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A calculation on the sliding of ice over a wavy surface using a Newtonian viscous approximation

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A glacier slides over its irregular rock bed by a combination of regelation and plastic deformation (Weertman 1957). An exact calculation for this combined process is made possible by using a model in which the flow properties of the ice are simplified as Newtonian viscous, rather than obeying a more realistic nonlinear flow law. The bed is represented as a smooth plane on which there are perturbations of general three-dimensional form but small slope, and the ice is assumed to maintain contact with the bed everywhere. The first-order solution for the velocity field leads to an expression for the drag, which is a second-order effect. It is found that the velocities due to regelation and to viscous flow are additive only when the bed consists of a single sine wave. In the general case the total drag is a summation of the drags due to each of the Fourier components of the bed relief taken separately. The total drag is expressible in terms of a single average property of the bed relief, namely, the product of its mean square amplitude and its autocorrelation function, or, alternatively, its power spectrum. Numerical illustrations are given for a Gaussian autocorrelation function.

1. Introduction

According to the Weertman theory (1957, 1964) a glacier sliding over its rock bed may move past obstacles by two mechanisms. The first is regelation, whereby the ice melts under pressure on the upstream side of the obstacles and refreezes on the downstream side. The second is plastic deformation. This paper provides an exact calculation of the drag from the combined processes by idealizing the flow properties of the ice as those of a Newtonian viscous material. The ice is pictured as sliding slowly over an undulating smooth rigid surface, which can exert only normal, not tangential, forces, and, at the same time, the ice is melting and freezing at the interface as it encounters variations of pressure. Inertia forces are negligible. The resistance to motion arises from the regelation process and also from the deformation that the ice undergoes in conforming to the shape of the surface. This is a model appropriate to ice at the melting point, where a water film provides the lubricant. If the ice is below the melting point there are two physical differences: there is no water to provide a lubricant, and regelation is suppressed. To adapt the model to these conditions it is simple to eliminate the regelation process. On the other hand, the lubrication condition is an essential part of the model, and therefore we do not expect our results to apply to ice below the melting point unless the molecular adhesion between ice and rock should turn out to be so small that it makes only a small contribution to the total drag.

Weertman estimated the drag arising from the combined plastic deformation and regelation mechanism by using the more realistic, nonlinear, Glen flow law

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for ice (strain-rate proportional to the 3rd or 4th power of the stress). Weertman's result is expressed in terms of the average dimensions and spacings of obstacles. Lliboutry (1959, 1968) has also made an estimate, again based on the Glen flow law, but taking first a sinusoidal profile, with amplitude small compared with the wavelength, and then considering a superposition of sine waves. If the ambient pressure is not sufficiently high, cavities will form in the lee of obstacles, and when this happens it may have an important effect on the drag (Lliboutry 1959, 1968; Weertman 1964). In this paper we deal with sliding without cavitation and make an exact calculation for a bed which is a slightly perturbed plane. This means that the irregularities in the bed can be of arbitrary three-dimensional form provided their slopes are always small. The price paid for an exact calculation with this degree of generality is that it is made for a Newtonian viscous material rather than for one obeying a nonlinear flow law. On the other hand, it has the advantage of bringing out very clearly the principles governing the superposition (*a*) of the regelation and the plastic deformation process and (*b*) of the drags from the different harmonic components of the bed surface. Both these are difficult points in calculations made with a nonlinear flow law and it is helpful to see their exact formulation in the linear flow law approximation. In addition, the analysis produces the following useful result. The drag can be expressed in two quite different forms. The first, and more obvious, is as an integral involving the two-dimensional Fourier components of the bed relief. The second form is as an integral involving the bed relief $z_0(x,y)$ itself. It is interesting that, in this second representation, the only feature of $z_0(x,y)$ that is relevant in determining the drag is the product of the mean square amplitude $\langle z_0^2\rangle$ and the autocorrelation function $c(X, Y)$, defined by

$$c(X,Y) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y\, z_0(x,y)\, z_0(x+X, y+Y)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y\,\{z_0(x,y)\}^2}.$$

Both $\langle z_0^2\rangle$ and $c(X, Y)$ express mean properties of the bed relief. It is, of course, very reasonable that certain average properties of the bed relief $z_0(x,y)$, rather than all its details, should determine the over-all drag. In the Newtonian viscous approximation one has the advantage of being able to see explicitly how this comes about and which average properties are relevant.

We start with a bed $z_0(x)$ which is wavy in the x direction only, and use a Fourier transform method. Then we extend the analysis to the bed $z_0(x,y)$ which is wavy in both the x and y directions.

2. Bed wavy in one direction only

We consider slow flow only and neglect inertia terms throughout. It may be verified at the end that this is permissible for vanishingly small Reynolds number, as is appropriate for a glacier, where the Reynolds number is in the range 10^{-11} to

10^{-17}. We also neglect gravity forces, since the scale of the perturbations of the bed is small. First consider a Newtonian viscous material of viscosity η moving steadily with uniform velocity U in the Ox direction between two boundary planes at $z = 0$ and $z = z_1$ (figure 1*a*). The boundary at $z = 0$ is smooth and stationary, while the boundary at $z = z_1$ is of the type that allows no slip of the fluid and is moving at velocity U, with the fluid. The lower boundary will correspond to the ice-rock

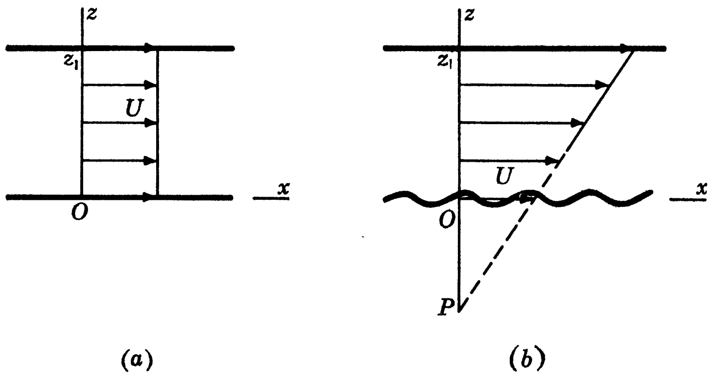

FIGURE 1 (*a*) Unperturbed flow, (*b*) flow after introducing a perturbation in the bed and a steady velocity on the plane $z = z_1$.

interface, the bed, while the upper one is introduced merely to make the boundary conditions of the problem specific. Now impose on the lower boundary (figure 1*b*) a small perturbation, so that its equation is

$$z = \epsilon f(x) = z_0(x), \text{ say,}$$

where ϵ is a dimensionless number small compared with 1, and let the mean z_0 be zero. At the same time impose a certain additional velocity on the upper boundary. Let the velocity components be u, w and consider the new steady motion. We choose z_1 large compared with any significant wavelength in the bed variation. The bed now exerts a drag on the material so that, at large z, u varies linearly with z. We fix the velocity imposed at the upper boundary by requiring that the average of u on the bed remains unchanged at the value U. The problem is to find the drag per unit area of the bed, which is η times the value of $\partial u/\partial z$ at large z.

This model would be appropriate in the absence of regelation. When regelation is present a certain amount of ice either melts or freezes on to the lower surface. This has the effect of changing the lower boundary condition in our model of the flow within the ice. Instead of there being no velocity component normal to the perturbed bed there is now a definite distribution of normal velocity, positive and negative, given by the freezing and melting rates. Although this complicates our mental picture of the flow it does not, as we shall see, complicate the mathematics to any great extent.

Turning now to the physics of the regelation process, our picture is that, owing to the depression of the melting point by pressure, there is melting at places of high pressure and freezing at places of low pressure. Water flows from one to the other via a thin film which separates the ice and the rock. At the same time the latent heat liberated at the freezing places flows through the ice and through the rock to the melting places. A thorough study of the analogous problem of regelation by a wire cutting through a block of ice (Nye 1967; Frank 1967; Nunn & Rowell 1967; Townsend & Vickery 1967; Shreve, private communication) shows that the details of this traditional picture need modification, perhaps substantial. There is evidence, however, that under pressures high enough to suppress cavitation and with materials of low conductivity, such as rock, the traditional picture may suffice, and we shall therefore adopt it in this paper. When flow in the ice occurs in conjunction with regelation there will be some modification in the thermal conduction process in the ice, in that the ice is now moving faster than it would otherwise have done. This effect of conduction in a moving medium is negligible when the velocity is that of pure regelation (Nye 1967) and will be assumed still negligible at the somewhat larger velocities which exist when regelation is combined with plastic flow.

A connecting link between the flow process and the regelation process is the distribution of pressure $p_n(x)$ on the bed, for this must, of course, be the same for both processes. In the same way the distribution of velocity normal to the bed $w_n(x)$ is common to both processes. Our procedure in calculation will be to start with an unknown distribution $w_n(x)$ and to calculate $p_n(x)$, first by going through the physics of regelation and then by solving the viscosity equations. Equating the two expressions for $p_n(x)$ will finally give us our result.

(i) *Regelation*

Consider unit length of bed parallel to Oy and a length $\mathrm{d}s$ measured in the xz plane. The volume of new ice frozen in this area per unit time is $w_n(x)\,\mathrm{d}s$, where $w_n(x)$ is the outward normal velocity component of the ice on the bed. The heat liberated is $Lw_n(x)\,\mathrm{d}s$, where L is the latent heat per unit volume of ice. To first order we can consider this heat source as $Lw_n(x)\,\mathrm{d}x$ lying in a strip of width $\mathrm{d}x$ in the plane $z = 0$. The temperature drop across the water film is taken as negligible (Nye 1967). Within the rock and the ice (ignoring internal melting and refreezing in the ice) the temperature θ obeys Laplace's equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2} = 0.$$

We take the Fourier transform thus

$$\int_{-\infty}^{\infty} \mathrm{d}x\,\frac{\partial^2\theta}{\partial x^2}\,\mathrm{e}^{-ikx}+\int_{-\infty}^{\infty} \mathrm{d}x\,\frac{\partial^2\theta}{\partial z^2}\,\mathrm{e}^{-ikx} = 0. \tag{1}$$

It is sufficient for our purpose if we allow the perturbation of the bed to be confined to a large but finite length of the x axis, namely $-\frac{1}{2}l < x < \frac{1}{2}l$, so that at sufficiently large distances from the origin all perturbations are zero. Then the first term in (1) may be integrated twice by parts to give

$$-k^2\overline{\theta}+\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}z^2} = 0, \quad \text{where} \quad \overline{\theta} = \int_{-\infty}^{\infty} \mathrm{d}x\,\theta(x,z)\,\mathrm{e}^{-ikx}.$$

The solution that satisfies the heat sources at $z = 0$ and does not diverge at $\pm\infty$ is

$$\bar{\theta} = \frac{L\bar{w}_n(k)}{2K|k|}\,\mathrm{e}^{-|kz|},$$

where $\bar{w}_n(k)$ is the Fourier transform of $w_n(x)$ and K is the mean thermal conductivity of ice and rock.

If θ is measured relative to the average melting point, we may write for the temperature on the boundary, $\theta(x) = -Cp_n(x)$, where C is a constant and $p_n(x)$ is the normal pressure (relative to the average pressure), or, taking transforms,

$$\bar{\theta}(k) = -C\bar{p}_n(k).$$

Bars will always denote Fourier transforms. Thus.

$$\bar{p}_n(k) = -\frac{L\bar{w}_n(k)}{2CK|k|}. \tag{2}$$

This is the relation we need, from the physics of regelation, between the distribution of normal velocity on the bed, represented here by its Fourier transform $\bar{w}_n(k)$, and the distribution of normal pressure, represented by its Fourier transform $\bar{p}_n(k)$. The corresponding calculation for the case of pure regelation, without plastic deformation, over a bed consisting of a single sine wave has been done by Lliboutry (1968, pp. 39–40).

(ii) *Flow within the ice*

We now seek a connexion between the distributions of normal pressure and normal velocity at the bed by studying the slow viscous flow of the ice.

The fundamental hydrodynamical equations for the slow steady flow of an incompressible viscous fluid in two dimensions, without body forces and with inertia terms neglected, are

$$\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial p}{\partial x}, \quad \eta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial p}{\partial z}, \tag{3}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0, \tag{4}$$

where η is the viscosity, u and w are the velocity components and p is the pressure. Operating on the first equation with $\partial/\partial x$ and on the second with $\partial/\partial z$, adding and using the third equation, gives

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2} = 0, \tag{5}$$

We write for the perturbed solution

$$\left.\begin{aligned} u &= U+\epsilon u_1(x,z)+\epsilon^2 u_2(x,z)+\dots, \\ w &= \qquad \epsilon w_1(x,z)+\epsilon^2 w_2(x,z)+\dots, \\ p &= \qquad \epsilon p_1(x,z)+\epsilon^2 p_2(x,z)+\dots. \end{aligned}\right\} \tag{6}$$

Boundary conditions

There are two conditions to be satisfied on the bed: the normal component of the velocity is given as $w_n(x)$ and the tangential component of the surface traction is zero. The first condition gives

$$-u\sin\alpha + w\cos\alpha = w_n \quad \text{on} \quad z = z_0, \tag{7}$$

where $\tan\alpha = \mathrm{d}z_0/\mathrm{d}x$. Since $w_n(x)$ is zero when ϵ is zero we write $w_n(x) = \epsilon W(x)$. Equation (7) must be expressed as a condition on $z = 0$. Using (6) and remembering that $z_0(x) = \epsilon f(x)$, we have, to order ϵ,

$$-Uf'(x) + w_1(x, 0) = W(x). \tag{8}$$

We have here assumed that α is small, of order ϵ.

To write down the second boundary condition denote the stress components by $\sigma_x, \sigma_z, \tau_{xz}$ and define the stress deviator components σ_x', σ_z' by

$$\sigma_x' = \sigma_x + p, \quad \sigma_z' = \sigma_z + p,$$

where $p = -\frac{1}{2}(\sigma_x + \sigma_y)$. Thus $\sigma_x' + \sigma_z' = 0$. Consider the tractions (apart from hydrostatic pressure) on a small triangular element with hypotenuse along the bed at an angle α to the x axis, and with the other two sides parallel to Ox and Oz. The condition that there is no tangential traction on the hypothenuse is readily obtained by resolving forces parallel to it:

$$\tau_{xz} = \sigma_x' \tan 2\alpha \quad \text{on} \quad z = z_0. \tag{9}$$

To express this as a condition on $z = 0$, first write

$$\tau_{xz}(x, z) = \epsilon\tau_1(x, z) + O(\epsilon^2).$$

Now both σ_x' and α are $O(\epsilon)$. Hence $\tau_1(x, z_0) = 0$, and, since z_0 is $O(\epsilon)$, we have

$$\tau_1(x, 0) = 0,$$

as the second boundary condition on $z = 0$.

In terms of velocity perturbations this is

$$\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} = 0 \quad \text{on} \quad z = 0. \tag{10}$$

$\partial w_1/\partial x$ on $z = 0$ may be obtained by differentiating the first condition (8). Thus the two conditions may be finally written

$$\left.\begin{aligned} w_1 &= Uf'(x) + W(x) \\ \partial u_1/\partial z &= -Uf''(x) - W'(x) \end{aligned}\right\} \quad \text{on} \quad z = 0. \tag{11}$$

Solution of the equations

When the solution (6) is substituted into the differential equations (3), (4) and (5), there result, for the first-order perturbation, the equations

$$\eta\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial z^2}\right) = \frac{\partial p_1}{\partial x}, \quad \eta\left(\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial z^2}\right) = \frac{\partial p_1}{\partial z}, \tag{12}$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z} = 0, \tag{13}$$

and

$$\frac{\partial^2 p_1}{\partial x^2}+\frac{\partial^2 p_1}{\partial z^2} = 0. \tag{14}$$

Taking the Fourier transform of (14) and integrating by parts as in equation (1), gives

$$-k^2\overline{p}_1+\frac{\mathrm{d}^2\overline{p}_1}{\mathrm{d}z^2} = 0, \quad \text{where} \quad \overline{p}_1 = \int_{-\infty}^{\infty} \mathrm{d}x\, p_1(x,z)\,\mathrm{e}^{-\mathrm{i}kx}.$$

The required solution is

$$\overline{p}_1 = A_1\mathrm{e}^{-|k|z}, \tag{15}$$

where A_1 is an arbitrary constant, since the solution proportional to $\mathrm{e}^{+|k|z}$ diverges at infinity. Taking the Fourier transforms of (12) and (13), and using (15), we have

$$\eta\,\mathrm{d}^2\overline{u}_1/\mathrm{d}z^2-\eta k^2\overline{u}_1 = \mathrm{i}kA_1\mathrm{e}^{-|k|z}, \tag{16}$$

$$\eta\,\mathrm{d}^2\overline{w}_1/\mathrm{d}z^2-\eta k^2\overline{w}_1 = -|k|A_1\mathrm{e}^{-|k|z}, \tag{17}$$

$$\mathrm{i}k\overline{u}_1+\mathrm{d}\overline{w}_1/\mathrm{d}z = 0. \tag{18}$$

The general solutions of (16) and (17) are

$$\overline{u}_1 = -\frac{\mathrm{i}A_1}{2\eta}\frac{k}{|k|}z\mathrm{e}^{-|k|z}+B_1\mathrm{e}^{-|k|z}, \tag{19}$$

$$\overline{w}_1 = \frac{A_1}{2\eta}z\mathrm{e}^{-|k|z}+C_1\mathrm{e}^{-|k|z}, \tag{20}$$

where B_1, C_1 are arbitrary constants. The boundary conditions on u_1 and w_1 are given by equations (11), which when transformed become

$$\left.\begin{aligned}\overline{w}_1 &= \mathrm{i}Uk\overline{f}+\overline{W}\\ \partial\overline{u}_1/\partial z &= -\mathrm{i}k(\mathrm{i}Uk\overline{f}+\overline{W})\end{aligned}\right\} \quad \text{on} \quad z = 0.$$

These two boundary conditions are insufficient to fix the three constants A_1, B_1, C_1, because in deriving (14) we differentiated the original equations and thereby introduced unwanted solutions. We must make sure that the solutions are compatible with (18) and this adds the necessary third condition on the constants. The solution satisfying the boundary conditions is then found to be

$$\left.\begin{aligned}\bar{u}_1 &= -\mathrm{i}k(\mathrm{i}U\bar{f}k+\bar{W})\,z\,\mathrm{e}^{-|k|z}, &(21)\\ \bar{w}_1 &= (\mathrm{i}U\bar{f}k+\bar{W})\,(1+|k|z)\,\mathrm{e}^{-|k|z}, &(22)\\ \bar{p}_1 &= 2\eta(\mathrm{i}U\bar{f}k+\bar{W})|k|\,\mathrm{e}^{-|k|z}, &(23)\end{aligned}\right\}$$

and, in particular, on $z = 0$,

$$\bar{p}_1(k) = 2\eta\{\mathrm{i}U\bar{f}(k)\,k+\bar{W}(k)\}\,|k|. \tag{24}$$

To find the normal pressure p_n on $z = 0$ we note that

$$p_n = -\sigma_z = p-\sigma'_z = p-2\eta\,\partial w/\partial z.$$

It is readily found from equation (22) that $\partial\bar{w}_1/\partial z = 0$ on $z = 0$. Hence, to order ϵ, $\partial w/\partial z = 0$ on $z = 0$, and, therefore, to order ϵ, $p_n = p$. Thus equation (24) provides the relation from viscosity theory that we have been seeking between the pressure distribution and the normal velocity distribution on $z = 0$; to order ϵ the Fourier transforms of these two distributions are $\epsilon\bar{p}_1(k)$ and $\epsilon\bar{W}(k)$. The corresponding relation obtained from regelation physics is (2), which may be written

$$\bar{p}_1(k) = -\frac{L\bar{W}(k)}{2CK|k|} \quad (z = 0). \tag{25}$$

Solving (24) and (25) as simultaneous equations for $\bar{p}_1(k)$ and $\bar{W}(k)$ gives

$$\bar{p}_1(k) = \frac{2\mathrm{i}\eta U\bar{f}k_*^2\,k|k|}{k_*^2+k^2} \quad (z = 0), \quad \bar{W}(k) = -\frac{\mathrm{i}U\bar{f}k^3}{k_*^2+k^2}, \tag{26}$$

where $k_*^2 = L/4CK\eta$. With $L = 70\,\mathrm{cal\,cm^{-3}}$, $C = 0{\cdot}007\,^\circ\mathrm{C\,bar^{-1}}$, $K = 0{\cdot}005\,\mathrm{cal\,cm^{-1}\,^\circ C^{-1}\,s^{-1}}$ and $\eta = 1$ bar yr, we find $k_* = 0{\cdot}1\,\mathrm{cm^{-1}}$, which corresponds to a wavelength of 50 cm. The expression thus found for $\bar{W}(k)$ may now be substituted into equations (21), (22) and (23) to give the flow solution to order ϵ, thus

$$\left.\begin{aligned}\bar{u}_1 &= U\bar{f}\frac{k_*^2k^2}{k_*^2+k^2}\,z\,\mathrm{e}^{-|k|z},\\ \bar{w}_1 &= \mathrm{i}U\bar{f}\frac{k_*^2k}{k_*^2+k^2}\,(1+|k|z)\,\mathrm{e}^{-|k|z},\\ \bar{p}_1 &= 2\mathrm{i}\eta U\bar{f}\frac{k_*^2k|k|}{k_*^2+k^2}\,\mathrm{e}^{-|k|z}.\end{aligned}\right\} \tag{27}$$

Consider for a moment the case where the bed is the single sine wave

$$z_0(x) = \epsilon a\sin k_0x.$$

It is clear that changing the sign of ϵ gives a bed that is identical except for a displacement, and therefore gives the same drag. Thus the drag will depend on even powers of ϵ, and it is in fact of order ϵ^2. It is possible to calculate it for the general case $z_0(x) = \epsilon f(x)$ by carrying the perturbation analysis as far as terms in ϵ^2 and then finding the average value of the stress τ_{xz} at fixed z. I have done this, but it is much more direct to make use of the fact that, physically, the drag on the bed arises because the normal pressure on the upstream side of the bumps is greater than that on the downstream side. To find the drag one simply forms the appropriate integral for the component of the normal pressure on the bed acting in the flow direction. The advantage of this method is that it only requires a knowledge of $p(x, z_0)$ to order ϵ, and thus it is not necessary to carry the perturbation analysis any further than we have already done. It also makes it clear that although the drag is of $O(\epsilon^2)$ it arises directly from the first-order perturbation, $O(\epsilon)$, in the flow, and this in turn explains why the drags arising from the different Fourier components of the bed are simply additive, as we shall see in a moment.

The force in the x direction on the bed, taking unit width in the y direction, is

$$F = \int_{-\infty}^{\infty} \mathrm{d}x\, p(x, z_0)\, \frac{\mathrm{d}z_0}{\mathrm{d}x} = \int_{-\infty}^{\infty} \mathrm{d}x\, \epsilon p_1(x, 0)\, \epsilon f'(x), \tag{28}$$

to order ϵ^2. By a reverse Fourier transformation,

$$f = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{d}k\, \bar{f}(k)\, \mathrm{e}^{ikx},$$

and hence, keeping $z = 0$ from now on,

$$F = (\epsilon^2/2\pi) \int_{-\infty}^{\infty} \mathrm{d}x\, p_1(x) \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{i}k\bar{f}(k)\, \mathrm{e}^{ikx},$$

or

$$F = (\mathrm{i}\epsilon^2/2\pi) \int_{-\infty}^{\infty} \mathrm{d}k\, k\bar{f}(k)\, \bar{p}_1^*(k), \tag{29}$$

where the asterisk denotes the complex conjugate. This is a general formula for the drag force into which we may substitute the expression for $\bar{p}_1^*(k)$ given by the complex conjugate of the first of equations (26). Thus

$$F = \frac{\epsilon^2 \eta U k_*^2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\bar{f}(k) \bar{f}^*(k)\, k^2 |k|}{k_*^2 + k^2}$$

or, since the integrand is even in k,

$$F = \frac{2\eta U k_*^2}{\pi} \int_0^{\infty} \mathrm{d}k\, \frac{|\bar{z}_0(k)|^2 k^3}{k_*^2 + k^2}. \tag{30}$$

Equation (30) expresses the drag force F in terms of the Fourier transform $\bar{z}_0(k)$ of the bed surface $z_0(x)$.

If the regelation process were absent (say $K \to 0$), we should have $k_*^2 \to \infty$ and

$$F = (2\eta U/\pi) \int_0^{\infty} \mathrm{d}k |\bar{z}_0(k)|^2 k^3.$$

If, on the other hand, movement were entirely by regelation (say $\eta \to \infty$), we should have $k_*^2 \to 0$ and, using $k_*^2 = L/4CK\eta$,

$$F = (LU/2\pi CK) \int_0^{\infty} \mathrm{d}k |\bar{z}_0(k)|^2 k.$$

Let us recall that the length of the perturbed part of the bed is l, and therefore the drag per unit area of the perturbed bed is F/l. To avoid introducing l explicitly into the expression for the drag per unit area we may introduce the mean square amplitude of the bed relief $\langle z_0^2 \rangle$. Angle brackets, $\langle\,\rangle$, will always denote the average value over the area of the perturbation. Then

$$\langle z_0^2 \rangle = \frac{1}{l}\int_{-\infty}^{\infty} \mathrm{d}x \{z_0(x)\}^2 = \frac{1}{2\pi l}\int_{-\infty}^{\infty} \mathrm{d}k |\bar{z}_0(k)|^2 = \frac{1}{\pi l}\int_0^{\infty} \mathrm{d}k |\bar{z}_0(k)|^2,$$

by Parseval's theorem. Then the drag per unit area $\langle \tau_{xz} \rangle$ may be written in the general case as

$$\langle \tau_{xz} \rangle = 2\eta U \langle z_0^2 \rangle \frac{k_*^2 \displaystyle\int_0^{\infty} \mathrm{d}k \frac{|\bar{z}_0(k)|^2 k^3}{k_*^2 + k^2}}{\displaystyle\int_0^{\infty} \mathrm{d}k |\bar{z}_0(k)|^2}. \tag{31}$$

These results could have been obtained, though less compactly, by assuming a periodic bed and using Fourier series rather than integrals. For example, if $z_0(x)$ is given by the sine series

$$z_0(x) = \epsilon f(x) = \epsilon(a_1 \sin k_0 x + a_2 \sin 2k_0 x + \ldots),$$

the leading terms of the solution are

$$\left.\begin{aligned}
u &= U + \epsilon U z \sum_{n=1}^{\infty} a_n \beta_n (nk_0)^2 \mathrm{e}^{-nk_0 z} \sin nk_0 x + \epsilon^2 \\
&\qquad\times \left[Uz \left\{ \sum_{n=1}^{\infty} a_n^2 \beta_n (nk_0)^3 \right\} + \ldots \right] + O(\epsilon^3), \\
w &= \epsilon U \sum_{n=1}^{\infty} a_n \beta_n (nk_0)(1 + nk_0 z) \mathrm{e}^{-nk_0 z} \cos nk_0 x + O(\epsilon^2), \\
p &= 2\epsilon\eta U \sum_{n=1}^{\infty} a_n \beta_n (nk_0)^2 \mathrm{e}^{-nk_0 z} \cos nk_0 x + O(\epsilon^2), \\
\tau_{xz} &= -2\epsilon\eta U z \sum_{n=1}^{\infty} a_n \beta_n (nk_0)^3 \mathrm{e}^{-nk_0 z} \sin nk_0 x + \epsilon^2 \\
&\qquad\times \left[\eta U \left\{ \sum_{n=1}^{\infty} a_n^2 \beta_n (nk_0)^3 \right\} + \ldots \right] + O(\epsilon^3),
\end{aligned}\right\} \tag{32}$$

where $\beta_n = k_*^2/\{k_*^2 + (nk_0)^2\}$ and dots denote harmonics, the drag per unit area being

$$\eta U \sum_{n=1}^{\infty} (\epsilon a_n)^2 \frac{k_*^2 (nk_0)^3}{k_*^2 + (nk_0)^2}. \tag{33}$$

This shows clearly that the drag arises from the term in u of order ϵ^2, which has a non-fluctuating component proportional to z. The expression for τ_{xz} shows that it has fluctuations of order ϵ, which vanish on $z = 0$; the steady term, which gives the drag, appears in order ϵ^2. The drags from the various harmonic components of the bed are additive and each contributes a term proportional to the square of its amplitude. The relative phases of the Fourier components are of no importance. The reason for this additive property lies in the steps leading from equation (28) to equation (30). The drag is proportional to the integral of the product of $p_1(x, 0)$ and $f'(x)$. When both functions are expanded into Fourier series the cross-terms give zero, after integration, by the orthogonality property of Fourier components. Thus the pressure fluctuations given by the mth Fourier component of the bed profile have no net interaction with the nth Fourier component of the bed slope, if $m \neq n$. (The precise place in the analysis where this orthogonality property of the Fourier components is invoked is the equation following (28) where the formula for the reverse Fourier transform is used.)

The argument about the non-interaction of different Fourier components only holds because to obtain the drag the product of $p_1(x, 0)$ and $f'(x)$ is integrated over a whole period, or from $-\infty$ to $+\infty$ in the case of the Fourier integral representation. From another point of view we may think of the drag as the mean value, the zero-order harmonic, of τ_{xz}. From this standpoint interaction between the harmonics begins, naturally enough, in the ϵ^2 terms. Thus, *all* harmonic components of the bed contribute to the *zero-order* harmonic of the drag, which is what we have calculated. Similarly, they all interact, but in a more complicated way, to contribute to each higher harmonic of the drag; these are the terms, indicated by the dots in equations (32), that we have not evaluated.

The single sine wave

$$z_0(x) = A \sin k_0 x$$

would evidently give a drag per unit area of amount

$$\langle \tau_{xz} \rangle = \eta U A^2 \frac{k_*^2 k_0^3}{k_*^2 + k_0^2}. \tag{34}$$

If we now think of U as the mean slip velocity on $z = 0$ produced by a given applied shear stress $\langle \tau_{xz} \rangle$ (applied, say, at the level $z = z_1$ in figure 1*b*) we may use this equation to write U as the sum of two terms

$$U = \frac{\langle \tau_{xz} \rangle}{\eta A^2} \left(\frac{1}{k_0^3} + \frac{1}{k_*^2 k_0} \right) = \frac{\langle \tau_{xz} \rangle}{\eta A^2 k_0^3} + \frac{4 \langle \tau_{xz} \rangle CK}{A^2 L k_0}.$$

The first term is the velocity that would be produced if regelation were absent ($K \to 0$), and the second is the velocity that would be produced if there were no deformation in the material ($\eta \to \infty$). Thus the first is the velocity due to pure viscosity and the second is the velocity due to pure regelation. The actual velocity is the sum of the two.

The form of the expression (33) shows that this decomposition of the velocity into two parts can only be made when the bed is a single sinusoid. When the bed is a sum of sinusoids one cannot simply add the regelation velocity to the viscous flow velocity. The reason why a single sinusoid has this property, while other curves do not, may be explained in the following way. Flow over a sinusoid by pure regelation gives a sinusoidal variation of pressure on the boundary. Likewise, flow over a sinusoid by pure viscous flow gives a sinusoidal pressure variation on the boundary. The two velocity fields may therefore be combined, since they have a common boundary pressure, without disturbing one another. But if the bed is anything other than a sinusoid (for example a sinusoid and a higher harmonic) the forms of the pressure distributions demanded by the two processes acting alone are different. So the two flows cannot be combined without mutual disturbance.

The superposition rule is therefore as follows. Decompose the bed into harmonics and for each harmonic add the velocity due to pure regelation to the velocity due to pure viscous flow. This gives the total velocity produced by a given applied shear stress—and hence the drag produced by a given velocity. The total drag for a given velocity is the sum of the drags due to the various harmonic components of the bed. This is also the conclusion reached by Lliboutry (1968), on more intuitive grounds, for the case when the flow law is non-linear.

We have assumed throughout that no cavities form between the bed and the flowing material, and we may suppose, roughly, that cavities are liable to form if the normal stress at the interface becomes tensile. To ensure that this does not happen the ambient pressure p_0, say, must be sufficiently high that the total pressure $p_0+p_n > 0$ everywhere on the bed (p_n is the normal pressure due to the viscous flow and regelation processes and takes both positive and negative values). We have already seen that, to order ϵ, $p_n = p$ on $z = 0$. The condition is thus

$$p_0 > -p.$$

For the single sine wave $z_0(x) = A \sin k_0 x$ this gives $p_0 > 2\eta U A k_*^2 k_0^2/(k_*^2 + k_0^2)$.

(iii) *The drag as an integral over the bed*

Having seen how the drag may be expressed as a summation or an integral over the Fourier components of the bed, we now try to express it as an integral over the bed itself. In equation (30) the quantity $|\bar{z}_0(k)|^2$ is the power spectrum of the bed relief, and this suggests that we should introduce the (dimensionless) autocorrelation function $c(X)$ of the bed defined by

$$c(X) = \int_{-\infty}^{\infty} \mathrm{d}x\, z_0(x)\, z_0(x+X) \Big/ \int_{-\infty}^{\infty} \mathrm{d}x \{z_0(x)\}^2. \tag{35}$$

$c(X)$ is symmetrical about $X = 0$, where it takes the value 1, and falls to zero at large X since the bed relief at widely separated points is uncorrelated. Taking the Fourier transform we have

$$\bar{c}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, z_0(x) \int_{-\infty}^{\infty} \mathrm{d}X\, z_0(x+X)\, \mathrm{e}^{-\mathrm{i}kX} \Big/ \int_{-\infty}^{\infty} \mathrm{d}x \{z_0(x)\}^2.$$

With the substitution $x' = x+X$ the numerator on the right-hand side reduces to

$$\int_{-\infty}^{\infty} \mathrm{d}x z_0(x)\, \mathrm{e}^{\mathrm{i}kx} \int_{-\infty}^{\infty} \mathrm{d}x' z_0(x')\, \mathrm{e}^{-\mathrm{i}kx'} = \bar{z}_0^*(k)\, \bar{z}_0(k) = |\bar{z}_0(k)|^2$$

(this is really an application of the convolution theorem). Thus we may substitute into equation (30) and divide by l to obtain

$$\begin{aligned}\langle\tau_{xz}\rangle &= 2\pi^{-1}\eta U\langle z_0^2\rangle\, k_*^2 \int_0^\infty \mathrm{d}k\bar{c}(k)\; k^3(k_*^2+k^2)^{-1} \\ &= 2\pi^{-1}\eta U\langle z_0^2\rangle\, k_*^2 \int_0^\infty \mathrm{d}k\, k^3(k_*^2+k^2)^{-1} \int_{-\infty}^\infty \mathrm{d}X c(X)\, \mathrm{e}^{-\mathrm{i}kX}.\end{aligned}$$

We must not rearrange these integrals and try to carry out the integration over k because this leads to divergence at $k = \infty$. Instead a factor of k is first removed by integrating by parts with respect to X, to obtain

$$\langle\tau_{xz}\rangle = -2\mathrm{i}\pi^{-1}\eta U\langle z_0^2\rangle\, k_*^2 \int_0^\infty \mathrm{d}k\, k^2(k_*^2+k^2)^{-1} \int_{-\infty}^\infty \mathrm{d}X c'(X)\, \mathrm{e}^{-\mathrm{i}kX};$$

the first term in the integration by parts vanishes at both limits, because $c(X)$ is zero at infinity. Write the exponential as $\cos kX - \mathrm{i}\sin kX$, and then, since $c'(X)$ is odd in X, the expression becomes

$$\begin{aligned}\langle\tau_{xz}\rangle &= -4\pi^{-1}\eta U\langle z_0^2\rangle\, k_*^2 \int_0^\infty \mathrm{d}k\, k^2(k_*^2+k^2)^{-1} \int_0^\infty \mathrm{d}X c'(X)\sin kX \qquad (36)\\ &= -4\pi^{-1}\eta U\langle z_0^2\rangle\, k_*^2 \int_0^\infty \mathrm{d}X c'(X) \int_0^\infty \mathrm{d}k\{1-k_*^2(k_*^2+k^2)^{-1}\}\sin kX.\end{aligned}$$

In integrating $\sin kX$ with respect to k the upper limit contributes zero in view of the subsequent integration over X. Therefore we finally obtain

$$\langle\tau_{xz}\rangle = -4\pi^{-1}\eta U\langle z_0^2\rangle\, k_*^2 \int_0^\infty \mathrm{d}X c'(X)\,\{X^{-1}-k_* F_1(k_* X)\}, \qquad (37)$$

where

$$F_1(k_* X) = k_* \int_0^\infty \mathrm{d}k(k_*^2+k^2)^{-1}\sin kX = \tfrac{1}{2}\{\mathrm{e}^{-k_* X}\mathrm{Ei}(k_* X)+\mathrm{e}^{k_* X}E_1(k_* X)\},$$

$\mathrm{Ei}\,(k_* X)$ and $E_1(k_* X)$ being exponential-integral functions defined and tabulated by Abramowitz & Stegun (1964, pp. 228-43, especially 5.1.30 and 5.1.31).

Equation (37) is the result we want, for it shows that the drag is proportional to the mean square amplitude of the bed relief $\langle z_0^2\rangle$ and to an integral that involves no property of the bed other than the dimensionless autocorrelation function $c(X)$. The expression $\{X^{-1}-k_* F_1(k_* X)\}$ is a weighting factor to be applied to $c'(X)$; it behaves as X^{-1} for small X, falls to zero at $X = 1{\cdot}86k_*^{-1}$, reaches a minimum value of $-0{\cdot}045k_*$ at $X = 2{\cdot}85k_*^{-1}$, and then approaches zero for large X as $-2k_*^{-2}X^{-3}$. In fact only one mean property of the bed relief is involved rather than two, namely the product $\langle z_0^2\rangle\, c(X)$. This is equal to the autocovariance, which may be defined for our purpose as

$$l^{-1}\int_{-\infty}^\infty \mathrm{d}x\, z_0(x)\, z_0(x+X),$$

and all the results could be expressed in terms of this function. It is a little more convenient, however, to continue in terms of $\langle z_0^2\rangle$ and $c(X)$.

As $k_* \to 0$, which corresponds to pure regelation without viscous flow,

$$k_* F_1(k_* X) \to 0.$$

Hence, for pure regelation, since $k_*^2 = L/4CK\eta$,

$$\langle \tau_{xz} \rangle = -\frac{UL\langle z_0^2 \rangle}{\pi CK} \int_0^\infty \mathrm{d}X \frac{c'(X)}{X}. \tag{38}$$

Near the origin $c'(X)$ is proportional to $-X$ and therefore the integrand remains finite. The behaviour at the other limit $k_* \to \infty$, which corresponds to viscous flow without regelation, is best obtained, not from equation (37), but by returning to equation (36), which takes the form

$$\langle \tau_{xz} \rangle = -4\pi^{-1}\eta U \langle z_0^2 \rangle \int_0^\infty \mathrm{d}k\, k^2 \int_0^\infty \mathrm{d}X c'(X) \sin kX.$$

The k^2 factor must be removed by integrating by parts twice with respect to X, and then, carrying out the integration with respect to k:

$$\langle \tau_{xz} \rangle = -4\pi^{-1}\eta U \langle z_0^2 \rangle \int_0^\infty \mathrm{d}X X^{-1} c''''(X) \Big[\cos kX\Big]_{k=0}^{k=\infty}.$$

The upper limit of k contributes zero in view of the integration over X. From the lower limit we therefore have

$$\langle \tau_{xz} \rangle = 4\pi^{-1}\eta U \langle z_0^2 \rangle \int_0^\infty \mathrm{d}X X^{-1} c''''(X) \tag{39}$$

as the required expression for the drag in the absence of regelation—the counterpart of (38).

The meaning of the various formulae for the drag may now be illustrated by examples.

(a) *Single sine wave*

When the bed is a single sine wave of amplitude A and wave number k_0, the drag is given by equation (34). If k_0 is much smaller than k_*, so that the waves are very long, the formula reduces to

$$\langle \tau_{xz} \rangle = \eta U A^2 k_0^3.$$

The drag is entirely determined by viscosity, the heat conduction path for the regelation process being so long that regelation plays no part. In the short-wave approximation, on the other hand, k_0 is much larger than k_* and we have

$$\langle \tau_{xz} \rangle = \eta U A^2 k_*^2 k_0 = (UL/4CK)\, A^2 k_0.$$

Here the drag is entirely determined by regelation, viscous flow being absent.

Formula (34) may also be written as

$$\langle \tau_{xz} \rangle = \eta U k_*^2 (A k_0)^2 k_0 (k_*^2 + k_0^2)^{-1}. \tag{40}$$

The function $k_0(k_*^2 + k_0^2)^{-1}$ has a maximum at $k_0 = k_*$. Therefore, if one considers a number of sine waves each with the same ratio of amplitude to wavelength, so that

Ak_0 remains the same, the wave with $k_0 = k_*$ will give the greatest drag. Both Weertman (1957) and Lliboutry (1968) consider models with this feature, namely, in varying the obstacle size they keep the ratio of amplitude to wavelength (the 'roughness') constant. In such a model k_* represents an optimum wave number, at which the regelation velocity and viscous flow velocities are equal. For shorter waves regelation predominates, because the thermal conduction path is small, while for longer waves viscous flow predominates. The most effective sine wave for producing drag is thus the one with $k_0 = k_*$. This lead Weertman to the idea that on a bed of general shape the drag would arise predominantly from obstacles of this size—the 'controlling obstacle size'. A difficulty with the further development of this idea is to know what range of obstacle sizes in the neighbourhood of the controlling size is effective in producing drag. As soon as one examines this question it becomes clear that no simple answer can be given unless the spectral distribution of the obstacles is first specified. To meet this point Lliboutry (1968, p. 48) adopts a specific model consisting of four sine waves, and it is instructive to consider this in the context of the present linear model.

(*b*) *Four sine waves*

Suppose the bed consists of four sine waves of wave numbers $k_0, sk_0, s^2k_0, s^3k_0$ and amplitudes a_0, $s^{-1}a_0$, $s^{-2}a_0$, $s^{-3}a_0$ respectively, s being a number. Thus the four waves have the same 'roughness' and their wave numbers are in geometrical progression. Lliboutry takes $s = 10^{1\cdot 5} = 31\cdot 6$. Using formula (40) shows that the drag contributions from the four waves are in the ratios

$$\frac{1}{k_*^2+k_0^2} : \frac{s}{k_*^2+(sk_0)^2} : \frac{s^2}{k_*^2+(s^2k_0)^2} : \frac{s^3}{k_*^2+(s^3k_0)^2}.$$

If the wave number of any of the four waves is k_*, that wave will contribute the greatest drag. Thus, for example, if $s \gg 1$ and if the second wave is at the optimum, so that $sk_0 = k_*$, the ratios are approximately $2s^{-1}:1:2s^{-1}:2s^{-2}$. If the waves are spaced out so that $s = 31\cdot 6$, this implies that 89 % of the total drag is contributed by the wave whose wave number is k_*.

On the other hand, if the waves are more closely spaced, clearly the wave with $k = k_*$ contributes proportionately less. Finally, when we pass to the limit of a continuous spectrum it seems no longer possible to speak of k_* as defining a controlling obstacle size. One reason for this is that there is no clear analogue of the assumption of constant 'roughness' in the case of a continuous spectrum.

The role played by k_* when the bed has a continuous spectrum may be illustrated by two examples.

(*c*) *Uniform power spectrum with sharp cut-off*

Suppose the power spectrum of the bed is

$$|\bar{z}_0(k)|^2 = \begin{cases} \text{constant} & (|k| \leqslant k_m), \\ 0 & (|k| > k_m), \end{cases}$$

where k_m is a cut-off frequency. Formula (31) then gives by straightforward integration

$$\langle \tau_{xz} \rangle = \eta U \langle z_0^2 \rangle k_*^3 \{\gamma - \gamma^{-1} \ln (1+\gamma^2)\},$$

where $\gamma = k_m/k_*$. If $\gamma \ll 1$, the spectrum contains only low frequencies and we find

$$\langle \tau_{xz} \rangle = \tfrac{1}{2} \eta U \langle z_0^2 \rangle k_m^3.$$

The thermal parameters do not appear and the drag is due entirely to viscosity. If $\gamma \gg 1$, so that the spectrum cuts off only at very high frequency, we find

$$\langle \tau_{xz} \rangle = \eta U \langle z_0^2 \rangle k_*^2 k_m = (UL/4CK) \langle z_0^2 \rangle k_m.$$

Here the drag is due entirely to regelation (since η does not appear). Thus $\langle \tau_{xz} \rangle$ is proportional to k_m^3 for small k_m and proportional to k_m for large k_m. For intermediate $k_m (k_m \sim k_*)$ there is a smooth transition between these two types of behaviour. One notices in particular that with a wide, flat spectrum covering the neighbourhood of the wave number k_*, namely the case $k_m \gg k_*$, the formula for the drag becomes independent of k_*. Thus k_* does not play the part of a controlling obstacle size if the spectrum is of this nature.

This power spectrum corresponds to an autocorrelation function

$$c(X) = \frac{\sin k_m X}{k_m X},$$

a function that oscillates with decreasing amplitude as X increases. If the autocorrelation function of a glacier bed were measured it would probably be a little like this but its oscillations would no doubt fall off much faster, and quite possibly it would not oscillate at all. The best guess we can make is perhaps a Gaussian,

$$c(X) = \exp(-\tfrac{1}{4} k_0^2 X^2), \tag{41}$$

say, where k_0^{-1} is a length.

(*d*) *Gaussian autocorrelation function*

If we adopt (41) the power spectrum $|\bar{z}_0(k)|^2$, being proportional to the Fourier transform of $c(X)$, will also be Gaussian, centred on the origin; it is proportional to $\exp(-k^2/k_0^2)$. Formula (31) then gives

$$\langle \tau_{xz} \rangle = 2\eta U \langle z_0^2 \rangle k_*^2 \int_0^\infty \mathrm{d}k\, k^3 (k_*^2 + k^2)^{-1} \exp(-k^2/k_0^2) \Big/ \int_0^\infty \mathrm{d}k \exp(-k^2/k_0^2).$$

The integral in the numerator may be written as

$$\int_0^\infty \mathrm{d}k\, k \exp(-k^2/k_0^2) - k_*^2 \int_0^\infty \mathrm{d}k\, k (k_*^2 + k^2)^{-1} \exp(-k^2/k_0^2)$$
$$= \tfrac{1}{2} k_0^2 - \tfrac{1}{2} k_*^2 \exp(k_*^2/k_0^2)\, E_1(k_*^2/k_0^2),$$

where $E_1(x)$ is the same exponential-integral function as was used earlier (Abramowitz & Stegun, 1964, p. 230, 5.1.28). The denominator equals $\pi^{\frac{1}{2}} k_0$. Hence

$$\langle \tau_{xz} \rangle = \eta U \langle z_0^2 \rangle k_*^3 F_2(\kappa), \tag{42}$$

where $$F_2(\kappa) = 2\pi^{-\frac{1}{2}} \{\kappa - \kappa^{-1} \exp(\kappa^{-2})\, E_1(\kappa^{-2})\} \quad (\kappa = k_0/k_*).$$

For $\kappa \ll 1$, $F_2(\kappa) = 2\pi^{-\frac{1}{2}}\kappa^3$, and for $\kappa \gg 1$, $F_2(\kappa) = 2\pi^{-\frac{1}{2}}\kappa$. Thus for small k_0

$$\langle\tau_{xz}\rangle = 2\pi^{-\frac{1}{2}}\eta U\langle z_0^2\rangle k_0^3, \tag{43}$$

and for large k_0

$$\langle\tau_{xz}\rangle = 2\pi^{-\frac{1}{2}}\eta U\langle z_0^2\rangle k_*^2 k_0 = (UL/2\pi^{\frac{1}{2}}CK)\,k_0.$$

It is hardly surprising that this behaviour at the extremes is identical, apart from numerical factors, with that of the previous example of a flat spectrum with a sharp cut-off. In fact the difference in general behaviour between the drag given by examples (*c*) and (*d*) is quite small, and we may draw the conclusion that the presence of oscillations in $c(X)$ in example (*c*) is not particularly important. The value of the drag is essentially determined by the fall-off distance of the autocorrelation function, k_m^{-1} in example (*c*) and $2k_0^{-1}$ in example (*d*).

In working out the above examples it has been more convenient to use the formula (31) for the drag expressed as an integral over k, rather than the formula (37) which gives the drag as an integral over X, the space variable of the autocorrelation function; but there is no fundamental significance in this. On the contrary, from a physical point of view the integral over X could be regarded as more meaningful, in that the autocorrelation function of a glacier bed is a more direct description than its power spectrum; from this standpoint the formula that contains the power spectrum is used merely as a device for calculation. Whether one accepts this view is ultimately a matter of taste; mathematically, of course, the two formulae are equivalent.

(iv) *The stagnation depth and application to glaciers*

An interesting general feature of the whole problem is illustrated in figure 1*b*. Consider the point P where the averaged velocity profile meets the z axis and let its depth below the surface be D, which we shall call the stagnation depth. From the theory of simple shearing

$$\langle\tau_{xz}\rangle = \eta U/D.$$

But $\langle\tau_{xz}\rangle$ is proportional to ηU in all the formulae, and it therefore follows that the stagnation depth is independent of the shear stress, the velocity and the viscosity. It is determined entirely by geometrical factors, that is, by the topography of the bed and the fixed constant k_*.

For example, the formula (34) for a single sine wave gives

$$D = \frac{1}{A^2k_0}\left(\frac{1}{k_0^2}+\frac{1}{k_*^2}\right),$$

and the most realistic example, the bed with a Gaussian autocorrelation function, gives, from equation (42),

$$D^{-1} = \langle z_0^2\rangle k_*^3 F_2(k_0/k_*).$$

One may guess that for many glacier beds $\frac{1}{2}k_0$, which in this last formula determines the rate of fall-off of the autocorrelation function, is considerably smaller than k_*. That is, the distance over which the bed is autocorrelated, $2k_0^{-1}$, is considerably greater than k_*^{-1}, which was estimated as 8 cm. Where this guess is correct $k_0/k_* \ll 1$, and the appropriate formula is (43), from which

$$D^{-1} = 2\pi^{-\frac{1}{2}}\langle z_0^2\rangle k_0^3, \tag{44}$$

Now temperate glaciers flow by a combination of shearing within the ice and sliding on the bed, in varying proportions. The significance of D is that when D is small compared with the thickness of the glacier the motion is mostly by shearing, whereas when D is large compared with the thickness of the glacier the motion is mostly by sliding. If we use the values $\langle z_0^2\rangle^{\frac{1}{2}} = 10\,\text{cm}$, $k_0^{-1} = 10\,\text{m}$ in (44) we find $D \sim 100\,\text{km}$, which means that motion is almost entirely by sliding. But if $\langle z_0^2\rangle^{\frac{1}{2}} = 1\,\text{m}$ and $k_0^{-1} = 5\,\text{m}$, $D \sim 100\,\text{m}$ which is of the same order of magnitude as the thickness of a glacier. In this case, then, motion would be shared more equally between shearing and sliding. From the scanty observational evidence available it seems that both situations —that is predominantly sliding motion and shared motion— actually occur in temperate glaciers.

Kamb & LaChapelle (1964), who have made direct observations of the process of glacier sliding over bedrock, refer to a detailed theoretical study by Kamb to be reported later. They emphasize that the natural distance scale for transition from regelation slip to plastic slip expresses itself basically in terms of the wavelength of the irregularities rather than in terms of their amplitude. That view is in accord with the conclusions of this paper, where the relevant parameter is the wave number k_*. It would be interesting to know how the other conclusions of this paper stand in relation to the Kamb theory, which one hopes will be published.

The application of the model described here to an actual glacier involves very revealing difficulties of interpretation that are described in a separate paper (Nye 1969). (*Note added in proof.* Further analysis shows that the power spectra of examples (*c*) and (*d*) fall off too sharply at high frequencies to represent a real glacier bed adequately, and that consequently formula (44) and the conclusions drawn from it are wrong. A power spectrum more appropriate to a real glacier bed, and its consequences, are given in Nye (1969).)

3. Bed wavy in both directions

The above analysis for a bed that is wavy only in the x direction may be modified quite simply to cover the general case of a bed that is wavy in both the x and the y directions.

Let the bed now be

$$z = \epsilon f(x, y) = z_0(x, y), \text{ say,}$$

with the mean z_0 equal to zero, and take the x axis parallel to the unperturbed flow, which is $u = U$, $v = 0$, $w = 0$. Consider first how the drag force on the bed is related to the pressure distribution, following the previous treatment on p. 453.

Let (x, y, z_0) be the normal pressure on the bed at the point (x, y, z_0). The drag force $\mathrm{d}\mathbf{F}$ over the element $\mathrm{d}x\,\mathrm{d}y$ in the (x, y) plane is

$$\mathrm{d}\mathbf{F} = p(x, y, z_0)\,\mathrm{grad}\,z_0\,\mathrm{d}x\,\mathrm{d}y,$$

$\mathrm{d}\mathbf{F}$ and grad z_0 being two-dimensional vectors in the (x, y) plane. To first order in ϵ we write $p(x, y, z_0) = \epsilon p_1(x, y, 0)$, as before, and so

$$\mathbf{F} = \epsilon^2 \int\int \mathrm{d}x\,\mathrm{d}y p_1(x, y, 0)\,\mathrm{grad} f. \tag{45}$$

Keeping $z = 0$ from now on, we have, by a reverse Fourier transformation,

$$f = (4\pi^2)^{-1} \int \mathrm{d}\mathbf{k} \bar{f}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}},$$

where $\mathbf{k} = (k_x, k_y)$ is the wave vector of the bed relief, $\mathrm{d}\mathbf{k}$ is the area element in $\mathbf{k}$-space, $\mathbf{r} = (x, y)$ is the position vector in the (x, y) plane and the integral is over the whole of $\mathbf{k}$-space. Then taking the gradient and substituting in (45) gives

$$\mathbf{F} = \mathrm{i}\epsilon^2(4\pi^2)^{-1} \int \mathrm{d}\mathbf{r} p_1(\mathbf{r}) \int \mathrm{d}\mathbf{k}\mathbf{k}\bar{f}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{46}$$

$\mathrm{d}\mathbf{r}$ being written for $\mathrm{d}x\,\mathrm{d}y$; and hence

$$\mathbf{F} = \mathrm{i}\epsilon^2(4\pi^2)^{-1} \int \mathrm{d}\mathbf{k}\mathbf{k}\bar{f}(\mathbf{k})\,\bar{p}_1^*(\mathbf{k}), \tag{47}$$

a formula analogous to (29), giving the drag in terms of the Fourier transform of the pressure distribution.

Now $\bar{p}_1(\mathbf{k})$ may be obtained directly from the previous analysis by the following reasoning. In the first calculation the $\mathbf{k}$ vectors were all parallel to the unperturbed velocity vector $\mathbf{U}$. The new feature here is that there are $\mathbf{k}$ vectors not parallel to $\mathbf{U}$. Consider a bed that contains only one Fourier component, with $\mathbf{k}$ not parallel to $\mathbf{U}$. If the whole bed is now imagined to have a velocity parallel to the corrugations it obviously makes no difference to the solution. Therefore the pressure distribution will be determined by the component of $\mathbf{U}$ in the direction of $\mathbf{k}$; the component of $\mathbf{U}$ perpendicular to $\mathbf{k}$ has no relevance. This means that the previous expression for $\bar{p}_1(k)$ will hold for $\bar{p}_1(\mathbf{k})$ provided we use not U but the component of $\mathbf{U}$ in the direction of $\mathbf{k}$, that is $(\mathbf{U}\,.\,\mathbf{k})/k$. Making this substitution in the first of equations (26) gives

$$\bar{p}_1(\mathbf{k}) = 2\mathrm{i}\eta\bar{f}(\mathbf{k})\,(\mathbf{U}\,.\,\mathbf{k})\,k k_*^2(k_*^2 + k^2)^{-1}$$

as the required expression. Using this in equation (47), and noting that

$$\epsilon^2\bar{f}(\mathbf{k})\bar{f}^*(\mathbf{k}) = |\bar{z}_0(\mathbf{k})|^2,$$

we have the formula for the vector drag in terms of the power spectrum of the bed relief

$$\mathbf{F} = (2\pi^2)^{-1}\eta k_*^2 \int \mathrm{d}\mathbf{k}|\bar{z}_0(\mathbf{k})|^2(\mathbf{U}\,.\,\mathbf{k})\,k\mathbf{k}(k_*^2 + k^2)^{-1}. \tag{48}$$

Since $\mathbf{U} = (U, 0)$ the drag may be written in component form as

$$F_x = (2\pi^2)^{-1}\eta U k_*^2 \int \mathrm{d}\mathbf{k}|\bar{z}_0(\mathbf{k})|^2\,k_x^2 k(k_*^2 + k^2)^{-1}$$

for the longitudinal drag, and

$$F_y = (2\pi^2)^{-1}\eta U k_*^2 \int \mathrm{d}\mathbf{k} |\bar{z}_0(\mathbf{k})|^2 k_x k_y k(k_*^2 + k^2)^{-1}$$

for the transverse drag, that is, the drag component perpendicular to the velocity.

(i) *The drag in terms of the autocorrelation function*

The next step is to turn the integral over $\mathbf{k}$-space into one over (x, y) space, the previous analysis being used as a guide. Thus, we first define a two-dimensional autocorrelation function $c(\mathbf{R})$ of $z_0(\mathbf{r})$ by

$$c(\mathbf{R}) = \int \mathrm{d}\mathbf{r}\, z_0(\mathbf{r})\, z_0(\mathbf{r}+\mathbf{R}) \Big/ \int \mathrm{d}\mathbf{r}\{z_0(\mathbf{r})\}^2,$$

where $\mathbf{r} = (x, y)$ and $\mathbf{R} = (X, Y)$ and the integrals are over the whole of the $\mathbf{r}$ plane. The two-dimensional Fourier transform is

$$\bar{c}(\mathbf{k}) = \int \mathrm{d}\mathbf{r}\, z_0(\mathbf{r}) \int \mathrm{d}\mathbf{R}\, z_0(\mathbf{r}+\mathbf{R})\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{R}} \Big/ \int \mathrm{d}\mathbf{r}\{z_0(\mathbf{r})\}^2.$$

Substitute $\mathbf{r}' = \mathbf{r}+\mathbf{R}$ into the numerator to give $\bar{z}_0^*(\mathbf{k})\,\bar{z}_0(\mathbf{k}) = |\bar{z}_0(\mathbf{k})|^2$, and use the resulting expression for $|\bar{z}_0(\mathbf{k})|^2$ in equation (48). Dividing by the area of the perturbed region we obtain

$$\langle \boldsymbol{\tau} \rangle = (2\pi^2)^{-1}\eta \langle z_0^2 \rangle k_*^2 \int \mathrm{d}\mathbf{k}\, \bar{c}(\mathbf{k})\, (\mathbf{U}\,.\,\mathbf{k})\, k\mathbf{k}(k_*^2 + k^2)^{-1},$$

where $\boldsymbol{\tau} = (\tau_{xz}, \tau_{yz})$. $\langle \boldsymbol{\tau} \rangle$ is therefore the vector drag per unit area.

The longitudinal drag component is

$$\begin{aligned}\langle \tau_{xz} \rangle &= (2\pi^2)^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int \mathrm{d}\mathbf{k}\bar{c}(\mathbf{k})\, k_x^2 k(k_*^2 + k^2)^{-1} \\ &= (2\pi^2)^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int \mathrm{d}\mathbf{k} k_x^2 k(k_*^2 + k^2)^{-1} \int_{-\infty}^{\infty} \mathrm{d}Y \int_{-\infty}^{\infty} \mathrm{d}X\, c(X, Y) \exp\{-\mathrm{i}(k_x X + k_y Y)\}.\end{aligned}$$

Remove the k_x^2 factor by integrating twice by parts with respect to X, thus

$$\langle \tau_{xz} \rangle = -(2\pi^2)^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int \mathrm{d}\mathbf{k} \frac{k}{k_*^2 + k^2} \int_{-\infty}^{\infty} \mathrm{d}Y \int_{-\infty}^{\infty} \mathrm{d}X \frac{\partial^2 c(X, Y)}{\partial X^2} \exp\{-\mathrm{i}(k_x X + k_y Y)\}.$$

Now take (k, ϕ) as polar coordinates for $\mathbf{k}$, measuring ϕ from the direction of $\mathbf{R}$, so that $\mathrm{d}\mathbf{k} = k\,\mathrm{d}k\,\mathrm{d}\phi$. Then

$$\begin{aligned}\langle \tau_{xz} \rangle &= -(2\pi^2)^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int_0^{\infty} \mathrm{d}k \frac{k^2}{k_*^2 + k^2} \int \mathrm{d}\mathbf{R} \frac{\partial^2 c(\mathbf{R})}{\partial X^2} \int_0^{2\pi} \mathrm{d}\phi \exp\{-\mathrm{i}kR\cos\phi\} \\ &= -\pi^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int_0^{\infty} \mathrm{d}k \left(1 - \frac{k_*^2}{k_*^2 + k^2}\right) \int \mathrm{d}\mathbf{R} \frac{\partial^2 c(\mathbf{R})}{\partial X^2} J_0(kR) \qquad (49)\end{aligned}$$

(J_0 being a Bessel function)

$$\begin{aligned}&= -\pi^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int \mathrm{d}\mathbf{R} \frac{\partial^2 c(\mathbf{R})}{\partial X^2} \left\{ \int_0^{\infty} \mathrm{d}k J_0(kR) - k_*^2 \int_0^{\infty} \mathrm{d}k \frac{J_0(kR)}{k_*^2 + k^2} \right\} \\ &= -\pi^{-1}\eta U \langle z_0^2 \rangle k_*^2 \int \mathrm{d}\mathbf{R} \frac{\partial^2 c(\mathbf{R})}{\partial X^2} \left\{ \frac{1}{R} - k_* F_3(k_* R) \right\}, \qquad (50)\end{aligned}$$

where

$$F_3(k_* R) = k_* \int_0^{\infty} \mathrm{d}k J_0(kR)\,(k_*^2 + k^2)^{-1} = \tfrac{1}{2}\pi\{I_0(k_* R) - \mathbf{L}_0(k_* R)\}$$

(Abramowitz & Stegun 1964, p. 488, 11.4.45). $I_0(k_* R)$ is a Bessel function and $\mathbf{L}_0(k_* R)$ is a modified Struve function, $I_0(k_* R) - \mathbf{L}_0(k_* R)$ being tabulated by Abramowitz & Stegun (1964, table 12.1).

Equation (50) is the analogue of equation (37) for the bed wavy only in the x-direction. As $k_* \to 0$ (pure regelation) $k_* F_3(k_* R) \to 0$ and we are left with the first term. Thus, since $k_*^2 = L/4CK\eta$,

$$\langle \tau_{xz} \rangle = -\frac{UL\langle z_0^2 \rangle}{4\pi CK} \int \mathrm{d}\mathbf{R} \frac{\partial^2 c(\mathbf{R})}{\partial X^2} \frac{1}{R}. \tag{51}$$

The behaviour at the other limit, $k_* \to \infty$ (pure viscous flow), is best obtained by returning to equation (49). Express $\mathrm{d}\mathbf{R}$ as $R\,\mathrm{d}\theta\,\mathrm{d}R$, where θ is an azimuth measured from an arbitrary zero, and group the terms so as to give

$$\langle \tau_{xz} \rangle = -\pi^{-1}\eta U \langle z_0^2 \rangle \int_0^\infty \mathrm{d}k k^2 \int_0^{2\pi} d\theta \int_0^\infty dR \frac{\partial^2 c(\mathbf{R})}{\partial X^2} R J_0(kR). \tag{52}$$

$\int_0^\infty \mathrm{d}k\, k^2 J_0(kR)$ diverges at large k and so we must not try to carry out the integration over k at this point. Nor can we proceed by expressing $k^2 J_0(kR)$ in terms of derivatives of $J_0(kR)$ by using Bessel's equation, for this leads to divergence later at $R = 0$. To avoid these difficulties first note the following relation which may be derived from Bessel's equation for $J_0(kR)$:

$$-\frac{1}{k^2}\left(R\frac{\partial}{\partial R}\right) J_0(kR) = \int \mathrm{d}R\, R J_0(kR).$$

This relation allows us to integrate in (52) by parts with respect to R, taking the two members as $\partial^2 c(\mathbf{R})/\partial X^2$ and $RJ_0(kR)$. We get first

$$\frac{1}{k^2}\left[\frac{\partial^2 c(\mathbf{R})}{\partial X^2} R \frac{\partial J_0(kR)}{\partial R}\right]_{R=0}^{R=\infty},$$

which vanishes at the upper limit because $\partial^2 c(\mathbf{R})/\partial X^2$ is zero. It also vanishes at the lower limit because, near $R = 0$, $J_0(kR) = 1 - \frac{1}{4}(kR)^2 + \ldots$, and so

$$R\,\partial J_0(kR)/\partial R = -\tfrac{1}{2}(kR)^2 + \ldots.$$

We are left with

$$\langle \tau_{xz} \rangle = -\pi^{-1}\eta U \langle z_0^2 \rangle \int_0^\infty \mathrm{d}k \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}R \left(\frac{\partial}{\partial R}\frac{\partial^2 c(\mathbf{R})}{\partial X^2}\right) \cdot \left(R\frac{\partial}{\partial R}\right) J_0(kR);$$

the k^2 has cancelled, and divergence at $R = 0$ has been avoided, as we shall see. The only term now depending on k is $J_0(kR)$. So, carrying out the integration over k, which gives R^{-1}, and carrying out the indicated differentiation with respect to R, we get a factor $-R^{-1}\,\mathrm{d}\theta\,\mathrm{d}R = -R^{-2}\,\mathrm{d}\mathbf{R}$. Thus, we finally reach the relatively simple formula

$$\langle \tau_{xz} \rangle = \frac{\eta U \langle z_0^2 \rangle}{\pi} \int \mathrm{d}\mathbf{R} \frac{1}{R^2}\frac{\partial}{\partial R}\frac{\partial^2 c(\mathbf{R})}{\partial X^2} \tag{53}$$

for the drag in the pure viscous case.

The expression of the transverse drag $\langle \tau_{yz} \rangle$ in terms of $c(\mathbf{R})$ follows precisely similar lines except that $\partial^2 c(\mathbf{R})/\partial X^2$ is replaced by $\partial^2 c(\mathbf{R})/\partial X\,\partial Y$ throughout, thus

$$\langle \tau_{yz} \rangle = \frac{\eta U \langle z_0^2 \rangle}{\pi} \int \mathrm{d}\mathbf{R} \frac{1}{R^2} \frac{\partial}{\partial R} \frac{\partial^2 c(\mathbf{R})}{\partial X\,\partial Y}.$$

Isotropic case

If the autocorrelation function of the surface is isotropic, the transverse component of the drag vanishes, and we use equation (50) for the longitudinal component, writing $c(\mathbf{R}) = c(R)$. It is found that, since $c(R)$ is independent of θ,

$$\frac{\partial^2 c(R)}{\partial X^2} = \left(\cos^2\theta \frac{\mathrm{d}^2}{\mathrm{d}R^2} + \sin^2\theta \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R} \right) c(R).$$

Putting this in (50) and carrying out the integration over θ gives the formula

$$\langle \tau_{xz} \rangle = -\eta U \langle z_0^2 \rangle k_*^2 \int_0^\infty \mathrm{d}R \left\{ \frac{1}{R} - k_* F_3(k_* R) \right\} \frac{\mathrm{d}}{\mathrm{d}R} \left(R \frac{\mathrm{d}}{\mathrm{d}R} \right) c(R).$$

In the limit of pure viscous flow ($k_* \to \infty$) for an isotropic bed we use equation (53) and obtain

$$\langle \tau_{xz} \rangle = \eta U \langle z_0^2 \rangle \int_0^\infty \mathrm{d}R \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R} \left\{ \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R} \left(R \frac{\mathrm{d}}{\mathrm{d}R} \right) \right\} c(R). \tag{54}$$

Example

Consider the isotropic, Gaussian autocorrelation function $c(R) = \exp\{-R^2/R_0^2\}$, where R_0 is a constant and suppose, as suggested previously for a realistic example, $R_0 \gg k_*^{-1}$. Formula (54) then gives, without difficulty (and with no divergence at $R = 0$)

$$\langle \tau_{xz} \rangle = 6\pi^{\frac{1}{2}} \eta U \langle z_0^2 \rangle R_0^{-3}.$$

If $2k_0^{-1}$ in the corresponding one-dimensional formula (43) is identified with R_0, the ratio of the drag of the isotropically wavy bed to the drag of the bed that is wavy in only one direction is $3\pi/8 = 1{\cdot}18$. Thus, introducing waviness in the transverse direction, but keeping the mean square amplitude the same, has increased the drag by 18 %.

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