2.6 Two theorems

This section concerns the application of two theorems that were established by Helmholtz (see Lamb, 1932), concerning the flow of viscous fluid of constant density at very low Reynolds number. For a modern discussion, see Batchelor (1967). In principle, Helmholtz's general results for flow in arbitrary connected domains do cover interstitial flow in permeable media with arbitrary distributions of (positive) permeability, but direct proofs for Darcy flow are very much simpler. Despite their antiquity, the theorems seem to have been largely overlooked in this area, though they do offer the potential for many conceptual insights and practical applications.

2.6.1 The uniqueness theorem

The uniqueness theorem established below asserts that there cannot be more than one Darcy flow solution for *constant density flow* with given boundary conditions – the solution is unique. The immediate utility of this theorem is that, if one finds a solution to such a problem, there are no others. The solution cannot develop a bifurcation or an instability. The theorem is true no matter what the internal permeability distribution may be. Note, however, that this uniqueness applies to fluids of constant density; it *does not extend to situations in which buoyancy forces are involved.* In fact, we know by examples that it is not true in these situations. One simple such case involves a saturated permeable region with horizontal isopycnals (lines of constant density). One solution is a state of rest – the velocity is everywhere zero. When fresh water lies over more saline water, the density decreases with height and the state of rest is stable – if it is disturbed slightly, it will return to its initial state of rest. However, when the salinity and density *increase* with height, the state of rest is unstable; tiny perturbations amplify and the system moves to a new time-dependent or steady-state solution.

To prove the uniqueness for constant density Darcy flow, consider the flow through a medium in which the permeability $k(\mathbf{x})$ is an arbitrary (but positive) function of position. In the absence of buoyancy forces,

$$\mathbf{u} = -(k(\mathbf{x})/\mu)\nabla p, \tag{2.41}$$

where p is the reduced pressure. The incompressibility condition is $\nabla \cdot \mathbf{u} = 0$. Let us suppose that the theorem is false, that two different patterns of flow and pressure $(\mathbf{u}, p)$ and $(\mathbf{u}', p')$ are possible in a given region V, both satisfying (2.41) and the incompressibility condition, with assigned distributions of either pressure or normal component of transport velocity on the boundaries S;

$$p = p' \quad \text{or} \quad \mathbf{u} \cdot d\mathbf{S} = \mathbf{u}' \cdot d\mathbf{S} \text{ on the boundary surface } S. \tag{2.42}$$

We prove that the two solutions are, in fact, the same. Consider the following integral throughout the volume

$$\begin{aligned}
\int k(\mathbf{x})[\nabla p - \nabla p']^2 dV
&= \int k(\mathbf{x})[\nabla p - \nabla p'] \cdot [\nabla p - \nabla p'] dV, \\
&= -\mu \int (\mathbf{u} - \mathbf{u}') \cdot \nabla (p - p') dV, \text{ from (2.6.1)}, \\
&= -\mu \int \nabla \cdot [(\mathbf{u} - \mathbf{u}')(p - p')] dV, \text{ from incompressibility}, \\
&= -\mu \int (p - p')(\mathbf{u} - \mathbf{u}') \cdot d\mathbf{S}, \text{ from the divergence theorem} \\
&= 0,
\end{aligned} \tag{2.43}$$

the last step expressing the identity of boundary conditions (2.42). The first integral (2.43) therefore vanishes and since $k(\mathbf{x})$ is everywhere positive, it follows that the integrand must be zero and consequently $\nabla p = \nabla p'$ everywhere. The two solutions are identical and the theorem is true. Solutions are unique.

2.6.2 The minimum dissipation theorem

The minimum dissipation theorem is not only conceptually important but also useful in some practical situations by providing a means of inferring, without detailed calculation, the general nature of flow patterns in perhaps complex geological situations. The theorem statement is as follows. In a region occupied by a permeable medium in which buoyancy effects are negligible, if the transport velocity distribution over the boundary of the region is prescribed, then the actual internal flow has a total rate of dissipation of energy that is less than any other conceivable, kinematically possible flow in the same region with the same boundary conditions. (A kinematically possible flow is any velocity field one might imagine that satisfies the incompressibility condition, but not necessarily the Darcy equation. The actual flow, of course, satisfies both.) For example, in a near-surface groundwater flow, rainwater infiltrates downward to the water table at a rate that can be regarded as given, moves a possibly significant distance as groundwater through a matrix with a complex distribution of permeability and ultimately discharges into streams or lakes. Whatever the internal permeability structure may be, this remarkable theorem asserts that the patterns of groundwater flow speed and direction are those which minimize the overall dissipation rate.

The proof is as follows. Let $\mathbf{u}(\mathbf{x})$, $p(\mathbf{x})$ represent the true solution with $\mathbf{u}$ prescribed on the boundary S. Consider a kinematically possible alternative flow that one might dream up, $\mathbf{u}+\mathbf{u}'$, $p+p'$, with the same velocities at the boundary, so that

$$\mathbf{u}' = 0 \text{ on } S. \tag{2.44}$$

This alternative flow also satisfies the incompressibility condition so that $\nabla \cdot \mathbf{u}' = 0$, but it may not satisfy the Darcy equation. From (2.36), the total rate of energy dissipation in the alternative flow is

$$\begin{aligned}\int \varepsilon dV &= \mu \int \{(\mathbf{u}+\mathbf{u}')^2/k\} dV,\\ &= \mu \int \{(\mathbf{u}^2+\mathbf{u}'^2)/k\} dV + 2\mu \int (\mathbf{u}\cdot\mathbf{u}'/k) dV.\end{aligned}$$

Since the true solution does satisfy the Darcy equation $\mu\mathbf{u}/k = -\nabla p$, the last term above becomes

$$\begin{aligned}2\int (\nabla p)\cdot \mathbf{u}' dV &= 2\int \nabla\cdot(p\mathbf{u}')dV && \text{since } \nabla\cdot\mathbf{u}'=0,\\ &= 2\int p\mathbf{u}'\cdot dS && \text{from the divergence theorem,}\\ &= 0, && \text{from (2.44).}\end{aligned}$$

Accordingly, the total dissipation rate in the alternative flow is $\mu \int \{(\mathbf{u}^2+\mathbf{u}'^2)/k\}dV$, which is greater than that of the true flow, namely $\mu \int (\mathbf{u}^2/k)dV$. This establishes the theorem.

It is a very important and far-reaching result. Examination of the derivation above confirms that it remains true if the permeability $k(\mathbf{x})$ in the region is not uniform provided only that the velocity distribution across the boundaries is specified. Regions or lenses of low permeability are generally sites of proportionately low flow velocities, and because of the quadratic dependence of the dissipation on $\mathbf{u}$, they contribute relatively little to the overall dissipation. The flow occurs preferentially in the high-permeability, low-resistance regions, attesting to the accuracy of the old adage that "flow follows the path of least resistance." It is a robust theorem, finding a number of applications in later sections of this book.

2.7 The thermal energy balance

The distribution of temperature in the fabric is constrained by an equation describing the heat balance in each averaging volume – an expression of the first law of thermodynamics. Let C represent the specific heat at constant pressure; we will use subscripts "M" and "F" to refer to properties of the saturated matrix as a whole and to those of the fluid, respectively. The rate of change in time of the heat content in any element of unit volume is $(\rho C_M)\partial T/\partial t$, and this may come about as a result of