A_1
$E_7(a_4)$	$5, 7$	$B_4\hat{A}_1$	D_6A_1	Sym_2
D_6	$5, 7$	B_5	D_6	A_1
$E_6(a_1)$	$5, 7$	C_4	C_4	$T_1 \cdot 2$
E_6	$5, 7, 11$	F_4	F_4	A_1
$E_7(a_3)$	$5, 7$	B_5A_1	D_6A_1	Sym_2
$E_7(a_2)$	$5, 7, 11$	E_7	E_7	1
$E_7(a_1)$	$5, \ldots, 13$	E_7	E_7	1
E_7	$5, \ldots, 17$	E_7	E_7	1

Table 3. $E_6, |u| > p$

u	p	E	J	R
D_4	5	G_2	G_2	A_2
A_5	5	C_3	A_5	A_1
$D_5(a_1)$	5	$G_2\hat{A}_1$	D_5T_1	T_1
$E_6(a_3)$	5	C_3A_1	A_5A_1	Sym_2
D_5	$5, 7$	B_4	D_5T_1	T_1
$E_6(a_1)$	$5, 7$	C_4	E_6	1
E_6	$5, 7, 11$	F_4	E_6	1

Table 4. $F_4, |u| > p$

u	p	E	J	R
B_3	5	G_2	G_2	$\hat{A}_1$
C_3	5	C_3	C_3	A_1
$F_4(a_2)$	5	C_3A_1	C_3A_1	Sym_2
$F_4(a_1)$	$5, 7$	B_4	B_4	Sym_2
F_4	$5, 7, 11$	F_4	F_4	1

Table 5. $G_2, |u| > p$

u	p	E	J	R
G_2	5	G_2	G_2	1

Now $u \in E < L$ and we set $R = C_G(E)$. We compute $C_G(E_1)^0$ using the results in [**?**]. However, in doing so we note that the characteristic restrictions in [**?**] are slightly stronger than what is assumed here. In view of the fact that $|u_0| > p$ we see that the only difference occurs when $E_1 = G_2 < E_7$ with $p = 5$, which occurs when E_1 is contained in a subsystem group of type D_4 or A_6. Using the action of the subsystem group on $L(G)$ it is easy to argue that $C_G(E_1)^0 = C_3$ or A_1, respectively.

In all cases we have $C_G(E_1)^0 = R_2$ a specific reductive group. Indeed, R_2 is a product of small classical groups or G_2. Write $E = E_1E_2$, as before. If $E_2 > 1$ we compute $C_{R_2}(E_2)^0 = C_G(E)^0$. Now E_2 is a restricted A_1 in R_2, so this centralizer is reductive by Theorem 1.2(ii) of [**?**]. Moreover, a computation shows that R^0 is as indicated in the tables of [**?**]. It follows from results of Mizuno [**?**], [**?**], Shoji[**?**], and Chang [**?**] that for good primes the reductive part of the centralizer is as presented in the tables of [**?**].

We must verify that R/R^0 is also correct. We see from [**?**] and our assumption $|u_0| > p$ that the component group $C_G(u)/C_G(u)^0 \in \{1, S_2, S_3\}$. Recall that $u \in E < L$. We have chosen E so that the component group is easy to obtain. For example, in cases where $C_G(u)/C_G(u)^0 = S_3$, we see that L' has an element s of order 3 and E is also centralized by an involution in the Weyl group of G that acts as a graph automorphism of $C_G(s)$. For example, this is the case when $G = E_8, u = E_8(b_5)$ and $u \in A_1F_4 < A_2E_6 = C_G(s)$. In this way we see that R/R^0 contains a group isomorphic to $C_G(u)/C_G(u)^0$. On the other hand we have $R < C_G(u)$, so equality must hold.

Now that R has been explicitly determined it is relatively easy to calculate $J = C_G(R)$. At the outset $E \leq J \leq C_G(T_0) = L$. The information in the tables is then obtained from a direct check, in some cases using information in [**?**] to assist in the analysis. Namely, in some cases the tables in [**?**] give $C_G(R)$ explicitly. In other cases information on restrictions can be used to determine $\dim C_{L(G)}(R)$. This completes the proof of the lemma. ■

We complete this section with a result which provides additional information regarding the groups E, R, J listed in the above tables.

Proposition 5.5 *Assume G is of exceptional type and $|u| > p$. Let E, R, J be as in the tables of Lemma 5.4. Then*

i) $L(G) \downarrow ER$ is completely reducible.

ii) ER^0 and JR^0 are restricted subgroups of G (see [?])

iii) $C_{L(G)}(E) = L(R)$.

Proof It follows from the arguments of Lemma 5.4 (using [?]) that either $ER = M$ is a maximal subgroup of G (e.g. $G_2C_3 = M < E_7$) or ER is contained in some convenient maximal subgroup M (e.g. $B_5B_2 < D_8 = M < E_8$). We calculate $L(G) \downarrow M$. This can be achieved using 2.1, 2.3, and 2.4 of [?]. In cases where $ER < M$, next restrict to ER. This is usually straightforward using information in Section 2 of [?] or the well-known result that an irreducible spin module for D_n restricts to B_kB_{n-k-1} as the tensor product of the corresponding spin modules. Next, verify that all composition factors are restricted and the corresponding Weyl modules are irreducible. Consequently $L(G) \downarrow ER$ is completely reducible giving (i). Counting fixed points we have $\dim(C_{L(G)}(E)) = \dim(R)$, so that (iii) holds. In many cases $E = J$, giving (ii) as well. In the remaining cases, calculate $L(G) \downarrow JR$ as above to complete the proof. ∎

6 Theorem 4

Here we establish Theorem 4. We must identify a particular u-distinguished 1-dimensional torus $T < J$, where J is as in Theorem 3 and show that $C_G(u) \cap C_G(T) = C_G(J)$. In the proof of Theorem 3 we constructed a certain reductive subgroup $E \leq J$ such that $u \in E$ and $C_G(E) = C_G(J)$. Once T has been chosen the main difficulty is in showing that $C_G(u) \cap C_G(T)$ is reductive.

Write $C_G(u) = QR$, where $Q = R_u(C_G(u))$ and $R = C_G(E) = C_G(J)$ is reductive.

Lemma 6.1 *Theorem 4 holds if $|u| = p$.*

Proof Here E is a restricted A_1 containing u. Let B be the Borel subgroup of E containing u, let T be a maximal torus of B, and set $W = R_u(B)$. Proposition 6.1 of [?] shows that $C_G(u) = C_G(f)$ for f a generator of $L(W)$. It then follows from the Springer map and Lemma 2.2(iii) that e and f are G-conjugate. So replacing e by a conjugate, we may assume $e \in L(W)$. Now T acts on $L(W)$, so T normalizes $C_G(L(W)) = C_G(e) = C_G(u)$. Also, Proposition 3.2 of [?] implies T is u-distinguished, so T satisfies Theorem 1 and W is T-homocyclic.

We have $C_G(u) \cap C_G(T) = C_G(E)Q_0$, where Q_0 is trivial or a connected subgroup of Q. If $Q_0 = 1$, then there is nothing to prove. So assume $Q_0 > 1$. Then $L(Q_0) \leq C_{L(G)}(u) \cap C_{L(G)}(T) = C_{L(G)}(B)$.

We claim that $C_{L(G)}(B) = C_{L(G)}(E)$ and hence E acts trivially on $L(Q_0)$. Suppose for the moment that we have established the claim. Then for $s \in E$

semisimple we have $L(C_{Q_0}(s)) = C_{L(Q_0)}(s) = L(Q_0)$, so that s centralizes Q_0. Since E is generated by semisimple elements, we conclude that E centralizes Q_0, a contradiction. So this establishes the theorem. Thus it will suffice to establish the claim.

Theorem 1(iii) of [**?**] shows that $L(G) \downarrow E$ is a tilting module, with the possible exception of $G = A_n$ with $p|n+1$. Exclude the latter case for the moment. Then $L(G) \downarrow E$ is a direct sum of tilting modules of the form $T(r)$ for $r \leq 2p-2$. It will suffice to establish the claim for $T(r)$. Assume that $T(r)$ has a common fixed point for u and T.

If $r < p$, then $T(r)$ is a restricted irreducible module with a fixed point for B. This is only possible if $r = 0$ and so $T(r) < C_{L(G)}(E))$. Now suppose $r \geq p$. Write $r = p + c$ and $s = p - 2 - c$. Then $T(r)$ is uniserial of length 3 of shape $V(s)|V(r)|V(s)$ where $V(s)$ and $V(r)$ denote the irreducible modules of high weights s and r, respectively. It is shown in Lemma 2.3 of [**?**] that u has a 2-dimensional fixed point space on $T(r)$ and the fixed points are also fixed by W. So this fixed point space is T-invariant and has T weights r, s with the latter coming from the socle. So we must have $s = 0$ and again this fixed point is contained in $C_{L(G)}(E)$.

Finally, assume G is of type A_n with $p|n+1$. Let $\hat{G} = SL_{n+1}$ and let $\pi : \hat{G} \to G$ be the natural surjection. Taking preimages in $\hat{G}$ we may assume $G = SL_{n+1}$. Now regard $G < GL_{n+1}$. It is shown in Theorem 1(iii) of [**?**] that $L(GL_{n+1}) \downarrow E$ is a tilting module and we can repeat the above argument. ■

Assume from now on that $|u| > p$. We will require two preliminary lemmas.

Lemma 6.2 *Suppose V is a section of $L(G)$ invariant under both T and e. Assume also that*

i) The Jordan decomposition of e on V has the form $J_{s+1} \oplus J_{t+1} \oplus \cdots \oplus J_{w+1}$, with $s \geq t \geq \cdots \geq w$.

ii) The weights of T on V are $\{s, s-2, \ldots, -s\}, \{t, t-2, \ldots, -t\}, \ldots, \{w, w-2, \ldots, -w\}$.

Then the Jordan blocks in (i) can be chosen such that each Jordan block, J_{k+1} has a basis of T-weight vectors $v_k, v_{k-2}, \ldots v_{-k}$, where v_i has T-weight i and $[ev_i] = v_{i+2}$ ($[ev_k] = 0$).

Proof Choose a T-weight vector $v \in V$ of weight k. Let $\bar{ad}(e)$ denote the induced action of e on V. Then $\bar{ad}(e)(v)$ is a T-weight vector of weight $k+2$. In particular, if $k = s$, then $\bar{ad}(e)(v) = 0$. Since the degree of nilpotency of e on V is $s+1$, v can be chosen such that $k = -s$ and $I = \langle v, \bar{ad}(e)(v), \ldots, \bar{ad}(e)^s(v) \rangle$ is a T-invariant Jordan block of length $s+1$ with weights $s, s-2, \ldots, -s$.

Since I is a Jordan block of maximal length, it splits off under the action of e. Therefore, the hypotheses hold in V/I and we can use induction to decompose this quotient (which is isomorphic to a section of $L(G)$) into a sum of T-invariant Jordan blocks satisfying the conditions of the lemma. Suppose

J/I is one of these, generated by images of $j+I$ under $\bar{ad}(e)$, where j has T-weight $-r$. Consider $\bar{ad}(e)^{r+1}(j)$. If this is 0, then we can write J as the sum of two T-invariant Jordan blocks, I and $\langle j, \bar{ad}(e)(j), \ldots, \bar{ad}(e)^r(j)\rangle$. Otherwise, $\bar{ad}(e)^{r+1}(j) = i \in I$ is a nonzero T-weight vector of weight $r+2$. It follows that $r < s$ and there is a vector l of weight $-r$ in I such that $i = \bar{ad}(e)^{r+1}(l)$. Now replacing j by $j-l$ we again obtain a decomposition of J. Doing this for all the summands of V/I we have the result. ■

Lemma 6.3 *Let V be a section of $L(G)$ invariant under T and e. Assume $k > l$ and that J_{k+1}, J_{l+1} are T- invariant Jordan blocks for e such that T has weights $k, \cdots, -k$ and $l, \cdots, -l$, respectively, on these blocks. Then $C_{J_{k+1} \otimes J_{l+1}}(T, e) = 0$.*

Proof Write $J = J_{k+1} \otimes J_{l+1}$. Let $\tilde{J}_{l+1} < J_{k+1}$ denote the subspace spanned by weight vectors for weights $l, \cdots, -l$. Then a weight consideration shows that all fixed points of T on J lie within the subspace $\tilde{J}_{l+1} \otimes J_{l+1}$. Suppose $v = \sum v_s \otimes w_{-s}$ is in the fixed space of e, where the vectors v_s, w_{-s} are vectors of weight $s, -s$, respectively. Then $0 = [ev] = \sum([ev_s] \otimes w_{-s}) + (v_s \otimes [ew_{-s}])$. Choose s maximal with $v_s \neq 0$. Then $[ev_s] = v_{s+2} \neq 0 \in J_{k+1}$ of T-weight $s+2$. But then $[ev] \neq 0$, a contradiction. ■

Lemma 6.4 *Theorem 4 holds if G is of classical type.*

Proof We take E as in Lemma 5.1. Then in view of the fixed Springer map we have $e \in L(E)$. Choose $T \leq E$ a u-distinguished 1-dimensional torus of E corresponding to e. Then T is also u-distinguished in G. Now $C_G(u) \cap C_G(T) = C_G(E)Q_0$, where $Q_0 < Q$ is either trivial or a connected subgroup of Q. If $Q_0 = 1$, then there is nothing to prove. So assume $Q_0 \neq 1$. As Q_0 is a unipotent group and the projection map from the simply connected cover of G to G is an isomorphism when restricted to unipotent groups, we may assume that G is the corresponding isometry group.

We have $0 < L(Q_0) \leq C_{L(G)}(u) \cap C_{L(G)}(T)$. Now u and e correspond under the Springer correspondence, so that $C_G(u) = C_G(e)$. If G is symplectic or orthogonal, then as p is a good prime, this implies that $C_{L(G)}(u) = L(C_G(u)) = L(C_G(e)) = C_{L(G)}(e)$ and so $L(Q_0) \leq C_{L(G)}(e) \cap C_{L(G)}(T)$. If G is of type A_n, then the same conclusion holds since $e = u - 1$ and $L(Q_0)$ consists of matrices.

Assume $G = Sp(V)$ or $SO(V)$. Write $V = \sum_{1 \leq i \leq k} V_i$, an orthogonal decomposition as described in the proof of Lemma 5.1. Here u acts on V_i as the sum of r_i Jordan blocks of size i. Also $V_i = W_i \otimes Z_{r_i}$ and there is a containment $I(V_i) \geq I(W_i) \circ I(Z_{r_i})$ such that $u \in G \cap \prod I(W_i)$.

Here $L(G) = L(I(V_1)) \oplus \cdots \oplus L(I(V_k)) \oplus \sum_{i<j}(V_i \otimes V_j)$. Fix i and let u_i, e_i, T_i denote the projections of u, e, T to $I(W_i), L(I(W_i)), I(W_i)$, respectively. Then u_i and also e_i act on V_i as the sum of Jordan blocks of size i. These blocks can be chosen so that each is invariant under T_i and T_i has weights $i-1, i-3, \ldots, -(i-1)$ on the block. Fix $i \neq j$. Then Lemma 6.3 implies that $L(Q_0) \cap (V_i \otimes V_j) = 0$.

Consider $L(I(V_i))$ for $1 \leq i \leq k$. View this as a subspace of $L(GL(V_i)) \cong V_i \otimes V_i^*$. Viewed as a module for T and $e, Hom(K, V_i \otimes V_i^*) \cong Hom(V_i, V_i) \cong Hom(W_i^{r_i}, W_i^{r_i}) \cong Hom(W_i, W_i)^{{r_i}^2} \cong K^{r_i^2}$. We get the same result viewing V_i as a module for $I(W_i)$. It follows that any common fixed points of e and T on $L(I(V_i))$ are also fixed points of $I(W_i)$. A dimension count now gives $C_{L(GL(V_i))}(e) \cap C_{L(GL(V_i))}(T) = C_{L(GL(V_i))}(I(W_i)) = L(GL(Z_{r_i}))$. Intersecting this with $L(I(V_i))$ gives $L(I(Z_{r_i}))$. As $R = G \cap \prod I(Z_{r_i})$, we have $L(Q_0) \leq C_{L(G)}(e) \cap C_{L(G)}(T) = L(R)$. But R is reductive and normalizes Q_0, so this forces $Q_0 = 1$.

Finally, assume $G = SL(V)$. Here we work in $GL(V)$ where the Lie algebra is $V \otimes V^*$, and argue as above that common fixed points of e and T are also fixed by R, completing the proof. ■

Lemma 6.5 *Theorem 4 holds if G is of exceptional type.*

Proof By Theorem 3 we have $u \in J \leq L \leq G$ where L is a Levi subgroup such that u is a distinguished unipotent element in L. Here $L = C_G(T_0)$, where T_0 is a maximal torus of $C_G(u)$. Also, $C_G(u) = QR$, where $R = C_G(J) = C_G(E)$ and $J = C_G(R)$. Choose $e \in L(G)$ such that $C_G(u) = C_G(e)$.

We first argue that $e \in L(J)$. First note that $C_G(e) > R$, so that $e \in C_{L(G)}(R)$. So it will suffice to show that $C_{L(G)}(R) = L(J)$. We may choose $T_0 \leq R$, so we immediately have $C_{L(G)}(R) \leq C_{L(G)}(T_0) = L(C_G(T_0)) = L(L)$. And as $L(T_0) \cap C_{L(G)}(R) = 0$, we have $C_{L(G)}(R) \leq L(L')$. In a few cases $J = L'$ and the assertion is obvious. In the other cases a direct check gives the assertion. Indeed, in most cases it is possible to choose a group of type Sym_2 or Sym_3 in $N_R(T_0)$ and note that $L(J)$ is the set of fixed points of this group on $L(L')$. Hence, $e \in L(J)$.

The tables of Lemma 5.4 give the possibilities for E and J. We next argue that there is a J-conjugate of E whose Lie algebra contains e. Of course, if $E = J$ this is obvious. If J is a product of classical groups, then using the special Springer correspondence within these classical groups it follows that $L(E)$ contains nilpotent elements with the same E-centralizer as u. Then Lemma 2.2(iii) implies that e is contained in a J-conjugate of E. The only remaining cases to settle are where $J = E_6$ with $E = F_4$ or C_4. But E_6 has precisely two semiregular classes of nilpotent elements and each class is stabilized by the graph automorphism. A Frattini argument implies that each semiregular nilpotent element is centralized by an involution in the coset of a graph automorphism. The assertion follows since there are two classes of involutions in the coset of a graph automorphsim with corresponding fixed points F_4 and C_4 and each of these groups has just one class of semiregular elements. So replacing E by a suitable conjugate we may assume $e \in L(E)$. However, with this change we may no longer have $u \in E$.

Now $C_E(e) = C_E(u_E)$, for some unipotent element u_E of E, so Lemma 2.6 shows that there is a u_E-distinguished torus, T, of E such that T acts on $\langle e \rangle$ acting by weight 2. Then T normalizes $C_G(e) = C_G(u)$.

We claim that T is a u-distinguished torus of G. To verify this we need only show that T determines the correct labelled diagram of L'. It will be sufficient to work with the simple factors of L' and the corresponding projection of T. For classical factors this is easy from the choice of E and the action on the natural module. Now consider a factor of L' of exceptional type. Here we refer to the tables of Lemma 5.4. If the projection of E to this factor is surjective, then the assertion is immediate. Consider one of the other cases (e.g. $E_7(a_4), E_6(a_1), E_6, ...$). For this particular assertion the characteristic is irrelevant - all that matters are the weights of T, which are independent of the characteristic. Consequently we can use the information in Table A, pp.65-66 of [**?**] which gives the corresponding labelled diagram for T. We also make use of the tables in [**?**] which list the labelled diagram for each of the relevant unipotent classes. This yields the assertion in all but one case in E_8 where we have chosen our subgroup E different from the one presented in [**?**]. This case is $E_8(b_6)$ and it is necessary to determine the weights of T on $L(G)$ to verify that they are the same as those obtained from the corresponding labelled diagram. However, this is straightforward using the construction of E together with information in [**?**] from which we can determine $L(G) \downarrow E$. This proves the claim so that we now have $T < E$, with T a u-distinguished 1-dimensional torus of G, and $e \in L(E)$ is the corresponding nilpotent element.

As $T \leq J = C_G(R)$, we have $C_G(u) \cap C_G(T) = RQ_0$, where Q_0 is a connected subgroup of Q. We must show that $Q_0 = 1$. Now $L(Q_0) \leq C_{L(G)}(u) \cap C_{L(G)}(T)$ and we next claim that $C_{L(G)}(u) \cap C_{L(G)}(T) = L(R)$. As $L(R) \cap L(Q_0) = 0$, this implies $Q_0 = 1$, as required. So it suffices to establish the claim which will be accomplished by making some reductions and then converting the assertion to one involving only $e \in L(E)$ and $T \leq E$, as above.

Write $L(G) = L(L) \oplus S$, with both summands invariant under L. To find common fixed points of u and T on $L(L)$, first note that $L = L'Z(L)^0 = L'T_0$. Then

$$C_{L(L)}(u) = L(T_0) \oplus C_{L(L')}(u).$$

Now p is a good prime for L' (since it is good for G) and u is distinguished in L', so

$$C_{L(L')}(u) = L(C_{L'}(u)) \leq L(R_u(P)).$$

However, T has only positive weights on $L(R_u(P))$, so we have

$$C_{L(L)}(u) \cap C_{L(L)}(T) = L(T_0).$$

We now look for common fixed points on S where the goal is to show

$$C_S(u) \cap C_S(T) \leq L(R).$$

Of course this is obvious if u is distinguished in G, for here $L = G$ and $S = 0$. So we now assume u is not distinguished.

We have $C_{L(G)}(u) = L(C_G(u)) = L(C_G(e)) = C_{L(G)}(e)$, so intersecting with S it will suffice to show

$$C_S(e) \cap C_S(T) \leq L(R).$$

Since $T \leq E$ and $e \in L(E)$, parts (i) and (iii) of Proposition 5.5 imply that we need only show

$$C_F(e) \cap C_F(T) = 0$$

for each nontrivial E-composition factor F in S. Lemmas 6.2 and 6.3 will be used to see that such an F has no common fixed points of e and T.

Recall from the construction in 5.4 that either $E = E_1$ is a simple group or $E = E_1E_2$, a product of two simple groups, with E_2 a restricted A_1. Let F be a nontrivial composition factor of E on V. Then either $F = F_i$, an irreducible representation of E_i or $F = F_1 \otimes F_2$, the tensor product of two nontrivial representations.

In the following table we list possible choices for E_1 and high weights μ_1 of F_1. We also indicate the weights of T_1 on F_1, where T_1 is the projection of T to E_1. We use the notation $(a); (b); \ldots; (r)$ to indicate the fact that T_1 has weights $a, a-2, \ldots, -a, b, b-2, \ldots, -b, \ldots, r, r-2, \ldots -r$. With one exception, these weights are obtained from 2.13 of [**?**]. The exception is where $E_1 = G_2$ for the dominant weight $2\lambda_1$. Here the corresponding irreducible module is a summand of codimension 1 in the symmetric square of the usual 7-dimensional module, so an easy computation yields the information indicated in the following table.

E_1	μ_1	T_1 weights
B_6	λ_1	(12)
	λ_6	$(21); (15); (11); (9); (3)$
B_5	λ_1	(10)
	λ_5	$(15); (9); (5)$
B_4	λ_1	(8)
	λ_4	$(10); (4)$
G_2	λ_1	(6)
	$2\lambda_1$	$(12); (8); (4)$
F_4	λ_4	$(16); (8)$
C_4	λ_1	(7)
	λ_2	$(12); (8); (4)$
	λ_3	$(15); (11); (9); (5); (3)$
C_3	λ_1	(5)
	λ_2	$(8); (4)$
	λ_3	$(9); (5); (3)$
E_7	λ_7	$(27); (17); (9)$

Recall that we are looking for common fixed points of T and e on F. We can immediately rule out two configurations in the above table. Observe that for the cases $(E_1, \mu_1) = (B_6, \lambda_6), (C_4, \lambda_3)$ there are no fixed points of T_i on the given module since all the weights are odd. Moreover, as u is not distinguished, these cases only occur when $E = E_1$ (the pair (C_4, λ_3) only occurs when $L = A_7 < E_8 = G$) so that we are not in a tensor product situation. So for these cases there is nothing to do. Similarly if $(E_1, \mu_1) = (E_7, \lambda_7)$, the weights are odd and a tensor product situation can only occur in the case u is distinguished, which we have already settled.

For the remaining cases we consider the action of e_i on F_i where e_i is the projection of e to $L(E_i)$. For e_2 this is relatively easy, since E_2 is either trivial or a restricted A_1. In fact for the cases considered, all composition factors of E_2 on S are restricted, so the Jordan blocks of e_2 are immediate from the high weights of the composition factors.

Next we describe the Jordan blocks of e_1 for the various weights given in the above table. The Jordan blocks of e_1 on F_1 are immediate in those cases where F_1 is the classical module for a classical group. Consider the spin module for B_5. Here we view $B_5 < D_6 < E_7$. The restricted 56-dimensional module for E_7 restricts to D_6 as two copies of the natural module plus a copy of the spin module. So from the Jordan form of e_1 on the orthogonal module and the 56-dimensional module, we can deduce the action on the spin module. The Jordan structure on the large module was calculated by Lawther (private communication) using a variation of his computer program that calculated the Jordan blocks for the corresponding unipotent element. Similarly for the action of B_4 on the spin module, where we use the embedding $B_4 < D_5 < E_6$ and the action on a restricted 27-dimensional module. Lawther also calculated the Jordan blocks of the regular nilpotent element of F_4 on the irreducible module with high weight λ_4.

The remaining C_3 and C_4 cases can be handled via direct calculation. Start with the known action on the symplectic module and then consider the wedge square and wedge cube (only for C_3) of this module. Splitting off either a trivial or natural module, we obtain the necessary information. Finally the case of G_2 on the irreducible module with high weight $2\lambda_1$ is settled, as above, by viewing this as a module of codimension 1 in the symmetric square of the usual orthogonal module.

The results from the above considerations show that in each case the Jordan blocks of e_1 on F_1 are just as in large characteristic or characteristic 0 and are compatible with the T_1-weights on this module, as in the hypothesis of Lemma 6.2.

Now consider the possibilities for F. If $F = F_1$, then the above table together with Lemma 6.2 show T and e have no common fixed points on F. And if $F = F_2$, then F is a nontrivial irreducible restricted module for E_2, a group of type A_1, so here too there are no common fixed points. In the remaining cases, $F = F_1 \otimes F_2$, a nontrivial tensor product, and the result follows from Lemmas 6.2 and 6.3. In checking this one verifies, using the information in Tables 1-5 of Lemma 5.4, that F_2 has high weight 1 or 2 on F_2, except when $E_1 = B_4$ or B_5, and in the latter cases F_2 has high weight at most 3. As an example, consider the case $E = B_5B_1 < D_7 < E_8$, where $F = F_1 \otimes F_2$, a tensor product of spin modules of dimension 32 and 2 respectively. By Lemma 6.2 we can write $F_1 = J_{16} \oplus J_{10} \oplus J_6$ where each summand is invariant under e_1 and T_1, with compatible T_1-weights. Similarly, $F_2 = J_2$. It follows from Lemma 6.3 that T

and e have no common fixed points on F, establishing the claim and completing the proof of the lemma. ■

At this point we have completed the proof of Theorem 4.

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