COURSE OUTLINE FOR MATH 684–686 FALL 2008 THROUGH SPRING 2009

Course title: Operator theory and C*-algebras.

References: [1], [2], [3], [4], [5]. (I will *not* ask people to buy all of these. I will primarily follow only one or two of them, and arrange to have all of them on reserve at the library.)

Summary:

- Some basic results of operator theory, especially Fredholm operators and index theory, spectrum, and the spectral theorem for selfadjoint operators on Hilbert space.
- Some basic results on Banach algebras, especially functional calculus and the Gelfand transform for commutative Banach algebras. (The Gelfand transform simultaneously generalizes Fourier series, the Fourier transform, the Poisson integral formula, and the fact that if X is compact then X can be recovered from the algebra $C(X)$ of continuous complex valued functions on X.)
- The basic theory of C*-algebras, including the basics of their representation theory.
- K-theory for Banach algebras and C*-algebras. (This is a generalization of the K-theory that algebraic topologists make from vector bundles. It is where "noncommutative" index theory lives. No previous knowledge of algebraic topology is required.)
- Topics from group C*-algebras and discrete crossed products, including connections with dynamical systems.

There are many related topics that I will not have time to treat, but I will at least try to mention the existence of some of them.

Prerequisite: Math 616–618 or equivalent.

Tentative course outline:

- A little review of Banach spaces and Hilbert spaces. Quotient spaces.
- The weak and weak* topologies, Alaoglu's Theorem, Krein-Milman Theorem.
- Basic definitions of linear operators; examples.
- Compact operators. Fredholm operators; Fredholm index. (I will keep things simple by proving things only for Fredholm operators between Hilbert spaces. Sections 11.2, 11.3, and 11.5 of Conway's book treat semi-Fredholm operators between Hilbert spaces, and Section 1.4 of Murphy's book treats Fredholm operators between Banach spaces.)
- Basic definitions of Banach algebras; many examples. (See Section 1.1 of Murphy's book and Sections 7.1 and 7.2 of Conway's book.)

- Spectrum, spectral radius, holomorphic functional calculus. (The proofs for holomorphic functional calculus are optional. See Section 1.2 of Murphy's book and Sections 7.3 through 7.6 of Conway's book.)
- Maximal ideal space of a commutative Banach algebra, Gelfand transform (unital and nonunital cases). (See Section 1.3 of Murphy's book and Section 7.8 of Conway's book.)
- C*-algebras, characterization of commutative C*-algebras.
- The basic general theory of C*-algebras: Continuous functional calculus, positivity, approximate identities, ideals and quotients, states and representations, double commutant and Kaplansky density theorems. (See Chapter 8 of Conway's book, Sections 2.1, 2.2, and 3.1 through 3.4 of Murphy's book, Chapter 1 of Arveson's book, and Chapter 1 of Davidson's book.)
- The spectral theorem and bounded Borel functional calculus.
- Type I C*-algebras and discussion (with most proofs omitted) of their representation theory. (See Arveson's book.)
- Discussion of very basic material on von Neumann algebras. (Most proofs omitted.)
- AF algebras. (See Sections 6.1 and 6.2 of Murphy's book and Chapter 3 of Davidson's book.) Discussion, without proofs, of AH algebras and their stable and real rank.
- The ordered K_0-group of a C*-algebra (or a Banach algebra) and the classification of AF algebras. (I do not expect to give complete proofs for the classification. See Sections 7.1 through 7.4 of Murphy's book, noting that his notation is not standard. Also see Chapter 4 of Davidson's book.)
- The K_1-group of a C*-algebra (or a Banach algebra) and Bott periodicity (using Atiyah's version of the Toeplitz operator proof). (See Section 7.5 of Murphy's book, but he gives a slicker proof of Bott periodicity than the one I am intending to use.) Index theory for Fredholm operators on Hilbert modules.
- The C*-algebra of a locally compact group and its relation to the unitary representation theory of the group. (Material selected from Chapter 7 of Davidson's book.)
- Crossed products by discrete groups, with emphasis on crossed products by minimal homeomorphisms. (Material selected from Chapters 6 and 8 of Davidson's book.)

References

[1] W. B. Arveson, *An Invitation to C*-algebras*, Springer-Verlag Graduate Texts in Math. no. 39, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[2] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer-Verlag Graduate Texts in Math. no. 96, Springer-Verlag, New York, Berlin, etc., 1990.

[3] K. R. Davidson, *C*-Algebras by Example*, Fields Institute Monographs no. 6, Amer. Math. Soc., Providence RI, 1996.

[4] G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, Boston, San Diego, New York, London, Sydney, Tokyo, Toronto, 1990.

[5] H. Lin, *An Introduction to the Classification of Amenable C*-algebras*, World Scientific, River Edge NJ, 2001.