RESEARCH STATEMENT

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My current research interests are in equivariant homotopy and cohomology theories, particularly for spaces with a $G = \mathbb{Z}/2$ action. Much work has been done (in [LMM] or [CW] for example) to extend many of the usual topological tools to the equivariant world (Homotopy Extension and Lifting Property, etc). In [Br], Bredon created equivariant homology and cohomology theories of G-spaces, now called Bredon homology and Bredon cohomology. In [LMM], a cohomology theory for G-spaces is constructed that is graded on $RO(G)$, the ring of virtual representation of G. For the case $\mathbb{Z}/2$, this produces a bigraded cohomology theory since any real representation V of $\mathbb{Z}/2$ can be decomposed as a direct sum of copies of the trivial 1-dimensional representation $\mathbb{R}^{1,0}$ and copies of the nontrivial 1-dimensional representation $\mathbb{R}^{1,1}$. The appropriate coefficient systems for this theory are Mackey functors, $\mathcal{M}$. If the virtual representation V has virtual dimension (p, q), then the Vth cohomology group is denoted $H^V(X, \mathcal{M}) = H^{p,q}(X, \mathcal{M})$. Most of my work has dealt with cohomology with coefficients in the Mackey functor $\underline{\mathbb{Z}/2}$ which has the form of the diagram below.

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[loop above, "id"] \arrow[r, bend left, "id"] & \mathbb{Z}/2 \arrow[l, bend left, "0"] \end{tikzcd}$$

Many of the usual tools for computing cohomolgy have their counterparts in the $RO(G)$-graded setting. These include a Mayer-Vietoris sequence, Künneth theorem, suspension isomorphisms, etc. Missing from the $RO(G)$ computational tool box was an equivariant version of the Serre spectral sequence associated to a fibration $F \to E \to B$. In my thesis, I create such a spectral sequence.

Theorem 0.1. *If $f\colon E \to X$ is a fibration of $\mathbb{Z}/2$ spaces, then for every $r \in \mathbb{Z}$ and every Mackey Functor $\mathcal{M}$ there is a spectral sequence with $E_2^{p,q} = H^{p,0}(X, \mathcal{H}^{q,r}(f, M)) \Rightarrow H^{p+q,r}(E, M)$.*

Under certain connectivity assumptions, the spectral sequence takes the following form:

Theorem 0.2. *If $f\colon E \to X$ is a fibration of $\mathbb{Z}/2$ spaces with X equivariantly 1-connected and fiber F, then for every $r \in \mathbb{Z}$ and every Mackey Functor $\mathcal{M}$ there is a spectral sequence with $E_2^{p,q} = H^{p,0}(X, \underline{H}^{q,r}(F, \mathcal{M})) \Rightarrow H^{p+q,r}(E, \mathcal{M})$.*

This spectral sequence is a generalization of the spectral sequence of [MS] for Bredon cohomology. In the case of the identity map $id\colon X \to X$, the above spectral sequence generalizes the Bredon spectral sequence in [Br].

Question 0.3. Are there weaker hypotheses on the fibration under which the local coefficient system $\mathcal{H}^{q,r}(f, M)$ reduces to the constant coefficient system $\underline{H}^{q,r}(F, \mathcal{M})$?

Here, the Mackey functor $\underline{H}^{q,r}(F,\mathcal{M})$ is given by $\underline{H}^{q,r}(F,\mathcal{M})(G/H) = H^{q,r}(G/H \times F,\mathcal{M})$, and the restriction and transfer maps are induced by those in the Mackey functor $\mathcal{M}$. This closely mimics the usual Leray-Serre spectral sequence of a fibration.

One of the applications of the non-equivariant Leray-Serre spectral sequence is to compute the singular cohomology ring of ΩS^n with, say, $\mathbb{Z}/2$ coefficients. In my thesis, I produce an analogous result using the spectral sequence above.

Proposition 0.4. *If $S^{p,q}$ is equivariantly 1-connected, then $H^{*,*}(\Omega S^{p,q},\underline{\mathbb{Z}/2})$ is an exterior algebra over $H^{*,*}(pt,\underline{\mathbb{Z}/2})$ on generators $a_1, a_2, \ldots,$ where $a_i \in H^{(p-1)\cdot q^{i-1},q^i}(\Omega S^{p,q},\underline{\mathbb{Z}/2})$.*

Here, the sphere $S^{p,q} \cong (\mathbb{R}^{p,q})^+$, the one point compactification of the representation $\mathbb{R}^{p,q}$.

In non-equivariant topology, the Leray-Serre spectral sequence gives rise to a description of characteristic classes of vector bundles. Consider the universal bundle $E_n \to G_n$ over the Grassmannian of n-planes in $\mathbb{R}^\infty$. Forming the associated projective bundle $\mathbb{P}(E_n) \to G_n$ yields a fiber bundle with fiber $\mathbb{RP}^\infty$. Applying the Leray-Serre spectral sequence to this projective bundle yields characteristic classes of E_n as the image of the cohomology classes $1, z, z^2, \cdots \in H^*_{sing}(\mathbb{RP}^\infty;\mathbb{Z}/2)$ under the transgressive differentials. Since this universal bundle classifies vector bundles, characteristic classes of arbitrary bundles can be defined as pullbacks of the characteristic classes, $c_i \in H^i(G_n;\mathbb{Z}/2)$, of the universal bundle. It would be nice to adapt this construction to the $\mathbb{Z}/2$ equivariant setting. However, the equivariant space $G_n((\mathbb{R}^{2,1})^\infty) = G_n(\mathcal{U}) = G_n$ is not 1-connected, and so the spectral sequence is not as easy to work with. There is an alternative method.

Theorem 0.5 (Equivariant Leray-Hirsch). *Let $E \to B$ be a $\mathbb{Z}/2$-fiber bundle over and equivariantly 1-connected space B. Suppose that for some ring Mackey Functor $\mathcal{M}$ and some point $\sigma\colon G/G \to B$ the following conditions are satisfied:*

(a) $H^{*,*}(F;\mathcal{M})$ is a finitely generated free $H^{*,*}(pt;\mathcal{M})$-module, and

(b) there exist classes $c_j \in H^{*,*}(E;\mathcal{M})$ whose restrictions $i^*(c_j)$ form a basis for $H^{*,*}(F;\mathcal{M})$ in the fiber F.

(Here $F = \sigma^*(B)$ is the fiber over σ and $i\colon F \to E$ is the inclusion.)

Then the map $\Phi\colon H^{*,*}(B;\mathcal{M}) \otimes_{H^{*,*}(pt;\mathcal{M})} H^{*,*}(F;\mathcal{M}) \to H^{*,*}(E;\mathcal{M})$ given by $\sum_{ij} b_i \otimes i^*(c_j) \mapsto \sum_{ij} p^*(b_i) \cup c_j$ is an isomorphism.

As shown in my thesis, the cohomology of the infinite projective space $\mathbb{RP}^\infty_{tw} = \mathbb{P}(\mathcal{U})$ is a free module over the cohomology of a point with $\underline{\mathbb{Z}/2}$ coefficients. If $E \to B$ is an equivariant vector bundle over a 1-connected space B, then the equivariant Leray-Hirsch theroem can be applied to the equivariant bundle $\mathbb{P}(E) \to B$ to obtain equivariant characteristic classes $c_{i,j} \in H^{i,j}(B,\underline{\mathbb{Z}/2})$ for equivariant bundles over equivariantly 1-connected spaces. This leads to many interesting questions.

Question 0.6. Are these all of the equivariant characteristic classes? In other words, can something be done to get characteristic classes for the universal bundle $E_n \to G_n$? Is $H^{*,*}(G_n,\underline{\mathbb{Z}/2})$ an algebra over the cohomology of a point generated by the equivariant characteristic classes $c_{i,j}$?

Question 0.7. More generally, for an arbitrary equivariant vector bundle $E \to B$, can $H^{*,*}(\mathbb{P}(E),\underline{\mathbb{Z}/2})$ be expressed as a polynomial algebra over $H^{*,*}(B,\underline{\mathbb{Z}/2})$, subject to some predictable relations?

Question 0.8. Do the equivariant characteristic classes $c_{i,j}$ satisfy some kind of equivariant analogue of the Whitney sum formula?

Related to the $\mathbb{Z}/2$-equivariant category is the category $Sm/\mathbb{R}$ of smooth schemes over $\mathbb{R}$. There is a functor $-(\mathbb{C})\colon Sm/\mathbb{R} \to Top_{\mathbb{Z}/2}$ which assigns to the scheme X the space of complex points $X(\mathbb{C})$ endowed with the conjugation action. I am interested to learn more about this motivic world, in particular more about the relationship between Voevodsky's motivic cohomology of a scheme X, [Voe], and the $RO(\mathbb{Z}/2)$-graded cohomology of $X(\mathbb{C})$.

Question 0.9. Which of the results of my thesis can be adapted to the motivic setting? In particular, can similar methods be used to understand characteristic classes of algebraic vector bundles with bilinear forms in the motivic setting?

References

[Br] G. E. Bredon. *Equivariant cohomology theories*, Springer Lecture Notes in Mathematics, vol. 34, Springer (1967).

[CW] Steven R. Costenoble and Stefan Waner. *Equivariant ordinary homology and cohomology*, http://www.citebase.org/abstract?id=oai:arXiv.org:math/0310237 (2003).

[LMM] L.G.Lewis, J. P.May and J. E. McClure. *Ordinary RO(G)-graded cohomology*, Bull. Amer. Math. Soc. 4 (1981), 208-212.

[MS] Moerdijk I., Svensson J.-A., *The Equivariant Serre Spectral Sequence*, Proc. AMS, Vol.118 (1993), 263-278.

[Voe] V. Voevodsky, *Motivic cohomology with Z/2-coefficients*, Publ. Math. Inst. Hautes Etudes Sci. 98 (2003), 59104.

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