OPERATOR ALGEBRAS ON L^p SPACES WHICH "LOOK LIKE" C*-ALGEBRAS

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1. Introduction

This talk is a survey of operator algebras on L^p spaces which "look like" C*-algebras. Its aim is to give a general impression of the field, and why I am hopeful that there is a rich theory waiting to be discovered. It does not aim to have you remember technical details, or even the precise statements of all the theorems.

This file contains more than will be in the talk. Warning: It has not been properly proofread.

Throughout, we assume $p \in [1, \infty)$ unless otherwise specified.

Definition 1.1. An L^p operator algebra is an L^p matrix normed Banach algebra which is completely isometrically isomorphic to a nondegenerate norm closed subalgebra of $L(L^p(X, \mu))$ for some measure space $(X, \mathcal{B}, \mu)$.

This definition is not the same as what is written in the papers so far.

In this talk, we will require the measure spaces to be σ-finite and have separable L^p spaces, and we will usually require that the algebras be unital. We mostly suppress mention of the matrix norms.

It is necessary to distinguish between (completely) isometric isomorphism of L^p operator algebras and just (complete) isomorphism (the existence of a (completely) bounded homomorphism from one algebra to the other with a (completely) bounded inverse).

We want L^p operator algebras which "look like" C*-algebras (even though there is no adjoint). As defined, all nonselfadjoint operator algebras on L^2 spaces are L^2 operator algebras, so additional restrictions are needed. It is not clear what conditions should be imposed. We have a number of examples which certainly seem to "look like" C*-algebras. Out of a number of possible conditions, here are the two which, at this early stage, seem to be most relevant, in that they have actually been used.

The "strong" condition is as follows.

Definition 1.2. A unital L^p operator algebra is p-incompressible if whenever $\varphi\colon A \to L(L^p(X, \mu))$ is a unital contractive homomorphism, then φ induces an isometric homomorphism $\overline{\varphi}\colon A/\mathrm{Ker}(\varphi) \to L(L^p(X, \mu))$.

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The correct version of the definition should probably have "completely contractive" and "completely isometric" in place of "contractive" and "isometric".

Warning: We don't know whether quotients of L^p operator algebras by closed ideals are again L^p operator algebras. The best known result is that they can be completely isometrically represented on subspaces of L^p spaces (Marius Junge, Habilitationsschrift).

Here is the "weak" condition.

Definition 1.3. A unital L^p operator algebra is isometrically generated if the closed linear span of the invertible isometries in A is norm dense in A.

(Suggestions for a better word?)

Isometrically generated doesn't imply incompressible. (See Corollary 6.8 below.) I hope that incompressible implies isometrically generated, or at least something close, but I don't know, and it may well be false.

An isometrically generated L^2 operator algebra is easily seen to be a C*-algebra. I don't know whether an incompressible L^2 operator algebra is isometrically generated. I do know (gotten from Garth Dales) that every C*-algebra A is *strongly incompressible* in the following sense: whenever B is a Banach algebra and $\varphi\colon A \to B$ is a contractive homomorphism, then φ induces an isometric homomorphism $\overline{\varphi}\colon A/\mathrm{Ker}(\varphi) \to B$. Also, the nonselfadjoint L^2 operator algebra of 2×2 upper triangular matrices is not incompressible. For every $p \in [1, \infty)$, the disk algebra

$$A(D) = \big\{ f \in C(\overline{D})\colon f|_D \text{ is holomorphic} \big\}$$

is an L^p operator algebra which is not p-incompressible.

One key technical theorem, which partially substitutes for the adjoint on a C*-algebra, is Lamperti's Theorem. It characterizes isometries on L^p spaces of σ-finite measure spaces, for $p \in (0, \infty) \setminus \{2\}$. Here is a special case, which illustrates that there are many fewer isometries than in C*-algebras. For $k \in \mathbb{Z}$ let $\delta_k \in l^p(\mathbb{Z})$ be the standard basis vector associated with k.

Proposition 1.4. Let $u \in L(l^p(\mathbb{Z}))$ be an invertible isometry. Then there is a permutation $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ and a function $\lambda\colon \mathbb{Z} \to S^1$ such that for all $k \in \mathbb{Z}$ we have $u\delta_k = \lambda(k)\delta_{\sigma(k)}$.

2. Matrix algebras

Set $l_n^p = l^p(\{1, 2, \ldots, n\})$, using counting measure on $\{1, 2, \ldots, n\}$. Then define $M_n^p = L(l_n^p)$, equipped with the operator norm. (We also call this the "spatial norm".) Clearly M_n^p an L^p operator algebra. By Proposition 1.4, for $p \neq 2$ its invertible isometries are exactly the $n \times n$ "complex permutation matrices", that is all $n \times n$ matrices which have exactly one nonzero entry in each row and in each column, and such that all nonzero entries have absolute value 1. We immediately get the following result.

Proposition 2.1. Let $p \in [1, \infty)$. Then M_n^p is isometrically generated.

We also have:

Theorem 2.2. Let $p \in [1, \infty)$. Then M_n^p is p-incompressible.

More is true. Let $p \in [1,\infty) \setminus \{2\}$. Then, up to conjugacy by an invertible isometry, every contractive unital homomorphism $\rho\colon M_n^p \to L(L^p(X,\mu))$ is of the form

$$a \mapsto a \otimes 1 \in L\big(l_n^p \otimes_p L^p(X_0,\mu_0)\big) = L\big(L^p(\{1,2,\ldots,n\} \times X_0)\big).$$

Moreover, the invertible isometry identifies the sets $\{k\} \times X_0$ with disjoint subsets of X.

The proof uses Lamperti's Theorem.

It follows that a contractive unital homomorphism $\rho\colon M_n^p \to L(L^p(X,\mu))$ is completely contractive.

Question 2.3. Is M_n^d strongly incompressible?

With suitable modifications, one can get similar results for the "obvious" direct limit $\overline{M}_\infty^p$. For $p \in (1,\infty)$ this direct limit is $K(l^p(\mathbb{Z}_{>0}))$, but for $p = 1$ it is strictly smaller.

Nobody has looked at $K(L^p([0,1]))$.

For $p \in [1,\infty) \setminus \{2\}$, the algebra $L(l^p(\mathbb{Z}_{>0}))$ is not isometrically generated. For $p = 1$, the closed linear span of the invertible isometries does not even contain all rank one operators.

3. $C(X)$

Let X be a compact metric space. Then $C(X)$, with its usual norm, is recognized as an L^p operator algebra by choosing a finite Borel measure μ on A with full support and letting $C(X)$ act in $L^p(X,\mu)$ via multiplication operators.

The algebra $C(X)$ is easily seen to be isometrically generated. Also, $C(X)$ is even strongly incompressible because it is a C*-algebra. Actually, more is true for representations on L^p spaces.

Theorem 3.1. Let $p \in [1,\infty) \setminus \{2\}$, let X be a compact metric space, let $(Y,\mathcal{C},\nu)$ be a σ-finite measure space, and let $\rho\colon C(X) \to L(L^p(Y,\nu))$ be a contractive unital homomorphism. Then there is a unital *-homomorphism $\varphi\colon C(X) \to L^\infty(Y,\nu)$ such that, for every $f \in C(X)$, the operator $\rho(f)$ is the multiplication operator by $\varphi(f)$.

The proof uses Lamperti's Theorem.

It is well known that $C(X)$ is an amenable Banach algebra. (One definition of amenability: existence of an "approximate diagonal".)

An L^p operator algebra A is *p-nuclear* (a definition of An, Lee, and Ruan) if its identity map is a pointwise norm limit of maps which factor as

$$A \xrightarrow{T} M_n^p \xrightarrow{S} A$$

(for varying n), in which S and T are completely contractive. (A result of Smith states that for C*-algebras, this condition is equivalent to nuclearity.) Although nobody has written it down, it should not be hard to prove directly that $C(X)$ is p-nuclear.

4. L^p UHF and AF algebras

A spatial L^p UHF algebra is the (completed) direct limit of a UHF type direct system $(M_{d(n)})_{n\in\mathbb{Z}_{\geq 0}}$ (that is, $d(0)|d(1)|\cdots$ and all the maps are unital) in which

$M_{d(n)}$ is given the spatial L^p operator norm and, in addition, all the maps of the system are isometric. Thus, we can write it as $\varinjlim M^p_{d(n)}$.

Theorem 4.1. A direct limit of L^p operator algebras with completely contractive maps is an L^p operator algebra.

Since we don't know an abstract characterization of L^p operator algebras, the C* proof doesn't work. One uses the fact that ultraproducts of L^p spaces are L^p spaces.

Theorem 4.2. A spatial L^p UHF algebra is simple and has a unique normalized trace.

The C* proof of simplicity does not work, because we do not know that a quotient of an L^p operator algebra by a closed ideal is an L^p operator algebra. One must work harder, using conditional expectations carefully.

The argument works with somewhat more general norms on $M_{d(n)}$, and gives some interesting L^p operator algebras which do not look like C*-algebras.

Theorem 4.3 (Glimm classification)**.** For fixed $p \in [1, \infty)$, two spatial L^p UHF algebras are isometrically isomorphic if and only if they have isomorphic scaled ordered K_0-groups.

Theorem 4.4. If A_j is a spatial L^{p_j} UHF algebra for $j = 1, 2$, and $p_1 \neq p_2$, then A_1 is not isomorphic to A_2.

The proof uses results on the structure of L^p spaces.

Theorem 4.5. A spatial L^p UHF algebra is p-incompressible and isometrically generated.

Theorem 4.6. A spatial L^p UHF algebra is amenable and p-nuclear.

The proofs of the last two results are easy.

It is very likely (work in progress, with Maria Grazia Viola) that there is a similar theory of spatial L^p AF algebras, including Elliott K-theoretic classification and with closed ideals in one to one correspondence with order ideals of the K_0-group. (Care is needed with the condition on nonunital maps in the system. It seems likely that not all isometric maps can be allowed.)

5. L^p analogs of Cuntz algebras

Let $p \in [1, \infty)$ and let $d \in \{2, 3, 4, \ldots\}$. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Recall that M^p_d is M_d equipped with the operator norm it gets by acting on $l^p\big(\{1, 2, \ldots, d\}\big)$. We consider the closed subalgebra of $L(L^p(X, \mu))$ generated by elements $s_1, s_2, \ldots, s_d, t_1, t_2, \ldots, t_d$ satisfying the algebraic relations

(1) $t_j s_j = 1$ for $j \in \{1, 2, \ldots, d\}$,
(2) $t_j s_k = 0$ for $j, k \in \{1, 2, \ldots, d\}$ with $j \neq k$,
(3) $\sum_{j=1}^{d} s_j t_j = 1$,

and the analytic relations

(4) s_j is an isometry for $j \in \{1, 2, \ldots, d\}$,
(5) the map from M^p_d to $L(L^p(X, \mu))$ which sends the standard matrix unit $e_{j,k}$ to $s_j t_k$ is isometric.

It is easy to write down examples of such operators using shifts on $l^p(\mathbb{Z}_{\geq 0})$.

Theorem 5.1 (Uniqueness)**.** Any two Banach algebras generated by elements $s_1, s_2, \ldots, s_d, t_1, t_2, \ldots, t_d$ as above are isometrically isomorphic.

Call this algebra $\mathcal{O}_d^p$. Then, moreover:

Theorem 5.2 (Further uniqueness)**.** $\mathcal{O}_d^p$ is p-incompressible.

For $p \neq 2$, the proofs depend on Lamperti's Theorem and additional ideas, and do not work in the case $p = 2$.

Theorem 5.3. $\mathcal{O}_d^p$ is simple.

This proof follows the C*-algebraic proof (with much more to check), and uses the fact that the d^∞ L^p UHF algebra is simple.

As for L^p UHF algebras, we can't get either of simplicity and uniqueness from the other.

Theorem 5.4. $\mathcal{O}_d^2$ is the usual Cuntz algebra.

Theorem 5.5. $\mathcal{O}_d^p$ is isometrically generated.

Theorem 5.6. $\mathcal{O}_d^p$ is purely infinite in a sense suitable for Banach algebras.

Theorem 5.7. $\mathcal{O}_d^p$ is amenable as a Banach algebra.

Problem 5.8. Is $\mathcal{O}_d^p$ p-nuclear?

Theorem 5.9. $\mathcal{O}_d^p$ has the same K-theory as when $p = 2$.

Theorem 5.10. $\mathcal{O}_d^p$ is an exchange ring.

(A C*-algebra is an exchange ring if and only if it has real rank zero. However, for $p \neq 2$ we don't know whether L^p UHF algebras are exchange rings.)

Theorem 5.11. For $p_1 \neq p_2$ and any d_1 and d_2, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $\mathcal{O}_{d_2}^{p_2}$.

The following is the original motivation for looking at L^p operator algebras, and is part of work in progress with Guillermo Cortiñas.

Conjecture 5.12. The comparison map $K_*^{\mathrm{alg}}(\mathcal{O}_d^p) \to K_*^{\mathrm{top}}(\mathcal{O}_d^p)$ is an isomorphism in all degrees.

Problem 5.13. Is $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p$ (spatial L^p tensor product) isomorphic to $\mathcal{O}_2^p$?

6. Group L^p operator algebras

Let G be a locally compact group. The *group L^p operator algebra* $F^p(G)$ is defined similarly to the group C*-algebra $C^*(G)$, but is the universal L^p operator algebra for isometric representations of G on L^p spaces. The *reduced group L^p operator algebra* $F^p_{\mathrm{r}}(G)$ is the L^p operator algebra specifically associated with the regular representation of G on $L^p(G)$. Both these algebras (and the crossed products described in the next section) are obtained from a general theory of crossed products of Banach algebras due to Dirksen, de Jeu, and Wortel. We assume that G is second countable (so that Haar measure is σ-finite), although here that isn't really necessary.

Theorem 6.1. If G is amenable, then the map $F^p(G) \to F^p_{\mathrm{r}}(G)$ is an isometric isomorphism.

Theorem 6.2. If $p \neq 1$ and the map $F^p(G) \to F^p_{\mathrm{r}}(G)$ is a isomorphism, then G is amenable.

My proof of Theorem 6.2 uses a generalization of Choi's multiplicatve domain result for completely positive maps of C*-algebras. But the result follows easily by entirely different methods from earlier work of Neufang and Runde on p-Figà-Talamanca-Herz algebras. (See Gardella-Thiel.)

For $p = 1$, the result is false. For every locally compact group G, we have $F^1(G) = F^1_{\mathrm{r}}(G) = L^1(G)$.

Theorem 6.3 (An-Lee-Ruan)**.** If G is discrete and amenable, then $F^p(G)$ is p-nuclear.

Combining several results above and some not yet stated (and omitting some definitions), one gets the following collection of equivalent conditions for amenability of a discrete group.

Theorem 6.4. Let $p \in (1, \infty)$, and let G be a countable discrete group. Then the following are equivalent:

(1) G is amenable.
(2) The map $F^p(G) \to F^p_{\mathrm{r}}(G)$ is an isometric isomorphism.
(3) There is a nonzero homomorphism $F^p_{\mathrm{r}}(G) \to \mathbb{C}$.
(4) $F^p_{\mathrm{r}}(G)$ is an amenable Banach algebra.
(5) $F^p(G)$ is an amenable Banach algebra.
(6) For any separable L^p operator algebra B, the map $F^p_{\mathrm{r}}(G) \otimes_{\max,p} B \to F^p_{\mathrm{r}}(G) \otimes_{\min,p} B$ is a completely isometric isomorphism.
(7) The canonical map $F^p_{\mathrm{r}}(G) \otimes_{\max,p} F^p_{\mathrm{r}}(G) \to F^p_{\mathrm{r}}(G) \otimes_{\min,p} F^p_{\mathrm{r}}(G)$ is an isomorphism (not necessarily isometric).

Theorem 6.5 (Pooya-Hejazian)**.** If G is a Powers group (such as F_n), then $F^p_{\mathrm{r}}(G)$ is simple and has a unique normalized trace.

The comparison for different values of p takes a different form than in the situations above.

Theorem 6.6 (Gardella-Thiel)**.** Let G be discrete and amenable, and suppose that $1 \leq p_1 \leq p_2 \leq 2$. Then the obvious map on group elements extends to an injective contractive homomorphism $F^{p_1}(G) \to F^{p_2}(G)$. If G is infinite and $p_1 < p_2$ then this map is not surjective.

To get at exponents greater than 2, use the duality $L^p(G)' \cong L^q(G)$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ to see that $F^q(G)$ is antiisomorphic to $F^p(G)$. Since a group algebra is antiisomorphic to itself (using $g \mapsto g^{-1}$), this gives $F^q(G) \cong F^p(G)$.

Theorem 6.7. If G is discrete abelian, then the maximal ideal space of $F^p(G)$ is canonically isomorphic to $\widehat{G}$, and the map $F^p(G) \to C\big(\widehat{G}\big)$ is injective and has dense range.

Presumably one doesn't need G to be discrete, but nobody has checked yet.

If G is infinite, then this map is not surjective (by Theorem 6.6). Since $C\big(\widehat{G}\big)$ is an L^p operator algebra, we therefore get:

Corollary 6.8. If G is discrete, abelian, and infinite, and $p \in [1, \infty) \setminus \{2\}$, then $F^p(G)$ is not p-incompressible.

However, it is trivially true that $F^p(G)$ and $F^p_{\mathrm{r}}(G)$ are isometrically generated for any discrete group.

Question 6.9. Suppose $p \in (1, 2)$ and G is discrete, abelian, and infinite. Does it follow that $F^p(G)$ has spectral synthesis?

This means: Does $F^p(G)$ have "the same" closed ideals as $C\big(\widehat{G}\big)$? For $p = 2$ this is trivially true, and for $p = 1$ it is known to be false.

7. L^p OPERATOR CROSSED PRODUCTS

Let G be a locally compact group, let A be an L^p operator algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action via complete isometries. Then there are full and reduced L^p operator crossed products $F^p(G, A, \alpha)$ and $F^p_{\mathrm{r}}(G, A, \alpha)$. Like the full and reduced group L^p operator algebras, these can be obtained from the general theory of crossed products of Banach algebras of Dirksen, de Jeu, and Wortel.

Theorem 7.1. If G is amenable, then $F^p(G, A, \alpha) \to F^p_{\mathrm{r}}(G, A, \alpha)$ is an isometric isomorphism.

Theorem 7.2. If G is discrete amenable and A is an amenable Banach algebra, then $F^p(G, A, \alpha)$ is an amenable Banach algebra.

If G is discrete and A is isometrically generated, then $F^p(G, A, \alpha)$ and $F^p_{\mathrm{r}}(G, A, \alpha)$ are clearly isometrically generated. In general, we don't know anything about p-incompressibility.

Theorem 7.3. If G is discrete, there is a canonical Banach conditional expectation $E\colon F^p_{\mathrm{r}}(G, A, \alpha) \to A$, and it is faithful in a suitable sense.

Theorem 7.4. Assume $p \neq 1$. If G is discrete, $A = C(X)$, and the action of G on X is minimal and essentially free, then $F^p_{\mathrm{r}}(G, A)$ is simple.

We don't know what happens for $p = 1$. The requirement $p \neq 1$ comes from the use of the L^p version of the Choi multiplicative domain theorem. That theorem is actually false for $p = 1$.

Theorem 7.5. If G is discrete, $A = C(X)$, and the action of G on X is free, then all normalized traces on $F^p_{\mathrm{r}}(G, A, \alpha)$ come from invariant probability measures on X.

Gardella and Thiel have looked specifically at L^p versions of irrational rotation algebras. They consider crossed products by rotation on the circle, and also crossed products by rotation on $F^p(\mathbb{Z})$. There are analogs of Rieffel projections in both kinds of crossed products, so that one gets (using Theorem 7.6 below) the same scaled ordered K_0-group as in the C* case. The crossed product by rotation on $F^p(\mathbb{Z})$ is clearly not p-incompressible.

If G is abelian, then there is a dual action of $\widehat{G}$ on $F^p(G, A, \alpha)$. But little is known towards Takai duality, mainly because nobody has looked. For discrete but not necessarily abelian G, and for $p \neq 1$, we do know that $F^p(G, C_0(G)) \cong K(l^p(G))$ (and we know the answer for $p = 1$). We don't know what happens if G is not discrete.

Theorem 7.6. Let $p \in [1, \infty)$, let A be a L^p operator algebra, and let $\alpha \in \mathrm{Aut}(A)$ be isometric. Then we have the following natural six term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-(\alpha^{-1})_*} & K_0(A) & \longrightarrow & K_0\big(F^p(\mathbb{Z}, A, \alpha)\big) \\ {\scriptstyle\partial}\big\uparrow & & & & \big\downarrow{\scriptstyle\partial} \\ K_1\big(F^p(\mathbb{Z}, A, \alpha)\big) & \longleftarrow & K_1(A) & \xleftarrow{\mathrm{id}-(\alpha^{-1})_*} & K_1(A). \end{array} \tag{7.1}$$

The computation of $K_*(\mathcal{O}_d^p)$ was done by realizing its stabilization as a crossed product by an automorphism of a stabilized L^p UHF algebra. This was the original motivation for looking at L^p operator crossed products. (Cuntz's original calculation of $K_*(\mathcal{O}_d)$ does not work for $p \neq 2$.)

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